Chapter 8: The Orthogonality of Al-Salam–Carlitz Polynomials for Complex Parameters


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Chapter 8

The Orthogonality of Al-Salam–Carlitz Polynomials for Complex Parameters

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In this chapter, we study the orthogonality conditions satisfied by Al-Salam–Carlitz polynomials $U^{(a)}_n(x;q)$ when the parameters $a$ and $q$ are not necessarily real nor “classical”, i.e., the linear functional $u$ with respect to such a polynomial sequence is quasi-definite and not positive definite. We establish orthogonality on a simple contour in the complex plane which depends on the parameters. In all cases we show that the orthogonality conditions characterize the Al-Salam–Carlitz polynomials $U^{(a)}_n(x;q)$ of degree $n$ up to a constant factor. We

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also obtain a generalization of the unique generating function for these polynomials.

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1. Introduction

The Al-Salam–Carlitz polynomials \(U_n(a)(x; q)\) were introduced by Al-Salam and Carlitz in [1] as follows:

\[
U_n(a)(x; q) := (-a)^n q^{n \binom{n}{2}} \sum_{k=0}^{n} \left(\frac{q^{-n}; q}{(q; q)_k}\right) \frac{(x^{-1}; q)_k q^k x^k}{a^k}.
\] (1.1)

In fact, these polynomials have a Rodrigues-type formula [2, (3.24.10)]

\[
U_n(a)(x; q) = a^n q^{n \binom{n}{2}} (1 - q^n) \frac{n!}{q^n w(x; a; q)} D_{q^{-1}} f(x; a; q),
\]

where

\[
w(x; a; q) := (qx; q)_\infty (qx/a; q)_\infty,
\]

the \(q\)-Pochhammer symbol (\(q\)-shifted factorial) is defined as

\[
(z; q)_0 := 1, \quad (z; q)_n := \prod_{k=0}^{n-1} (1 - zq^k),
\]

\[
(z; q)_\infty := \prod_{k=0}^{\infty} (1 - zq^k), \quad |z| < 1,
\]

and the \(q\)-derivative operator is defined by

\[
D_q f(z) := \begin{cases} 
\frac{f(qz) - f(z)}{(q - 1)z} & \text{if } q \neq 1 \text{ and } z \neq 0, \\
f'(z) & \text{if } q = 1 \text{ or } z = 0.
\end{cases}
\]

**Remark 1.1.** Observe that by the definition of the \(q\)-derivative

\[
D_{q^{-1}} f(z) = D_q f(qz), \quad \text{and } D_n^{n-1} f(z) := D_n^{-1}(D_{q^{-1}} f(z)), \quad n = 2, 3, \ldots
\]
The expression (1.1) shows us that $U_n^{(a)}(x; q)$ is an analytic function for any complex-valued parameters $a$ and $q$, and thus can be considered for general $a, q \in \mathbb{C} \setminus \{0\}$.

The classical Al-Salam–Carlitz polynomials correspond to parameters $a < 0$ and $0 < q < 1$. For these parameters, the Al-Salam–Carlitz polynomials are orthogonal on $[a, 1]$ with respect to the weight function $w$.

More specifically, for $a < 0$ and $0 < q < 1$ [2, (14.24.2)],

$$\int_a^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q)(qx, qx/a; q)_\infty d_q x = d_n^2 \delta_{n,m},$$

where

$$d_n^2 := (-a)^n(1 - q)(q; q)_\infty(a; q)_\infty(q/a; q)_\infty q^n,$$

and the $q$-Jackson integral [2, (1.15.7)] is defined as

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

where

$$\int_0^a f(x) d_q x := a(1 - q) \sum_{n=0}^\infty f(aq^n)q^n.$$

Taking into account the previous orthogonality relation, it is a direct result that if $a$ and $q$ are classical, i.e., $a, q \in \mathbb{R}$, with $a \neq 1$, $0 < q < 1$, all the zeros of $U_n^{(a)}(x; q)$ are simple and belong to the interval $[a, 1]$. This is no longer valid for general $a$ and $q$ complex. In this paper, we show that for general $a, q$ complex numbers, but excluding some special cases, the Al-Salam–Carlitz polynomials $U_n^{(a)}(x; q)$ may still be characterized by orthogonality relations. The case $a < 0$ and $0 < q < 1$ or $0 < aq < 1$ and $q > 1$ is classical, i.e., the linear functional $\mathbf{u}$ with respect to such a polynomial sequence is orthogonal, which is positive definite and in such a case there exists a weight function $\omega(x)$ so that

$$\langle \mathbf{u}, p \rangle = \int_a^1 p(x) \omega(x) dx, \quad p \in \mathbb{P}[x].$$

Note that this is the key for the study of many properties of Al-Salam–Carlitz polynomials I and II. Thus, our goal is to establish orthogonality conditions for most of the remaining cases for which the linear form $\mathbf{u}$ is quasi-definite, i.e., for all $n, m \in \mathbb{N}_0$

$$\langle \mathbf{u}, p_n p_m \rangle = k_n \delta_{n,m}, \quad k_n \neq 0.$$

We believe that these new orthogonality conditions can be useful in the study of the zeros of Al-Salam–Carlitz polynomials. For general
Fig. 1. Zeros of $U_{10}^{(1+i)}(x; \frac{4}{3} \exp(\pi i/6))$.

2. Orthogonality in the Complex Plane

**Theorem 2.1.** Let $a, q \in \mathbb{C}$, $a \neq 0, 1$, $0 < |q| < 1$. The Al-Salam–Carlitz polynomials are the unique polynomials (up to a multiplicative constant) satisfying the property of orthogonality

$\int_1^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q) w(x; a; q) d_q x = d_n^2 \delta_{n,m}. \quad (2.1)$

**Remark 2.2.** If $0 < |q| < 1$, the lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{aq^k : k \in \mathbb{N}_0\}$ is a set of points which are located inside on a single contour that goes from 1 to 0, and then from 0 to $a$, through the spirals

$S_1: z(t) = |q|^t \exp(it \arg q), \quad S_2: z(t) = |a||q|^t \exp(it \arg q + i \arg a),$

where $0 < |q| < 1$, $t \in [0, \infty)$, which we can see in Fig. 2. Taking into account (2.1), we need to avoid the $a = 1$ case. For the $a = 0$ case, we cannot apply Favard’s result [3], because in such a case this polynomial sequence fulfills the recurrence relation (see [2])

$U_{n+1}^{(0)}(x; q) = (x - q^n)U_n^{(0)}(x; q), \quad U_0^{(0)}(x; q) = 1.$
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Proof of Theorem 2.1. Let $0 < |q| < 1$, and $a \in \mathbb{C}$, $a \neq 0, 1$. We are going to express the $q$-Jackson integral (2.1) as the difference of the two infinite sums and apply the identity

$$\sum_{k=0}^{M} f(q^k) \mathcal{D}_{q^{-1}} g(q^k) q^k = \frac{f(q^M) g(q^M) - f(q^{-1}) g(q^{-1})}{q^{-1} - 1} - \sum_{k=0}^{M} g(q^{k-1}) \mathcal{D}_{q^{-1}} f(q^k) q^k. \quad (2.2)$$

Let $n \geq m$. Then, for one side, since $w(q^{-1}; a; q) = 0$, and using the identities [2, (14.24.7) and (14.24.9)], one has

$$\sum_{k=0}^{\infty} U_{m}^{(a)}(q^k; q) U_{n}^{(a)}(q^k; q) w(q^k; a; q) q^k$$

$$= \frac{a(1-q)}{q^{2-n}} \lim_{M \to \infty} \sum_{k=0}^{M} \mathcal{D}_{q^{-1}} [w(q^k; a; q) U_{n-1}^{(a)}(q^k; q)] U_{m}^{(a)}(q^k; q) q^k$$

$$= aq^{n-1} \lim_{M \to \infty} U_{m}^{(a)}(q^M; q) U_{n-1}^{(a)}(q^M; q) w(q^M; a; q)$$

$$+ aq^{n-1}(q^m - 1) \lim_{M \to \infty} \sum_{k=0}^{M-1} w(q^k; a; q) U_{n-1}^{(a)}(q^k; q) U_{m-1}^{(a)}(q^k; q) q^k.$$
Following an analogous process as before, and since \( w(aq^{-1}; a; q) = 0 \), we have

\[
\sum_{k=0}^{\infty} U_n^{(a)}(aq^{k}; q)U_n^{(a)}(aq^{k}; q)w(aq^{k}; a; q)aq^k
\]

\[
= aq^{n-1} \lim_{M \to \infty} U_n^{(a)}(aq^{M}; q)U_n^{(a)}(aq^{M}; q)w(aq^{M}; a; q)
\]

\[
+ aq^{n-1}(q^{m} - 1) \lim_{M \to \infty} \sum_{k=0}^{M-1} w(aq^{k}; a; q)U_n^{(a)}(aq^{k}; q)U_{n-1}^{(a)}(aq^{k}; q)aq^k.
\]

Therefore, if \( m < n \), and since \( m \) is finite, one can first repeat the previous process \( m + 1 \) times obtaining

\[
\sum_{k=0}^{\infty} U_n^{(a)}(aq^{k}; q)U_n^{(a)}(aq^{k}; q)w(aq^{k}; a; q)q^k
\]

\[
= \lim_{M \to \infty} \sum_{\nu=1}^{m+1} (-aq^n)\nu q^{-\nu+1/2}(q^{-m+n+1}; q)\nu
\]

\[
\times U_n^{(a)}(aq^{M}; q)U_n^{(a)}(aq^{M}; q)w(aq^{M}; a; q),
\]

and

\[
\sum_{k=0}^{\infty} U_n^{(a)}(aq^{k}; q)U_n^{(a)}(aq^{k}; q)w(aq^{k}; a; q)aq^k
\]

\[
= \lim_{M \to \infty} \sum_{\nu=1}^{m+1} (-aq^n)\nu q^{-\nu+1/2}(q^{-m+n+1}; q)\nu
\]

\[
\times U_n^{(a)}(aq^{M}; q)U_n^{(a)}(aq^{M}; q)w(aq^{M}; a; q).
\]

Hence, since the difference of both limits, term by term, goes to 0 since \( |q| < 1 \), then

\[
\int_a^1 U_n^{(a)}(x; q)U_n^{(a)}(x; q)w(x; a; q)dx = 0.
\]

For \( n = m \), following the same idea, we have

\[
\int_a^1 U_n^{(a)}(x; q)U_n^{(a)}(x; q)w(x; a; q)dx
\]

\[
= a(q^n - 1) \sum_{k=0}^{\infty} (w(aq^{k}; a; q)(U_n^{(a)}(aq^{k}; q))^2 q^k
\]

\[
- aw(aq^{k}; a; q)(U_{n-1}^{(a)}(aq^{k}; q))^2 q^k.
\]
Let us denote

\[ \delta_{n,m} = \frac{1}{(q^{n+1}; q)_{\infty} \frac{q^{n}}{(q; q)_{\infty}}} \]

and the property of orthogonality given by

\[ \int_{a}^{1} U^{(a)}_{m}(x; q^{1}) U^{(a)}_{n}(x; q^{1}) d_{q}(x) = (-a)^n (1 - q^{1-n}) (q^{1-n}; q^{1})_{\infty} \times (a; q^{1})_{\infty} q^{-1} \delta_{m,n}. \]

\[ \int_{a}^{1} U^{(a)}_{n}(x; q_{1}) U^{(a)}_{n}(x; q_{1}) w(x; a; q_{1}) d_{q_{1}}(x) = (-a)^n (q; q)_{\infty} (a; q_{1})_{\infty} q^{-1} \delta_{n,m}. \]

\[ \int_{a}^{1} U^{(a)}_{n}(x; q_{1}) U^{(a)}_{n}(x; q_{1}) w(x; a; q_{1}) d_{q_{1}}(x) = (-a)^n (q; q)_{\infty} (a; q_{1})_{\infty} q^{-1} \delta_{n,m}. \]

Due to the normality of this polynomial sequence, i.e., \( \deg U^{(a)}_{n}(x; q) = n \) for all \( n \in \mathbb{N}_{0} \), the uniqueness is straightforward, thus the result holds. □

From this result, and taking into account that the squared norm for the Al-Salam–Carlitz polynomials is known, we obtained the following consequence for which we could not find any reference.

**Corollary 2.3.** Let \( a, q \in \mathbb{C} \setminus \{0\}, |q| < 1 \). Then

\[ \sum_{k=0}^{\infty} \frac{(q^{k+1}/a; q)_{\infty} - a(aq^{k+1}; q)_{\infty}}{(q; q)_{k}} q^{k} = (a; q)_{\infty} (q/a; q)_{\infty}. \]

The following case, which is just the Al-Salam–Carlitz polynomials for the \( |q| > 1 \) case, is commonly called the Al-Salam–Carlitz II polynomials.

**Theorem 2.4.** Let \( a, q \in \mathbb{C}, a \neq 0,1, |q| > 1 \). Then, the Al-Salam–Carlitz polynomials are unique (up to a multiplicative constant) satisfying the property of orthogonality given by

\[ \int_{a}^{1} U^{(a)}_{n}(x; q^{1}) U^{(a)}_{m}(x; q^{1}) (q^{1-n} x; q^{-1})_{\infty} q^{-1} x/a; q^{-1} \times (a; q^{-1})_{\infty} q^{-1} \delta_{m,n}. \]
that \( w(aq; a; p) = w(q; a; p) = 0 \) and \([2, (14.24.9)]\), for \( m < n \) one has
\[
\sum_{k=0}^{\infty} aw(ap^k; a; p)U_n^{(a)}(ap^k; p)U_n^{(a)}(ap^k; p)p^k
\]
\[= ap^{n-1} \lim_{M \to \infty} U_m^{(a)}(ap^M; p)U_{n-1}^{(a)}(ap^M; p)w(ap^M; a; p)\]
\[+ ap^{n-1}(1 - p^n) \lim_{M \to \infty} \sum_{k=0}^{M-1} aw(ap^k; a; p)U_{n-1}^{(a)}(ap^k; p)U_{m-1}^{(a)}(ap^k; p)p^k.\]

Following the same idea from the previous result, we have
\[
\sum_{k=0}^{\infty} w(p^k; a; p)U_m^{(a)}(p^k; p)U_n^{(a)}(p^k; p)p^k
\]
\[= ap^{n-1} \lim_{M \to \infty} U_m^{(a)}(p^M; p)U_{n-1}^{(a)}(p^M; p)w(p^M; a; p)\]
\[+ ap^{n-1}(1 - p^n) \lim_{M \to \infty} \sum_{k=0}^{M-1} w(p^k; a; p)U_{n-1}^{(a)}(p^k; p)U_{m-1}^{(a)}(p^k; p)p^k.\]

Therefore, the property of orthogonality holds for \( m < n \). Next, if \( n = m \), we have
\[
\int_a^1 U_n^{(a)}(x; p)U_n^{(a)}(x; p)w(x; a; p) dx
\]
\[= \frac{ap^n - 1}{p^{n-1}} \sum_{k=0}^{\infty} (aw(ap^k; a; p)(U_{n-1}^{(a)}(ap^k; p))^2p^k
\[-w(p^k; a; p)(U_{n-1}^{(a)}(p^k; p))^2p^k)
\[= (-a)^n (p; p)_n(p; p)_n(\sum_{k=0}^{\infty} aw(ap^k; a; p)p^k - w(p^k; a; p)p^k)
\[= (-a)^n (q^{-1}; q^{-1})_n(p; p)_\infty(p; p)_\infty(\sum_{k=0}^{\infty} q^{k}(a(p^{k+1}; a; p)_{\infty} - (p^{k+1}/a; p)_{\infty} )
\[= (-a)^n(q^{-1}; q^{-1})_n(p; p)_\infty(a; p)_\infty(p/a; p)_\infty(p; p)_n.\]

Using the same argument as in Theorem 2.1, the uniqueness holds, so the claim follows. \(\square\)

Remark 2.5. Observe that in the previous theorems if \( a = q^m \), with \( m \in \mathbb{Z} \), \( a \neq 0 \), after some logical cancellations, the set of points where we need
to calculate the \( q \)-integral is easy to compute. For example, if \( 0 < a q < 1 \) and \( 0 < q < 1 \), one obtains the sum [2, (14.25.2), p. 537].

**Remark 2.6.** The \( a = 1 \) case is special because it is not considered in the literature. In fact, the linear form associated with the Al-Salam–Carlitz polynomials \( u \) is quasi-definite and fulfills the Pearson-type distributional equations

\[
\mathcal{D}_q[(x - 1)^2u] = \frac{x - 2}{1 - q} u \quad \text{and} \quad \mathcal{D}_q^{-1}[q^{-1}u] = \frac{x - 2}{1 - q} u.
\]

Moreover, the Al-Salam–Carlitz polynomials fulfill the three-term recurrence relation [2, (14.24.3)]

\[
xU_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (a + 1)q^nU_n^{(a)}(x; q) - aq^{n-1}(1 - q^n)U_{n-1}^{(a)}(x; q),
\]

(2.4)

where \( n = 0, 1, \ldots \), with initial conditions \( U_0^{(a)}(x; q) = 1 \), \( U_1^{(a)}(x; q) = x - a - 1 \).

Therefore, we believe that it will be interesting to study such a case for its peculiarity because the coefficient \( q^{n-1}(1 - q^n) \neq 0 \) for all \( n \), so one can apply Favard’s result.

### 2.1. The \(|q| = 1\) case

In this section, we only consider the case where \( q \) is a root of unity. Let \( N \) be a positive integer such that \( q^N = 1 \); then, due to the recurrence relation (2.4) and following the same idea that the authors did in [4, Section 4.2], we apply the following process:

1. The sequence \( \left(U_n^{(a)}(x; q)\right)_{n=0}^{N-1} \) is orthogonal with respect to the Gaussian quadrature

\[
\langle v, p \rangle := \sum_{s=1}^{N} \gamma_s^{(a)} \cdots \gamma_{N-1}^{(a)} \frac{p(x_s)}{(U_{N-1}^{(a)}(x_s))^2},
\]

where \( \{x_1, x_2, \ldots, x_N\} \) are the zeros of \( U_N^{(a)}(x; q) \) for such value of \( q \).

2. Since \( \langle v, U_n^{(a)}(x; q)U_n^{(a)}(x; q) \rangle = 0 \), we need to modify such a linear form. Next, we can prove that the sequence \( \left(U_n^{(a)}(x; q)\right)_{n=0}^{2N-1} \) is orthogonal with respect to the bilinear form

\[
\langle p, r \rangle_2 = \langle v, p q \rangle + \langle v, q \mathcal{D}_q^{-1}[p q]N r \rangle,
\]

since \( \mathcal{D}_qU_n^{(a)}(x; q) = (q^n - 1)/(q - 1)U_{n-1}^{(a)}(x; q) \).
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(3) Since $\langle U_{2N}^{(a)}(x; q), U_{2N}^{(a)}(x; q) \rangle_2 = 0$, and taking into account the above results, we consider the linear form

$$\langle p, r \rangle_2 = \langle v, pq \rangle + \langle v, D^N_q p D^N_q r \rangle + \langle v, D^2_{N_q} p D^2_{N_q} r \rangle.$$ (4)

Therefore one can obtain a sequence of bilinear forms such that the Al-Salam–Carlitz polynomials are orthogonal with respect to them.

3. A Generalized Generating Function for Al-Salam–Carlitz Polynomials

For this section, we are going to assume $|q| > 1$, or $0 < |p| < 1$. Indeed, by starting with the generating functions for Al-Salam–Carlitz polynomials [2, (14.25.11) and (14.25.12)], we derive generalizations using the connection relation for these polynomials.

**Theorem 3.1.** Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$. Then

$$U_n^{(a)}(x; p) = (-1)^n p_{n}^{-\text{Iz}}(z) \sum_{k=0}^{n} \frac{(-1)^k a^{n-k} (b/a; p)_{n-k} p^{(k)}(z)}{(p; p)_n (p; p)_k} U_k^{(b)}(x; p).$$ (3.1)

**Proof.** If we consider the generating function for Al-Salam–Carlitz polynomials [2, (14.25.11)]

$$\frac{(xt; p)_\infty}{(t, at; p)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n p\text{Iz}(z)}{(p; p)_n} U_n^{(a)}(x; p)t^n,$$

and multiply both sides by $(bt; p)_\infty/(bt; p)_\infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n p\text{Iz}(z)}{(p; p)_n} U_n^{(a)}(x; p)t^n = \frac{(bt; p)_\infty}{(at; p)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n p\text{Iz}(z)}{(p; p)_n} U_n^{(b)}(x; p)t^n. \tag{3.2}$$

If we now apply the $q$-binomial theorem [2, (1.11.1)]

$$\frac{(az; p)_\infty}{(z; p)_\infty} = \sum_{k=0}^{\infty} \frac{(ap; p)_n z^n}{(p; p)_n}, \quad 0 < |p| < 1, \quad |z| < 1,$$
to (3.2), and then collect powers of $t$, we obtain
\[
\sum_{k=0}^{\infty} t^k \sum_{m=0}^{k} \frac{(-1)^m b^{-m} (b/a; p)_m}{(p; p)_m (p; p)_m} U_m^{(b)}(x; p)
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p; p)_n} U_n^{(a)}(x; p)t^n.
\]
Taking into account this expression, the result follows. \hfill \square

\textbf{Theorem 3.2.} Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$, $t \in \mathbb{C}$, $|at| < 1$. Then
\[
(at; p)_\infty \phi_1 \left( \frac{x}{at}; p, t \right) = \sum_{k=0}^{\infty} \frac{p^{k(1-k)}}{(p; p)_k} \phi_1 \left( \frac{b/a}{0}; p, atp^k \right) U_k^{(b)}(x; p)t^k,
\]
where
\[
\phi_\infty \left( a_1, a_2, \ldots, a_r \atop b_1, b_2, \ldots, b_s \right) := \sum_{k=0}^{\infty} \frac{(a_1; p)_k(a_2; p)_k \cdots (a_r; p)_k}{(b_1; p)_k(b_2; p)_k \cdots (b_s; p)_k} \frac{z^k}{(p; p)_k},
\]
is the unilateral basic hypergeometric series.

\textbf{Proof.} We start with a generating function for Al-Salam–Carlitz polynomials [2, (14.25.12)]
\[
(at; q)_\infty \phi_1 \left( \frac{x}{at}; q, t \right) = \sum_{k=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} V_n^{(a)}(x; q)t^n
\]
and (3.1) to obtain
\[
(at; p)_\infty \phi_1 \left( \frac{x}{at}; p, t \right)
\]
\[
= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{(-1)^k a^{-k} (b/a; p)_n k^\binom{k}{2}}{(p; p)_{n-k}(p; p)_k} U_k^{(b)}(x; p).
\]
We reverse the order of summations, shift the $n$ variable by a factor of $k$, and use the basic properties of the $q$-Pochhammer symbol, and [2, (1.10.1)]. Observe that we can reverse the order of summation since our sum is of the form

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x; p),$$

where

$$a_n = t^n, \quad c_{n,k} = \frac{(-1)^k a^{n-k} (b/a; p)_{n-k} p^{(k)}}{(p;p)_{n-k} (p;p)_k}.$$ 

In this case, one has

$$|a_n| \leq |t|^n, \quad |c_{n,k}| \leq K(1+n)^{\sigma_1} |a|^n,$$

and $|U_k^{(a)}(x; p)| \leq (1+n)^{\sigma_2}$, where $K_1$, $\sigma_1$, and $\sigma_2$ are positive constants independent of $n$. Therefore, if $|at| < 1$, then

$$\left| \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x; p) \right| < \infty,$$

and this completes the proof. \(\square\)

As we saw in Section 2, the orthogonality relation for Al-Salam–Carlitz polynomials for $|q| > 1$, $|p| < 1$, and $a \neq 0, 1$ is

$$\int_{\Gamma} U_n^{(a)}(x; p) U_m^{(a)}(x; p) w(x; a; p) dx = d_n^2 \delta_{n,m}.$$

Taking this result in mind, the following result follows.

**Theorem 3.3.** Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$, $|at| < 1$, $|p| < 1$, $m \in \mathbb{N}_0$. Then

$$\int_{\Gamma} U_n^{(a)}(x; p) w(x; a; p) dx = \left( -bt \right)^{m} q^{\left( \frac{3}{2} \right) \left( b \right)} (p/b; p)_{\infty} 1_{\phi 1} \left( \frac{b/a}{q}; q, atq^m \right).$$

**Proof.** From (3.3), we have $x \mapsto p^x$ and multiply both sides by $U_m^{(b)}(x; p) w(x; a; p)$, and by using the orthogonality relation (2.3), the desired result holds. \(\square\)
Note that the applications of connection relations to the rest of the known generating functions for Al-Salam–Carlitz polynomials [2, (14.24.11) and (14.25.12)] leave these generating functions invariant.

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