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Orthogonality of the big -1 Jacobi polynomials for non-standard parameters

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Abstract. The big -1 Jacobi polynomials $(Q_n^{(0)}(x;\alpha,\beta,c))_n$ have been classically defined for $\alpha,\beta\in(-1,\infty)$, $c\in(-1,1)$. We extend this family so that wider parameter values are allowed, i.e., the parameters may be non-standard. Assuming initial conditions $Q_0^{(0)}(x)=1$, $Q_{-1}^{(0)}(x)=0$, we consider the big -1 Jacobi polynomials as monic orthogonal polynomials which satisfy the three-term recurrence relation

$$xQ_n^{(0)}(x)=Q_{n+1}^{(0)}(x)+b_nQ_n^{(0)}(x)+u_nQ_{n-1}^{(0)}(x),\quad n=0,1,2,\ldots.$$

For standard parameters, the coefficients u_n are positive for all n. We discuss the situation when Favard's theorem cannot be directly applied, as there is some positive integer n such that $u_n=0$. We express the big -1 Jacobi polynomials for non-standard parameters as a product of two polynomials. Using this factorization, we obtain a bilinear form with respect to which these polynomials are orthogonal.

1. Introduction

In 2011, Vinet and Zhedanov in [**19**] obtained a new family of "classical" orthogonal polynomials which can be obtained from the little q-Jacobi polynomials in the limit $q\to-1$. By "classical" it is meant that these polynomials satisfy (in addition to a three-term recurrence relation) a nontrivial eigenvalue equation of the form

$$\mathcal{L}(p_n(x))=\lambda_np_n(x),$$

where $\mathcal{L}$ is a linear differential–difference operator which is of first order in the derivative ∂_x, and contains a reflection operator R which acts as $Rf(x)=f(-x)$. They referred to these polynomials as being in the continuous -1 hypergeometric orthogonal polynomial scheme. This was illustrated by using a limiting process from the Askey–Wilson polynomials to the Bannai–Ito polynomials. In [**20**] the same authors applied this limit by starting with the big q-Jacobi polynomials in order to obtain hypergeometric representations for the big -1 Jacobi polynomials.

The polynomials these authors have considered in this -1 hypergeometric orthogonal polynomial scheme are q-polynomials for standard parameters, i.e., ones for which the weight function related to these families is positive definite (see [**16**]). The main aim of this work is to investigate the property of orthogonality for the big -1 Jacobi polynomials for those values of the parameters, α and β, for which

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the coefficient u_n in the recurrence relation (1.2) relative to these polynomials can be equal to zero for some $n \in \mathbb{N}_0$.

In the last three decades, some of the classical orthogonal polynomials with non-classical parameters have been provided with certain non-standard orthogonality properties. In the pioneering work by Kwon and Littlejohn in 1995 [**10**], Sobolev orthogonality for the Laguerre polynomials $L_n^{(-N)}(x)$ with $N \in \mathbb{N}$ was obtained using the inner product

$$\langle f, g\rangle = \sum_{m=0}^{N-1}\sum_{j=0}^{m} B_{m,j}(N)\left(f^{(m)}(0)g^{(j)}(0) + f^{(j)}(0)g^{(m)}(0)\right) + \int_0^\infty f^{(N)}(x)g^{(N)}(x)\,\mathrm{e}^{-x}\,\mathrm{d}x,$$

where the superscripts represent differentiation and

$$B_{m,j}(N) := \begin{cases} \displaystyle\sum_{p=0}^{j}(-1)^{m+j}\binom{N-1-p}{m-p}\binom{N-1-p}{j-p}, & 0 \leqslant j \leqslant m \leqslant N-1, \\ \displaystyle\frac{1}{2}\sum_{p=0}^{m}\binom{N-1-p}{m-p}^2, & 0 \leqslant j = m \leqslant N-1. \end{cases}$$

In 1998, Álvarez de Morales and co-authors [**3**] found, using a different technique, the orthogonality for ultraspherical polynomials $C_n^{-N+\frac{1}{2}}$ with $N \in \mathbb{N}$. Later, in [**1**], M. Alfaro et al. considered the cases of the Jacobi polynomials in which both parameters, α and β, were negative integers, proving that in such a case the Jacobi polynomials satisfy a Sobolev orthogonality. In [**2**], M. Alfaro et al. considered the situation when the inner product can be written in the form

$$\langle f, g\rangle = F^t A G + \int f^{(N)}(x)g^{(N)}(x)\,\mathrm{d}\mu(x), \tag{1.1}$$

where F and G are vectors obtained by evaluating f and g and maybe their derivatives at some points, A is a symmetric real matrix, and $\mathrm{d}\mu$ is the orthogonality measure associated with the Nth derivative of either the Laguerre, ultraspherical or Jacobi polynomials.

Note that for the parameters considered in this situation there exists an $n = N \in \mathbb{N}_0$ for which the coefficient u_n in the recurrence relation

$$xp_n = p_{n+1} + b_n p_n + u_n p_{n-1}\,, \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

vanishes, i.e., $u_N = 0$. We say a non-standard parameter (or set of parameters) is degenerate if there exists at least one positive integer N for which $u_N = 0$.

The first term in the inner product (1.1) plays the role of the orthogonality for the set of polynomials of degree less than N, since if f, g are two polynomials with $\deg f$, $\deg g < N$, then $f^{(N)}(x) = g^{(N)}(x) = 0$, and one has

$$\langle f, g\rangle = F^t A G.$$

In this term, the points for the evaluation are the roots of p_N; this ensures that the inner product vanishes when one entry is p_n with $n \geqslant N$, since p_N is a factor of any p_n with $n \geqslant N$. The second term in (1.1) is relevant for polynomials with degree greater than N, and is needed in order to have an orthogonality characterizing the sequence $(p_n)_{n=0}^\infty$. The technique used in [**3**] is applicable to all classical orthogonal polynomials for which u_n vanishes at some $n = N \in \mathbb{N}_0$, and in fact it has been used, among others, in [**12–15**].

In 2009 [**5**], Costas-Santos and Sanchez-Lara studied this problem for classical discrete polynomials, i.e., discrete polynomials in the Askey scheme of generalized hypergeometric orthogonal polynomials. We consider classical discrete polynomials for which there exists N for which $u_N = 0$, i.e., for non-standard degenerate parameters. Note that this can only happen with the Racah, Hahn, dual Hahn, and Krawtchouk polynomials, which are the Wilson, continuous Hahn, continuous dual Hahn, and Meixner polynomials for which some parameters are equal to a negative integer. The corresponding three-term recurrence relation for these polynomials also presents one vanishing coefficient, i.e., $u_N = 0$, and the inner product found can be written as

$$\langle f, g\rangle = \int f(x)g(x)\,\mathrm{d}\mu_d(x) + \int f^{(N)}(x)g^{(N)}(x)\,\mathrm{d}\mu_c(x), \tag{1.3}$$

where $\mathrm{d}\mu_d$ is a discrete measure with a finite number of masses and $\mathrm{d}\mu_c$ is an absolutely continuous measure. For more details and further reading see [**5**, **6**, **9**, **17**, **18**] and references therein. The organization of the paper is as follows. In Section 2 we provide some preliminary material; in Section 3 we obtain the property of orthogonality for the big -1 Jacobi polynomials for non-standard degenerate parameters; and in Section 4 we summarize the key findings and contributions of the present study and describe potential future works in the -1 orthogonal polynomial framework.

2. Preliminary material and notations

We adopt the following notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$, and we use $\mathbb{P}, \mathbb{P}'$ to represent the linear space of polynomials with complex coefficients and its algebraic dual space. We denote by $\langle \mathbf{u}, p\rangle$ the duality bracket for $\mathbf{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$. The big -1 Jacobi polynomials have representations given in terms of the Gauss hypergeometric function defined as [**7**, (15.2.1)]

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.1}$$

where $|z| < 1$ and the shifted factorial is defined as [**7**, (5.2.4-5)] $(a)_n := (a)(a+1)\cdots(a+n-1)$, $a \in \mathbb{C}$, $n \in \mathbb{N}_0$. The shifted factorial is related to the ratio of two gamma functions [**7**, Chapter 5] for $a \in \mathbb{C} \setminus -\mathbb{N}_0$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.2}$$

which allows one to extend the definition to non-positive integer values of n. For a description of the properties of the gamma function, see [**7**, Chapter 5].

3. Big −1 Jacobi polynomials

The monic big −1 Jacobi polynomials can be defined in terms of the Gauss hypergeometric function by

(3.1)
$$Q_n^{(0)}(x) := \kappa_n \begin{cases} {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{\alpha+1}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right) \\ \quad + \dfrac{n(1-x)}{(c+1)(\alpha+1)}\, {}_2F_1\left(\begin{matrix} 1-\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{\alpha+3}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right) & \text{if } n \text{ is even,} \\ {}_2F_1\left(\begin{matrix} -\frac{n-1}{2}, \frac{n+\alpha+\beta+1}{2} \\ \frac{\alpha+1}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right) \\ \quad - \dfrac{(\alpha+\beta+n+1)(1-x)}{(1+c)(\alpha+1)}\, {}_2F_1\left(\begin{matrix} -\frac{n-1}{2}, \frac{n+\alpha+\beta+3}{2} \\ \frac{\alpha+3}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right) & \text{if } n \text{ is odd,} \end{cases}$$

where κ_n is defined by

$$\kappa_n := \begin{cases} \dfrac{(1-c^2)^{\frac{1}{2}n}(\frac{1}{2}(\alpha+1))_{\frac{1}{2}n}}{(\frac{1}{2}(n+\alpha+\beta+2))_{\frac{1}{2}n}} & \text{if } n \text{ is even,} \\ (c+1)\dfrac{(1-c^2)^{\frac{1}{2}(n-1)}(\frac{1}{2}(\alpha+1))_{\frac{1}{2}(n-1)}}{(\frac{1}{2}(n+\alpha+\beta+1))_{\frac{1}{2}(n-1)}} & \text{if } n \text{ is odd.} \end{cases} \tag{3.2}$$

The big −1 Jacobi polynomials $Q_n^{(0)}(x;\alpha,\beta,c)$ satisfy the three-term recurrence relation [**20**, (2.20), (2.21)]

$$xQ_n^{(0)}(x) = Q_{n+1}^{(0)}(x) + b_n Q_n^{(0)}(x) + u_n Q_{n-1}^{(0)}(x), \tag{3.3}$$

where

$$b_n := \begin{cases} -c + \dfrac{(c-1)n}{\alpha+\beta+2n} + \dfrac{(c+1)(\beta+n+1)}{\alpha+\beta+2n+2} & \text{if } n \text{ is even,} \\ c - \dfrac{(c-1)(n+1)}{\alpha+\beta+2n+2} - \dfrac{(c+1)(\beta+n)}{\alpha+\beta+2n} & \text{if } n \text{ is odd,} \end{cases} \tag{3.4}$$

and

$$u_n := \begin{cases} \dfrac{(c-1)^2 n(\alpha+\beta+n)}{(\alpha+\beta+2n)^2} & \text{if } n \text{ is even,} \\ \dfrac{(c+1)^2(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2} & \text{if } n \text{ is odd,} \end{cases} \tag{3.5}$$

with initial conditions $Q_{-1}^{(0)}(x) = 0$, $Q_0^{(0)}(x) = 1$. For any real $c \neq 1$ and real α, β satisfying the restriction $\alpha, \beta > -1$, the recurrence coefficients b_n, u_n are real and positive. Hence the big −1 Jacobi polynomials are positive definite orthogonal polynomials.

Remark 3.1. Observe that if α (resp. β) is a negative odd integer, namely $\alpha = -2N+1$ (resp. $\beta = -2N+1$) then $u_{2N-1} = 0$, i.e., α (resp. β) is a non-standard degenerate parameter. So we can apply the degenerate version of Favard's theorem (see [**6**, Theorem 2.2]), i.e., the result that there exist moment linear functionals

$\mathscr{L}_0$ and $\mathscr{L}_N$, such that the polynomial sequence is orthogonal with respect to the bilinear form

$$\langle p, r\rangle = \mathscr{L}_0(pr) + \mathscr{L}_N(\mathscr{T}^{(2N-1)}p\,\mathscr{T}^{(2N-1)}r), \qquad p, r \in \mathbb{P},$$

where $\mathscr{T}$ is a certain lowering operator.

By taking the former remark into account, the Gauss hypergeometric function with a suitable normalization factorizes as follows.

Lemma 3.2 (Factorization). *Let $n, N \in \mathbb{N}$, $a, x \in \mathbb{C}$. Then*

$$\begin{aligned}(-N+1)_{n+N}\,{}_2F_1\left(\begin{matrix}-n-N, a\\ -N+1\end{matrix}; x\right)\\ = (N+1)_n(-N+1)_N\,{}_2F_1\left(\begin{matrix}-N, a\\ -N+1\end{matrix}; x\right){}_2F_1\left(\begin{matrix}-n, a+N\\ N+1\end{matrix}; x\right).\end{aligned} \tag{3.6}$$

Proof. By definition one has

$$\begin{aligned}(-N+1)_{n+N}\,{}_2F_1\left(\begin{matrix}-n-N, a\\ -N+1\end{matrix}; x\right) &= (-N+1)_{n+N}\sum_{k=0}^{n+N}\frac{(-n-N, a)_k}{(-N+1, 1)_k x^k}\\ &= \sum_{k=0}^{n+N}\frac{(-n-N, a)_k}{(1)_k}(-N+k+1)_{n+N-k}x^k.\end{aligned}$$

Then setting $k = k + N$ one has
(3.7)

$$\begin{aligned}(-N+1)_{n+N}\,{}_2F_1\left(\begin{matrix}-n-N, a\\ -N+1\end{matrix}; x\right) &= \sum_{k=0}^{n}\frac{(-n-N, a)_{k+N}}{(1)_{k+N}}(k+1)_{n-k}x^{k+N}\\ &= \frac{(-n-N, a)_N n!}{N!}x^N\sum_{k=0}^{n}\frac{(-n, a+N)_k}{(N+1, 1)_k}x^k\\ &= (N+1)_n(a)_N(-x)^N\,{}_2F_1\left(\begin{matrix}-n, a+N\\ N+1\end{matrix}; x\right).\end{aligned}$$

Setting $n = 0$ in (3.7), one obtains

$$(-N+1)_N\,{}_2F_1\left(\begin{matrix}-N, a\\ -N+1\end{matrix}; x\right) = (a)_N(-x)^N. \tag{3.8}$$

Combining (3.7) and (3.8) produces the result. □

Now that we have obtained a factorization of the Gauss hypergeometric function, we will use it to obtain a factorization for the big −1 Jacobi polynomials when the parameter α (resp. β) is equal to $-2N-1$ for some positive integer N.

Lemma 3.3 (Factorization of big −1 Jacobi polynomials). *Let $m, N \in \mathbb{N}_0$, $\alpha, \beta, c \in \mathbb{C}$, $c \neq \pm 1$, with $\alpha = -2N-1$ or $\beta = -2N-1$. Then*

$$Q^{(0)}_{2N+1}(x; -2N-1, \beta, c) = (x^2-1)^N(x-1), \tag{3.9}$$

$$Q^{(0)}_{2N+1}(x; \alpha, -2N-1, c) = (x^2-c^2)^N(x+c), \tag{3.10}$$

and

$$Q^{(0)}_{2N+1+m}(x;-2N-1,\beta,c) = (-1)^m(x^2-1)^N(x-1)Q^{(0)}_m(-x;2N+1,\beta,-c), \tag{3.11}$$

$$Q^{(0)}_{2N+1+m}(x;\alpha,-2N-1,c) = (x^2-c^2)^N(x+c)Q^{(0)}_m(x;\alpha,2N+1,-c). \tag{3.12}$$

Proof. Identities (3.9) and (3.10) follow from (3.12) and (3.11) by setting $m=0$. We will first prove (3.11) and then (3.12). Consider $m = 2M$, $n = 2N+1+m$. Then one has

$$Q^{(0)}_{2N+1+m}(x;\alpha,-2N-1,c) = \kappa_n\left({}_2F_1\left(\begin{matrix}-N-M, M+\widehat{\beta}\\ -N\end{matrix};\frac{1-x^2}{1-c^2}\right) + \frac{(M+\widehat{\beta})(1-x)}{(c+1)N}\,{}_2F_1\left(\begin{matrix}-N-M, M+1+\widehat{\beta}\\ -N+1\end{matrix};\frac{1-x^2}{1-c^2}\right)\right),$$

where $\widehat{\beta} = (\beta+1)/2$. Next, by using Lemma 3.2, and after a tedious calculation comparing the final expression with the one for $Q_m(-x;2N+1,\beta,-c)$ and its corresponding normalization coefficient, the identity for the even m values holds. For $n = 2N+1+2M+1$, i.e., $m = 2M+1$, for $\alpha = -2N-1$, the proof is similar and will be omitted. Hence (3.11) follows. The identity (3.12) holds by applying the same procedure, which completes the proof. $\square$

A different way to guess and then obtain the factorization for the big -1 Jacobi polynomials for non-standard parameters throughout is to use the following two-step procedure. The first step is to use Lemma 3.2 in order to prove the identities (3.9), (3.10). Once we have that, it is straightforward to find, e.g., for $m \geqslant 0$, that

$$Q^{(0)}_{2N+1+m}(x;-2N-1,\beta,c) = Q^{(0)}_{2N+1}(x;-2N-1,\beta,c)P_m(x),$$

where $P_m \in \mathbb{P}$ of degree m. In the second step, we obtain a recurrence relation for these new polynomials, and comparing the recurrence coefficients, we prove this new polynomial sequence is related to the original one.

Lemma 3.4. *For any $N \in \mathbb{N}$, $\alpha, \beta, c \in \mathbb{C}$, $c \neq \pm 1$, with $\alpha = -2N-1$ (resp. $\beta = -2N-1$), the following recurrence relation holds for $n \geqslant 0$:*

$$xQ^{(0)}_n(x;2N+1,\beta,-c) = -Q^{(0)}_{n+1}(x;2N+1,\beta,-c) + \widehat{b}_nQ^{(0)}_n(x;2N+1,\beta,-c) + \widehat{u}_nQ^{(0)}_{n-1}(x;2N+1,\beta,-c), \tag{3.13}$$

where

$$\widehat{b}_n = -b_{n+2N+1}(-2N-1,\beta,c) = b_n(2N+1,\beta,-c), \tag{3.14}$$

$$\widehat{u}_n = u_{n+2N+1}(-2N-1,\beta,c) = u_n(2N+1,\beta,-c). \tag{3.15}$$

Respectively, the following recurrence relation holds for $n \geqslant 0$:

$$xQ^{(0)}_n(x;\alpha,2N+1,-c) = Q^{(0)}_{n+1}(x;\alpha,2N+1,-c) + \widetilde{b}_nQ^{(0)}_n(x;\alpha,2N+1,-c) + \widetilde{u}_nQ^{(0)}_{n-1}(x;\alpha,2N+1,-c), \tag{3.16}$$

where

$$\widetilde{b}_n = b_{n+2N+1}(\alpha,-2N-1,c) = b_n(\alpha,2N+1,-c), \tag{3.17}$$

$$\widetilde{u}_n = u_{n+2N+1}(\alpha,-2N-1,c) = u_n(\alpha,2N+1,-c). \tag{3.18}$$

PROOF. One starts by considering $\alpha = -2N-1$, so that $u_{2N+1} = 0$. In fact,

$$xQ^{(0)}_{2N+1}(x; 2N+1, \beta, c) = (x^2-1)^N(x-1).$$

If one considers $n \geqslant m + 2N + 1$, then

$$Q^{(0)}_{2N+1+m}(x; -2N-1, \beta, c) = Q^{(0)}_{2N+1}(x; -2N-1, \beta, c)(-1)^m Q^{(0)}_m(-x; 2N+1, \beta, -c),$$

and therefore the recurrence relation for these polynomials can be written as

$$\begin{aligned} x(-1)^m Q^{(0)}_m(-x; 2N+1, \beta, -c) &= (-1)^{m+1} Q^{(0)}_{m+1}(-x; 2N+1, \beta, -c) \\ &\quad + b_{m+2N+1}(-1)^m Q^{(0)}_m(-x; 2N+1, \beta, -c) \\ &\quad + u_{m+2N+1}(-1)^{m-1} Q^{(0)}_{m-1}(-x; 2N+1, \beta, -c). \end{aligned}$$

After some straightforward calculations the result follows. For $\beta = -2N-1$ the derivation is analogous, which completes the proof. □

Note that if $u_{2N+1} = 0$ for some $N \in \mathbb{N}$ then the orthogonality property for the big -1 Jacobi polynomials holds for polynomials of degree less or equal to $2N+1$. In the next result, we construct, thanks to the factorization of the big -1 polynomials, a new property of orthogonality that is valid for the big -1 Jacobi polynomials even when $n > 2N+1$, i.e., we extend the property of orthogonality for the big -1 polynomials to non-standard degenerate parameters.

THEOREM 3.5 (Orthogonality of big -1 Jacobi polynomials for non-standard parameters). *Let $N \in \mathbb{N}_0$, $\alpha, \beta, c \in \mathbb{C}$, $c \neq \pm 1$. Define the linear operator* $\mathbf{u}$ [**20**, Section 4] *by*

$$\langle \mathbf{u}, pq\rangle := \int_{[-c,-1]\cup[1,c]} p(x)q(x)\,\frac{x}{|x|}(x+1)(c-x)(x^2-1)^{\frac{1}{2}(\alpha-1)}(c^2-x^2)^{\frac{1}{2}(\beta-1)}\,\mathrm{d}x,$$

and the norm with respect to the linear functional $\mathbf{u} \in \mathbb{P}'$ *by* $h_n := h_n(\alpha, \beta, c)$, *where*

(3.19)
$$\begin{aligned} h_n(\alpha,\beta,c) &:= \langle \mathbf{u}, Q^{(0)}_n Q^{(0)}_n\rangle \\ &= \langle \mathbf{u}, 1\rangle \begin{cases} \dfrac{2(c^2-1)^n(\frac{1}{2}n)!(\frac{1}{2}(\alpha+1),\frac{1}{2}(\beta+1))_{\frac{1}{2}n}}{(\frac{1}{2}(\alpha+\beta)+1)_n(\frac{1}{2}(\alpha+\beta+n)+1)_{\frac{1}{2}n}} & \text{if } n \text{ is even,} \\ \dfrac{2(c-1)^{n-1}(c+1)^{n+1}(\frac{1}{2}(n-1))!(\frac{1}{2}(\alpha+1),\frac{1}{2}(\beta+1))_{\frac{1}{2}(n+1)}}{(\frac{1}{2}(\alpha+\beta)+1)_n(\frac{1}{2}(\alpha+\beta+n+1))_{\frac{1}{2}(n+1)}} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Then, the polynomial sequences

$$(3.20)\qquad (Q^{(0)}_n(x; -2N-1, \beta, c)), \qquad (Q^{(0)}_n(x; \alpha, -2N-1, c))$$

are orthogonal with respect to the bilinear forms

$$(3.21)\qquad \langle p, q\rangle_1 = \mathscr{L}_0(p,q) + \lambda_N(\beta)\langle \mathbf{u}_1, (\tau^{2N+1}_\alpha p)(\tau^{2N+1}_\alpha q)\rangle,$$

$$(3.22)\qquad \langle p, q\rangle_2 = \mathscr{L}_1(p,q) + \lambda_N(\alpha)\langle \mathbf{u}_2, (\tau^{2N+1}_\beta p)(\tau^{2N+1}_\beta q)\rangle,$$

respectively. Here τ_α, τ_β *are linear operators defined such that*

$$(3.23)\qquad \tau_\alpha[Q^{(0)}_n(x; \alpha, \beta, c)] = Q_{n-1}(-x; \alpha+2, \beta, -c),$$

$$(3.24)\qquad \tau_\beta[Q^{(0)}_n(x; \alpha, \beta, c)] = Q_{n-1}(x; \alpha, \beta+2, -c),$$

and

$$\langle \mathbf{u}_1, f(x,\alpha,\beta,c)\rangle := \langle \mathbf{u}, f(-x,-\alpha,\beta,-c)\rangle, \tag{3.25}$$

$$\langle \mathbf{u}_2, f(x,\alpha,\beta,c)\rangle := \langle \mathbf{u}, f(x,\alpha,-\beta,-c)\rangle. \tag{3.26}$$

$\mathscr{L}_0$ and $\mathscr{L}_1$ are two moment linear functionals, and

$$\lambda_N(\alpha) = \frac{1}{2} h_{2N+1}(-2N-1,\alpha,c), \qquad \lambda_N(\beta) = \frac{1}{2} h_{2N+1}(-2N-1,\beta,c).$$

PROOF. Consider the situation corresponding to (3.21), i.e., $\alpha = -2N-1$, with $N \in \mathbb{N}$, $\beta \in \mathbb{C}$. We need to prove the properties of orthogonality $\langle Q_n^{(0)}, Q_m^{(0)}\rangle_1 = 0$ for all $m < n$, and $\langle Q_n^{(0)}, Q_n^{(0)}\rangle_1 \neq 0$ for all n, in order to prove that $(Q_n^{(0)})$ is a monic orthogonal polynomial sequence with respect to $\langle\,,\rangle_1$. There are three cases: (i) $m \leqslant n < 2N+1$; (ii) $m < 2N+1 \leqslant n$; (iii) $2N+1 \leqslant n \leqslant m$. (i): If $m \leqslant n < 2N+1$ the coefficient u_k of the recurrence relation is non-zero for $k = 0,1,\ldots,2N$, so by the Favard result (see [**4**]), there exists a moment linear functional, namely $\mathscr{L}_0$, such that the big -1 Jacobi polynomials $(Q_j^{(0)})_{j=0}^{2N}$ are orthogonal with respect to it. Due to the fact that $\tau_\alpha^{2N+1}(Q_j) = 0$ for $j = 0,1,\ldots,2N$, therefore the property of orthogonality holds in this case, i.e.,

$$\langle Q_n^{(0)}, Q_m^{(0)}\rangle_1 = \langle \mathbf{u}, Q_n^{(0)} Q_m^{(0)}\rangle = h_n(-2N-1,\beta,c)\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker (delta) symbol. (ii): If $m < 2N+1 \leqslant n$, then $\mathscr{L}_0(Q_n^{(0)}, Q_m^{(0)}) = 0$; at the same time $\tau_\alpha^{2N+1}(Q_j) = 0$ for $j = 0,1,\ldots,2N$, therefore the property of orthogonality holds in this case. (iii): If $2N+1 \leqslant n \leqslant m$, then $\mathscr{L}_0(Q_n^{(0)}, Q_m^{(0)}) = 0$; and since

$$\tau_\alpha^{2N+1} Q_n^{(0)}(x;-2N-1,\beta,c) = Q_{n-(2N+1)}^{(0)}(-x;2N+1,\beta,-c)$$

is orthogonal with respect to the linear functional $\mathbf{u}$, one has

$$\begin{aligned}\langle Q_n^{(0)}, Q_m^{(0)}\rangle_1 &= \lambda_N(\beta)\langle \mathbf{u}, \tau_\alpha^{2N+1} Q_n^{(0)}(x;-2N-1,\beta,c)\tau_\alpha^{2N+1} Q_m^{(0)}(x;-2N-1,\beta,c)\rangle \\ &= \lambda_N(\beta)\langle \mathbf{u}, Q_{n-2N-1}^{(0)}(-x;2N+1,\beta,-c) Q_{m-2N-1}^{(0)}(-x;2N+1,\beta,-c)\rangle \\ &= h_n(-2N-1,\beta,c)\delta_{n,m},\end{aligned}$$

and the property of orthogonality holds in this situation.

Moreover, since the polynomials are monic, it is straightforward to check using the recurrence relation (3.3) that

$$h_n = \langle \mathbf{u}, x Q_n^{(0)}(x) Q_{n-1}^{(0)}(x)\rangle = u_n \langle \mathbf{u}, x Q_{n-1}^{(0)}(x) Q_{n-1}^{(0)}(x)\rangle = u_n h_{n-1}. \tag{3.27}$$

Furthermore, the norm squared (3.19) and the recurrence coefficient u_n appearing in (3.3) satisfy (3.27). The situation which corresponds to (3.22), i.e., $\beta = -2N-1$, with $N \in \mathbb{N}$, $\alpha \in \mathbb{C}$, is similar so we will omit its proof. Hence the result holds. □

REMARK 3.6. Note that if $\alpha = -2N-1$ and $\beta \in \mathbb{C}$, due to (3.9) one must consider the moment linear form [**11**]

$$\mathscr{L}_0(p,q) = \sum_{j=0}^{N} \lambda_j\, p^{(j)}(1)\, q^{(j)}(1) + \sum_{j=0}^{N-1} \mu_j\, p^{(j)}(-1)\, q^{(j)}(-1),$$

where λ_j, μ_j are positive. So, taking into account (3.11), one has

$$\mathscr{L}_0(Q_j^{(0)}(x), \pi(x)) = 0, \quad j \geqslant 2N+1,$$

for any $\pi \in \mathbb{P}$. Moreover, by construction, if $0 \leqslant j < 2N+1$ one has

$$\langle Q_j^{(0)}(x), Q_j^{(0)}(x)\rangle_1 = \mathscr{L}_0\left(Q_j^{(0)}(x)p, Q_j^{(0)}(x)\right) = \langle \mathbf{u}, Q_j^{(0)}Q_j^{(0)}\rangle = h_j \neq 0.$$

If $j \geqslant 2N+1$ then taking into account Lemma 3.4 one has

$$\begin{aligned}
&\langle Q_j^{(0)}(x), Q_j^{(0)}(x)\rangle_1 \\
&\quad = \lambda_N(\beta)\langle \mathbf{u}, Q_{j-2N-1}^{(0)}(-x; 2N+1, \beta, -c)Q_{j-2N-1}^{(0)}(-x; 2N+1, \beta, -c)\rangle \\
&\quad = \lambda_N(\beta)h_{j-2N-1}(2N+1, \beta, -c) = h_j(-2N-1, \beta, c).
\end{aligned}$$

Observe that the last identity must be considered as a formal limit:

$$\tfrac{1}{2}\lim_{\epsilon\to 0} h_{2N+1}(-2N-1+\epsilon, \beta, c)h_{j-2N-1}(2N+1+\epsilon, \beta, -c) = \lim_{\epsilon\to 0} h_j(-2N-1+\epsilon, \beta, c).$$

For $\beta = -2N-1$, the situation is analogous since in (3.10) one must consider the moment linear form [**11**]

$$\mathscr{L}_1(p,q) = \sum_{j=0}^{N} \kappa_j p^{(j)}(1)q^{(j)}(-c) + \sum_{j=0}^{N-1} \ell_j p^{(j)}(-1)q^{(j)}(c),$$

where κ_j, ℓ_j are positive. In a similar way

$$\frac{1}{2}\lim_{\epsilon\to 0} h_{2N+1}(\alpha, -2N-1+\epsilon, c)h_{j-2N-1}(\alpha, 2N+1+\epsilon, -c) = \lim_{\epsilon\to 0} h_j(\alpha, -2N-1+\epsilon, c).$$

4. Conclusion and future work

We have studied the big -1 Jacobi polynomials, which are a $q \to -1$ limit of the Bannai–Ito polynomials, when at least one of the parameters α, β is non-standard and the coefficient u_n of the three-term recurrence relation (3.3) is equal to zero for a certain index, namely N, i.e., $\alpha = -2N-1$ or $\beta = -2N-1$ where N is a positive integer. We have also obtained a Gauss hypergeometric representation, its factorization, and the property of orthogonality, using techniques similar to those used in [**5**].

In [**6**, Section 4], a similar procedure for the big q-Jacobi polynomials was applied for non-standard parameters for which the coefficient of u_n in the recurrence relation is equal to zero for some $n \in \mathbb{N}_0$. Along these lines we are investigating a method for obtaining the orthogonality property of the big -1 Jacobi polynomials for non-standard parameters, following a procedure analogous to the one followed by T. E. Pérez and M. A. Piñar in [**17**] for the generalized Laguerre polynomials. In [**6**] the authors also studied the q-Racah polynomials for non-standard parameters for which the u_n coefficient vanishes for some $n \in \mathbb{N}_0$, and since the Bannai–Ito polynomials can be obtained from the q-Racah polynomials ([**8**, §14.2]) by taking an analogous limit $q \to -1$ (see [**20**]), it makes sense to perform a similar study for the Bannai–Ito polynomials. We are also working on obtaining explicit hypergeometric representations, their factorizations, and the property of orthogonality for the Bannai–Ito polynomials and other families of the continuous -1 hypergeometric orthogonal polynomial scheme for non-standard parameters.

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