Sampling Architectures for Ultra-Wideband Signals

Stephen D. Casey
Department of Mathematics and Statistics,
American University, Washington, DC 20016 USA
Email: *scasey@american.edu*

Howard S. Cohl
Applied and Computational Mathematics Division,
National Institute of Standards and Technology,
Gaithersburg, MD 20899-8910 USA
Email: *howard.cohl@nist.gov*

***Abstract*—Ultra-wideband (UWB) signal processing is a technology that has tremendous potential to develop advances in communication and information technology. However, it also presents challenges to the signal processing community, and, in particular, to sampling theory. This article outlines a UWB signal processing system via a basis projection and a basis system designed specifically for UWB signals. The method first windows the signal and then decomposes the signal into a basis via a continuous-time inner product operation, computing the basis coefficients in parallel. The windows are key, and we develop windows that have variable partitioning length, variable roll-off and variable smoothness. They preserve orthogonality of any orthonormal system between adjacent blocks. In this paper, we develop new windows, and give an outline for a new architecture for the projection. We then use this projection with a basis system designed to work with UWB signals, implementing modified Gegenbauer functions designed specifically for these signals.**

I. INTRODUCTION

Ultra-wideband (UWB) signal processing is a technology with many features that promise potential advances in wireless communications, networking, radar, imaging, and positioning systems. This article outlines a UWB signal processing system via a basis projection and a basis system designed especially for UWB signals. The method first windows the signal and then decomposes the signal into an orthonormal basis via a continuous-time inner product operation, computing the basis coefficients in parallel. We call this procedure the *Projection Method*. The windows are key, and we develop windows that have variable partitioning length, variable roll-off and variable smoothness. They are designed to preserve orthogonality of any orthonormal system between adjacent blocks. We then use the Projection Method with a basis system designed to work with UWB signals. This system is a modified Gegenbauer system designed specifically for UWB signals. This system minimizes the Gibbs phenomenon, giving the point values of a piecewise smooth signal with essentially the same accuracy as a smooth approximation, making it the ideal system to use for the Projection Method as applied to these signals. The use of the Gegenbauer system for UWB signals is known in the engineering community. Justification for this system is given by numerical simulation [13], [16] (and references therein). In this article, we provide an outline of an analytic justification, and we give new methods for creating windows and new outlines for the system architecture that advance our previous work [4]. Mathematical definitions and computations for the paper follow those given in Benedetto [1].

A UWB communication system is a large bandwidth system based on the transmission of very short pulses with relatively low energy. These systems operate by running as signaling waveforms, baseband pulses of very short duration, rather than the traditional method using a sinusoidal carrier. The UWB technique has a fine time resolution which makes it a technology appropriate for accurate ranging. The large bandwidth of a UWB system is dominated by its pulse shape and duration. This large system bandwidth relative to the information bandwidth allows UWB systems to operate with a low power spectral density. Such a low power spectral density implies that the UWB signal may be kept near or below the noise floor of detection devices. For these reasons, UWB technology has many potential advantages, such as high data rate, low probability of interception and detection, system simplicity, low cost, reduced average power consumption, weak sensitivity to the near-far problem and immunity to interference.

However, UWB systems present challenges to current methods of signal processing. From a signal processing perspective, we can approach this problem by implementing an appropriate signal decomposition in the analog portion that provides parallel outputs for integrated digital conversion and processing [3]. This naturally leads to an architecture with windowed time segmentation and parallel analog basis expansion. The method represents a change of view in sampling, from that of a stationary view of a signal used in classical sampling to an “short-time windowed stationary” view. This viewpoint gives that the time and frequency space “tile” occupied by the signal is processed quickly. The windows give us the tools to partition time-frequency so that the UWB signal can be partitioned uniformly but also quickly and efficiently. With the blocks, the signal can be sampled in parallel [3].

II. THE PROJECTION METHOD

Classical sampling theory applies to functions that are square integrable and band-limited. A function in $L^2(\mathbb{R})$ whose Fourier transform $\widehat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i t\omega}dt$ is compactly supported and has several smoothness and growth properties given in the Paley-Wiener Theorem. The choice to have 2π in the exponent simplifies certain expressions, e.g., for $f, g \in L^1 \cap L^2(\mathbb{R})$, $\widehat{f}, \widehat{g} \in L^1 \cap L^2(\widehat{\mathbb{R}})$, we have *Plancherel-Parseval* – $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\widehat{\mathbb{R}})}$ $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$. The Paley-Wiener Space $\mathbb{PW}_\Omega$ is defined as $\mathbb{PW}_\Omega = \{f$ continuous :

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$f, \widehat{f} \in L^2, \text{supp}(\widehat{f}) \subset [-\Omega, \Omega]\}$. The Whittaker-Kotel'nikov-Shannon (W-K-S) Sampling Theorem applies to functions in $\mathbb{PW}_\Omega$.

Theorem 1 (W-K-S Sampling Theorem): Let $f \in \mathbb{PW}_\Omega$, $\text{sinc}_T(t) = \sin(\frac{\pi}{T}t)/\pi t$, and $\delta_{nT}(t) = \delta(t - nT)$.

1) If $T \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = T\left(\left[\sum_{n\in\mathbb{Z}} \delta_{nT}\right] \cdot f\right) * \text{sinc}_T(t) \,. \qquad (1)$$

2) If $T \leq 1/2\Omega$ and $f(nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

To demonstrate the Projection Method, let's start with a few "back of the envelope computations." Let χ_S denote the characteristic (or indicator) function of the set S. Let $T > 0$ and let $g(t)$ be a function such that $\text{supp}\, g \subseteq [0, T]$. The T-*periodization* of g is $[g]^\circ(t) = \sum_{n=-\infty}^{\infty} g(t - nT)$. Let f be a signal of finite energy in the Paley-Wiener class $\mathbb{PW}_\Omega$. For a block of time T, let $f(t) = \sum_{k\in\mathbb{Z}} f(t)\chi_{[(k)T,(k+1)T]}(t)$. If we take a given block $f_k(t) = f(t)\chi_{[(k)T,(k+1)T]}(t)$, we can T-periodically continue the function, getting $[f_k]^\circ(t) = [f(t)\chi_{[(k)T,(k+1)T]}(t)]^\circ$. Expanding $(f_k)^\circ(t)$ in a Fourier series, we get $[f_k]^\circ(t) = \sum_{n\in\mathbb{Z}} \widehat{[f_k]^\circ}[n]\exp(2\pi int/T)$. The original function f is Ω band-limited. However, the truncated block functions f_k are not. The cutoff functions have a "ringing" of order $O(1/\omega)$ in frequency. Using the original Ω band-limit gives us a lower bound on the number of non-zero Fourier coefficients $\widehat{[f_k]^\circ}[n]$ as follows. We have $\frac{n}{T} \leq \Omega$, i.e., $n \leq T \cdot \Omega$. So, choose $N = \lceil T \cdot \Omega \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. For this choice of N, we compute

$$\begin{aligned} f(t) &= \sum_{k\in\mathbb{Z}} f(t)\chi_{[(k)T,(k+1)T]}(t) \\ &= \sum_{k\in\mathbb{Z}} \left[[f_k]^\circ(t)\right]\chi_{[(k)T,(k+1)T]}(t) \\ &\approx \sum_{k\in\mathbb{Z}} \left[\sum_{n=-N}^{n=N} \widehat{[f_k]^\circ}[n]\exp(2\pi int/T)\right]\chi_{[(k)T,(k+1)T]}(t) \,. \end{aligned}$$

Note that for the standard basis (sines, cosines), we can, for a fixed value of N, adjust to a large bandwidth Ω by choosing small time blocks T. Also, after a given set of time blocks, we can deal with an increase or decrease in bandwidth Ω by again adjusting the time blocks, e.g., given an increase in Ω, decrease T, and vice versa. The quality of the signal, as expressed in the accuracy of the representation of f, depends on N, Ω, and T.

We develop windows which preserve orthogonality of any orthonormal (ON) system between adjacent blocks. The construction uses any orthonormal basis for $L^2(\mathbb{R})$ and is created by solving a Hermite interpolation problem with constraints. These ON preserving windows allow us to create a method of time-frequency analysis for a wide class of signals, giving an arbitrary degree of smoothness in time which in turn then gives an arbitrary degree of cut-off decay in frequency. Preserving orthogonality requires that the ON windows $\{\mathbb{W}_k(t)\}$ satisfy $\sum_k [\mathbb{W}_k(t)]^2 \equiv 1$.

Let $T, r \in \mathbb{R}$, and consider a signal block of length $T + 2r$ centered at the origin. Let $0 < r \ll T/2$. Define $\text{Cap}(t)$ as follows:

$$\begin{cases} 0 & |t| \geq \frac{T}{2} + r\,, \\ 1 & |t| \leq \frac{T}{2} - r\,, \\ \sin(\pi/(4r)(t + (T/2 + r))) & \frac{-T}{2} - r < t < \frac{-T}{2} + r\,, \\ \cos(\pi/(4r)(t - (T/2 - r))) & \frac{T}{2} - r < t < \frac{T}{2} + r\,. \end{cases} \qquad (2)$$

Given $\text{Cap}(t)$, we form a windowing system $\{\text{Cap}_k(t)\}$ such that $\text{supp}(\text{Cap}_k(t)) \subseteq [kT - r, (k+1)T + r]$ for all k. Note that the Cap window has a continuous roll-off at the endpoints, windows the signal in $[\frac{-T}{2} - r, \frac{T}{2} + r]$ and is identically 1 on $[\frac{-T}{2} + r, \frac{T}{2} - r]$. It has a $1/\omega^2$ decay in frequency space, and also has the property that for all $t \in \mathbb{R}$,

$$[\text{Cap}_k(t)]^2 + [\text{Cap}_{k+1}(t)]^2 = 1\,.$$

If we had a signal f with an absolutely convergent Fourier series, then $(f \cdot \text{Cap})_k\hat{}[n] = \sum_m f[n-m]\text{Cap}\hat{}[m] = \widehat{f} * \text{Cap}\hat{}[n]$. The Fourier transform of Cap is a linear combination of $\text{sinc}\,\omega$ and $\sin\omega$ functions and has an asymptotic $1/\omega^2$ decay.

The theory of splines gives us the tools to generalize this system. The idea is to cut up the time domain into perfectly aligned segments so that there is no loss of information. We also want the systems to be smooth, so as to provide control over decay in frequency, and adaptive, so as to adjust accordingly to changes in frequency band. Finally, we develop our systems so that the orthogonality of bases in adjacent and possible overlapping blocks is preserved.

Definition 1 (ON Window System): An **ON Window System** is a set of functions $\{\mathbb{W}_k(t)\}$ such that for all $k \in \mathbb{Z}$,

$$\begin{aligned} &(i.) && \text{supp}(\mathbb{W}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\ &(ii.) && \mathbb{W}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r]\,, \\ &(iii.) && \mathbb{W}_k \text{ is symmetric about its midpoint}\,, \\ &(iv.) && \sum [\mathbb{W}_k(t)]^2 \equiv 1\,, \\ &(v.) && \{\widehat{\mathbb{W}_k{}^\circ}[n]\} \in l^1\,. \end{aligned} \qquad (3)$$

Conditions $(i.)$ and $(ii.)$ are partition properties, in that they give an exact snapshot of the input function f on $[kT + r, (k+1)T - r]$ with smooth roll-off at the edges. Conditions $(iii.)$ and $(iv.)$ are needed to preserve orthogonality between adjacent blocks. Condition $(v.)$ is needed for the computation of Fourier coefficients. We generate our systems by translations and dilations of a given window $\mathbb{W}_I$, where $\text{supp}(\mathbb{W}_I) = [-\frac{T}{2} - r, \frac{T}{2} + r]$. Condition $(v.)$ is needed for the following reason. Let $I = T + 2r$ and let $\mathbb{PW}_\Omega$ denote the Paley-Wiener space for bandlimit Ω. Let $f \in \mathbb{PW}_\Omega$ and let $\{\mathbb{W}_k(t)\}$ be an ON window system with generating window $\mathbb{W}_I$. Then

$$\frac{1}{I}\int_{-T/2-r}^{T/2-r} [f \cdot \mathbb{W}_I]^\circ(t)\exp(-2\pi int/[I])\,dt = \widehat{f} * \widehat{\mathbb{W}_I}[n]\,. \quad (4)$$

Our general window function $\mathbb{W}_I$ is m-times differentiable, has $\text{supp}(\mathbb{W}_I) = [-\frac{T}{2} - r, \frac{T}{2} + r]$, and has values

$$\mathbb{W}_I = \begin{cases} 0 & |t| \geq T/2 + r\,, \\ 1 & |t| \leq T/2 - r\,, \\ \rho(\pm t) & T/2 - r < |t| < T/2 + r\,. \end{cases} \tag{5}$$

We solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{cases} (a.) & \rho(T/2 - r) = 1\,, \\ (b.) & \rho^{(n)}(T/2 - r) = 0\,,\ n = 1, 2, \dots, m\,, \\ (c.) & \rho^{(n)}(T/2 + r) = 0\,,\ n = 0, 1, 2, \dots, m\,. \end{cases}$$

with the conditions that $\rho \in C^m$ and

$$[\rho(t)]^2 + [\rho(-t)]^2 = 1 \text{ for } t \in [\pm(\tfrac{T}{2} - r), \pm(\tfrac{T}{2} + r)]\,. \tag{6}$$

We will modify the Cap window given above using B-splines to construct our functions. Let $0 < \alpha \ll \beta$ and consider $\chi_{[-\alpha,\alpha]}$. We want the $(m+2)$-fold convolution of $\chi_{[\alpha,\alpha]}$ to fit in the interval $[-\beta, \beta]$. Then we choose α so that $0 < (m+2)\alpha < \beta$, and let

$$\Psi(t) = \underbrace{\chi_{[-\alpha,\alpha]} * \chi_{[-\alpha,\alpha]} * \cdots * \chi_{[-\alpha,\alpha]}(t)}_{(m+2)-times}\,.$$

The C^m solution for ρ is given by a theorem of Schoenberg (see [18], pp. 7-8). Schoenberg solved the Hermite interpolation problem with endpoints -1 and 1. An interpolant that minimizes the Chebyshev norm is called the *perfect spline*. The perfect spline $S(t)$ for the Hermite problem with endpoints -1 and 1 such that $S(1) = 1$, $S^{(n)}(1) = 0$, $n = 1, 2, \dots, m$, and $S^{(n)}(-1) = 0$, $n = 0, 1, \dots, m$, is given by the integral of the function

$$M(t) = (-1)^m \sum_{j=0}^{m} \frac{\Psi(t - t_j)}{\phi'(t_j)}\,,$$

where Ψ is the $m+1$ convolution of characteristic functions, the knot points are $t_j = -\cos(\frac{\pi j}{m})$, $j = 0, 1, \dots, m$, and $\phi(t) = \prod_{j=0}^{k}(t - t_j)$. Given these knots, we have to choose α to fit the knot points. If m is even, the midpoint occurs at the $m/2$ knot point. If m is odd, the midpoint occurs at the midpoint between the $m/2$ and $(m+1)/2$ knot points. Let $\xi = l(t) = \frac{r}{2}(t-1)$, and let $\alpha(\xi) = S \circ l(\pm\xi)\,, |\xi| \leq r$. Let $A = \int_{-r}^{r} \alpha(\zeta)\, d\zeta$. Now, normalize α by letting $\beta(\xi) = \frac{\pi}{2A}\alpha(\xi)$, and let

$$\Theta(\tau) = \int_{-r}^{\tau} \beta(\xi)\, d\xi\,,\ |\tau| \leq r\,. \tag{7}$$

Define

$$\rho_{\text{up}}(\tau) = \sin(\Theta(\tau))\,,\ \rho_{\text{down}}(\tau) = \cos(\Theta(\tau))\,. \tag{8}$$

We define our C^m window $\mathbb{W}_I(t) = \mathbb{ON}_{C^m}(t)$ as follows:

$$\begin{cases} 0 & |t| \geq T/2 + r\,, \\ 1 & |t| \leq T/2 - r\,, \\ \rho_{\text{up}}(t + (T/2 + r) & -T/2 - r < t < -T/2 + r\,, \\ \rho_{\text{down}}(t - (T/2 - r))) & T/2 - r < t < T/2 + r\,. \end{cases} \tag{9}$$

We translate the window as needed. The resultant windowing system has variable partitioning length, variable roll-off, and variable smoothness. With each degree of smoothness, we get an additional degree of decay in frequency.

We designed the ON windows $\{\mathbb{W}_k(t)\}$ so that they preserve orthogonality of basis elements of overlapping blocks. Because of the partition properties of these systems, we need only check the orthogonality of adjacent overlapping blocks. The best way to think about the construction is to visualize how one would create the extension for a system of sines and cosines. We would extend the odd reflections about the left endpoint and the even reflections about the right. Let $\{\varphi_j(t)\}$ be an ON basis for $L^2[-\frac{T}{2}, \frac{T}{2}]$. Define

$$\widetilde{\varphi_j}(t) = \begin{cases} 0 & |t| \geq T/2 + r\,, \\ \varphi_j(t) & |t| \leq T/2 - r\,, \\ -\varphi_j(-T - t) & -T/2 - r < t < -T/2\,, \\ \varphi_j(T - t) & T/2 < t < T/2 + r\,. \end{cases} \tag{10}$$

Theorem 2: $\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi_j}\}$ is an ON basis for $L^2(\mathbb{R})$.

Proof : See [3]. □

This theorem gives a new method for analog to digital conversion. Unlike W-K-S Sampling, which examined the function at specific points and then uses those individual points to recreate the curve, the Projection Method breaks the signal into time blocks and then approximates their respective periodic expansions with a Fourier series. This process allows the system to individually evaluate each piece and base its calculation on the needed bandwidth. The individual Fourier series are then summed, recreating a close approximation of the original signal. It is important to note that instead of fixing T, the method allows us to fix any of the three parameters while allowing the other two to fluctuate. From the design point of view, the easiest and most practical parameter to fix is N. For situations in which the bandwidth does not need flexibility, it is possible to fix Ω and T by the equation $N = \lceil T \cdot \Omega \rceil$. However, if greater bandwidth Ω is need, choose shorter time blocks T. Theorem (2) is for any ON basis, giving us the freedom to choose a basis to optimize the analysis of a given class of signals, e.g., modified Gegenbauer (for UWB) or Walsh (for binary).

Given characteristics of the class of input signals, the choice of basis functions used can be tailored to optimal representation of the signal or a desired characteristic in the signal.

Theorem 3 (Projection Formula for ON Windowing): Let $\{\mathbb{W}_k(t)\}$ be ON windows, and let $\{\Psi_{k,n}\} = \{\mathbb{W}_k \widetilde{\varphi_n}\}$ be an ON basis that preserves orthogonality between adjacent windows. Let $f \in \mathbb{PW}_\Omega$ and $N = N(T, \Omega)$ be such that $\langle f, \Psi_{k,n} \rangle = 0$ for all $n > N$ and all k. Then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N}^{N} \langle f, \Psi_{k,n} \rangle \Psi_{k,n}(t) \right]. \tag{11}$$

Proof : See [3]. □

The analysis of the error generated by the Projection Method involves looking at the decay rates of the Fourier coefficients. If we are working with the standard basis, for $f \in C(\mathbb{T}_{2\Phi})$, we can define the modulus of continuity as $\mu(\delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|$. This measures the local oscillation of the signal. We say that f satisfies a Hölder condition with exponent α if there exists a constant K such that $|f(x+\delta) - f(x)| \leq K\delta^\alpha$. If f is m-times continuously differentiable and f^m satisfies a Hölder condition with exponent α, then there exists a constant K such that $|\widehat{f}[n]| \leq K\frac{1}{n^{m+\alpha}}$.

The sharp cut-offs $\chi_{[kT,(k+1)T]}$ have a decay of only $\mathcal{O}(1/\omega)$ in frequency. We designed the ON windows so that the windows have decay $\mathcal{O}(1/(\omega)^{m+2})$ in frequency. This makes the error on each block summable.

We assume $\mathbb{W}_k$ is C^m. Therefore, $\widehat{\mathbb{W}_k}(\omega) = \mathcal{O}(1/(\omega)^{m+2})$. We will analyze the error $\mathcal{E}_{k_\mathcal{P}}$ on a given block. Let $M = \|(f \cdot \mathbb{W}_k)\|_{L^2(\mathbb{R})}$. Then $\mathcal{E}_{k_\mathcal{P}}$

$$\begin{aligned} &= \sup\left|(f(t)\cdot\mathbb{W}_k) - \left[\sum_{n=-N}^{N}\langle f, \Psi_{n,k}\rangle\Psi_{n,k}(t)\right]\mathbb{W}_k(t)\right| \\ &= \sup\left[\sum_{|n|>N}\langle f, \Psi_{n,k}\rangle\Psi_{n,k}(t)\right]\mathbb{W}_k(t) \leq \sum_{|n|>N}\frac{M}{n^{m+2}}. \end{aligned}$$

III. MODIFIED GEGENBAUER SYSTEMS

The Gegenbauer polynomials are the symmetric specialization of the Jacobi polynomials (see [12], [17, Chapter 18]). They are used in a UWB communication system to construct pulses with narrow widths. The Gegenbauer waveform is used to modulate data, and has demonstrated superior performance to classic waveforms, e.g., Gaussian waveforms and the Hermite systems. The investigations in the engineering literature give justifications for their claims via numerical simulation [13], [16]. In this section, we develop a modified Gegenbauer system, and use it to construct a windowed basis system for UWB signals with exponentially small error on each block.

Using the spirit of [13], [16] (and references therein), we define an ON basis for $L^2[-\frac{T}{2}, \frac{T}{2}]$ using modified Gegenbauer functions, constructed from Gegenbauer polynomials. The Gegenbauer polynomials are modified so that they zero-out at the endpoints and normalized to create an ON system. This then allows UWB signals to be expanded in the Projection Method (11) using the modified Gegenbauer system.

The Gegenbauer polynomials $C_n^\nu : \mathbb{C} \to \mathbb{C}$ are orthogonal over $(-1, 1)$ with orthogonality relation given by [17, Table 18.3.1]

$$\int_{-1}^{1} C_n^\nu(x) C_m^\nu(x) w(x;\nu) dx = h_n^\nu \delta_{n,m}, \tag{12}$$

for $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, where

$$w(x;\nu) := (1-x^2)^{\nu-1/2}, \tag{13}$$

$$h_n^\nu := \frac{2^{1-2\nu}\pi\Gamma(2\nu+n)}{(\nu+n)\Gamma^2(\nu)n!}. \tag{14}$$

The gamma function $\Gamma : \mathbb{C} \setminus -\mathbb{N}_0 \to \mathbb{C}$ is defined in [17, Chapter 5], where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The Gegenbauer polynomials are defined using the Gauss hypergeometric function [17, (18.5.9)] as

$$C_n^\nu(x) := \frac{(2\nu)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, 2\nu+n \\ \nu+\frac{1}{2} \end{array}; \frac{1-x}{2}\right),$$

where the Pochhammer symbol $(\cdot)_n : \mathbb{C} \to \mathbb{C}$ for $n \in \mathbb{N}_0$ is defined by $(a)_n := (a)(a+1)\cdots(a+n-1)$, and the Gauss hypergeometric function is defined in [17, Chapter 15]. They have a Rodrigues-type formula [17, Table 18.5.1]

$$C_n^\nu(x) := \frac{(-1)^n(2\nu)_n}{2^n(\nu+\frac{1}{2})_n n!}\frac{1}{w(x;\nu)}\frac{d^n}{dx^n}w(x;\nu+n),$$

and can also be computed using three-term recurrence relations [17, Table 18.9.1] or using trigonometric [14, p. 220] series expressions. The Gegenbauer polynomials are given in terms of the more general Jacobi polynomials symmetric in parameters with [17, (18.7.1)]

$$C_n^\nu(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n}P_n^{(\nu-1/2,\nu-1/2)}(x).$$

Consider the modified Gegenbauer function $\mathcal{C}_n^\nu : [-\frac{T}{2}, \frac{T}{2}] \times (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{C}_n^\nu(t;T) := \sqrt{\frac{2\,w\left(\frac{2t}{T};\nu\right)}{Th_n^\nu}}\,C_n^\nu\left(\frac{2t}{T}\right). \tag{15}$$

It is easy to see from (12) that these functions form an ON basis for $L^2[-\frac{T}{2}, \frac{T}{2}]$ with $\nu \in (\frac{1}{2}, \infty)$, namely

$$\int_{-T/2}^{T/2} \mathcal{C}_n^\nu(t;T)\mathcal{C}_m^\nu(t;T)dt = \delta_{m,n}.$$

Note that we exclude the parameters $\nu \in (-\frac{1}{2}, \frac{1}{2}]$ in order to keep the endpoints $\pm\frac{T}{2}$ in the domain of integration. By using (13) and (14), one has

$$\begin{aligned} \mathcal{C}_n^\nu(t;T) = \frac{2^{2\nu-1/2}\Gamma(\nu)}{T^\nu}\sqrt{\frac{(n+\nu)n!}{\pi\Gamma(2\nu+n)}} \\ \times\left(\left(\tfrac{T}{2}\right)^2 - t^2\right)^{\nu/2-1/4} C_n^\nu\left(\frac{2t}{T}\right). \end{aligned} \tag{16}$$

The modified Gegenbauer system zeros out at the endpoints, which allows us to use it to create the windowed ON basis $\{\Psi_{k,n}\} = \{\mathbb{W}_k\,\mathcal{C}_n^\nu(t;T)\}$, where we window with $\mathbb{ON}_{C^m}$.

We use two theorems of Gottlieb and Shu [7] to give an outline of an analytic argument showing they minimize the Gibbs phenomenon. Given an integrable function f defined on $[-\frac{T}{2}, \frac{T}{2}]$, we can compute its modified Gegenbauer coefficients $\widehat{f^\nu}(l)$ by

$$\widehat{f^\nu}(l) = \int_{-T/2}^{T/2} f(t)\mathcal{C}_l^\nu(t;T)dt. \tag{17}$$

This then gives the Gegenbauer expansion for the first $j+1$ terms as

$$f_j^\nu(t) = \sum_{l=0}^{j}\widehat{f^\nu}(l)\mathcal{C}_l^\nu(t;T). \tag{18}$$

It is important to note that in order to get exponential decay in our error computation at the end of this section (24), ν must grow linearly with N. We assume $\nu = \alpha N$, for $\alpha > 0$. Now, let f be our original signal, f^m be the expansion of f into m-th degree modified Gegenbauer functions, and f_N^m be the expansion of f into m-th degree modified Gegenbauer functions truncated at N. We want to estimate $\|(f - f_N^m)\|$. By the triangle inequality,

$$\|(f - f_N^m)\| \leq \|(f - f^m)\| + \|(f^m - f_N^m)\| \,. \tag{19}$$

Assume $m = \beta N$, for $\beta > 0$. Let $TE(\nu, \alpha, \beta, N)$ be the truncation error in $\|(f^m - f_N^m)\|$ (Theorem 4.1, page 659 from [7]). One has

$$TE(\nu, \alpha, \beta, N) \leq C(q_T)^N \,, \tag{20}$$

where $q_T = \frac{(\beta+2\alpha)^{(\beta+2\alpha)}}{(2\alpha)^\alpha \beta^\beta}$. Note, q_T is the key to exponential decay in the truncation estimate. We have that if $\alpha = \beta < 2/27$, $q_T < 1$. (In fact, if $\alpha = \beta$, the function $g(x) = [(3x)^{(3x)}]/[2^x \cdot (x)^{(2x)}]$ is minimized at $x = 2/(27e)$ with value $e^{-2/(27e)}$.)

Next, we estimate the regularization error $\|(f - f^m)\|$. Let

$$RE(\nu, m) = \sup_{-\frac{T}{2} \leq t \leq \frac{T}{2}} \left| f(t) - \sum_{l=0}^{m} \widehat{f^\nu}(l) \mathcal{C}_l^\nu(t;T) \right| . \tag{21}$$

We modify Theorem 4.3, p. 662 from [7]. Here, we have to be careful, because the estimate in [7] involves $\frac{1}{C_l^\nu(1)}$ (Equation (4.21), p. 661). The modified Gegenbauer functions zero out at the endpoints. Therefore, we had to recompute the estimates. Adjusting the arguments in Gottlieb and Shu for the modified Gegenbauer functions, we arrive at $RE(\nu, m) \leq$

$$\max_{-\frac{T}{2} \leq t \leq \frac{T}{2}} |f(t)| \left[1 + \frac{2^{1-\nu}}{\sqrt{T}\Gamma(\nu + 1/2)} \sum_{l=0}^{m} \sqrt{\frac{(l+\nu)\Gamma(l+2\nu)}{\Gamma(l-1)}} \right].$$

Using Stirling's approximation formula [17, p. 141], and bounding $\sup_{-\frac{T}{2} \leq t \leq \frac{T}{2}} |f(t)|$ by M, we get that

$$RE(\nu, m) \leq \frac{M}{\sqrt{2\pi T}} \left[\frac{2^{1-\nu}}{(\nu - \frac{1}{2})^\nu} \right] \left[\frac{(m+2\nu)^{\frac{(m+2\nu)}{2}}}{m^{\frac{m}{2}}} \right] . \tag{22}$$

We can now follow the proof of Theorem 4.3, pp. 662-663 from [7]. If $\nu = \gamma m$, there exists $q_R < 1$ such that the error in this estimate satisfies, for $0 < r < 1$,

$$RE(\nu, m) \leq Cm(q_R)^m, \; q_R = \frac{(1+2\gamma)^{\frac{(1+2\gamma)}{2}}}{\gamma^{2\gamma}} r \,. \tag{23}$$

We close by computing $\mathcal{E}_{k_\mathcal{P}}$ in terms of the modified Gegenbauer system. The minimization of the Gibbs phenomenon, giving the point values of a piecewise smooth signal with essentially the same accuracy as a smooth approximation, makes this system the ideal system to use for the Projection Method as applied to UWB systems. Let $\sigma \in \mathbb{N}$ be the smoothness parameter, and assume $\mathbb{W}_k$ is C^σ, and so $\widehat{\mathbb{W}_k}(\omega) = \mathcal{O}(1/(\omega)^{\sigma+2})$. Now approximate the signal f with the windowed ON basis $\{\Psi_{k,n}\} = \{\widehat{\mathbb{W}_k \mathcal{C}_n^\nu(t;T)}\}$, where we window with $\mathbb{ON}_{C^\sigma}$. Let $q = \max\{q_T, q_R\}$. Note, $q < 1$. Then, the error $\mathcal{E}_{k_\mathcal{P}}$ on a given block is

$$\sup \left| (f(t) \cdot \mathbb{W}_k) - \left[\sum_{n=-N}^{N} \langle f, \Psi_{n,k} \rangle \Psi_{n,k}(t) \right] \mathbb{W}_k(t) \right| \tag{24}$$

$$\leq \sup \left[\sum_{|n|>N} \left| \langle f, \Psi_{n,k} \rangle \Psi_{n,k}(t) \right| \right] \mathbb{W}_k(t) \leq \sum_{|n|>N} \frac{e^{\log(q)N}}{n^{\sigma+2}} .$$

Since $q < 1$, $e^{\log(q)N}$ decays exponentially as N increases.

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