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Generalizations of generating functions for hypergeometric orthogonal polynomials with definite integrals

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ABSTRACT

We generalize generating functions for hypergeometric orthogonal polynomials, namely Jacobi, Gegenbauer, Laguerre, and Wilson polynomials. These generalizations of generating functions are accomplished through series rearrangement using connection relations with one free parameter for these orthogonal polynomials. We also use orthogonality relations to determine corresponding definite integrals.

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1. Introduction

In this paper, we apply connection relations (see for instance Andrews et al. [2, Section 7.1]; Askey [3, Lecture 7]) with one free parameter for the Jacobi, Gegenbauer, Laguerre, and Wilson polynomials (see Chapter 18 in [12]) to generalize generating functions for these orthogonal polynomials using series rearrangement. This is because connection relations with one free parameter only involve a summation over products of gamma functions and are straightforward to sum. We have already applied our series rearrangement technique using a connection relation with one free parameter to the generating function for Gegenbauer polynomials [12, (18.12.4)]

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^\infty \rho^n C_n^\nu(x). \tag{1}$$

The connection relation for Gegenbauer polynomials is given in Olver et al. [12, (18.18.16)] (see also Ismail [9, (9.1.2)]), namely

$$C_n^\nu(x)=\frac{1}{\mu}\sum_{k=0}^{\lfloor n/2\rfloor}(\mu+n-2k)\frac{(\nu-\mu)_k\,(\nu)_{n-k}}{k!(\mu+1)_{n-k}}C^\mu_{n-2k}(x). \tag{2}$$

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Inserting (2) into (1), we obtained a result [5, (1)] which generalizes (1), namely

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^{\infty}f_n^{(\nu,\mu)}(\rho)C_n^\mu(x), \tag{3}$$

where $f_n^{(\nu,\mu)}:\{z\in\mathbf{C}:0<|z|<1\}\setminus(-1,0)\to\mathbf{C}$ is defined by

$$f_n^{(\nu,\mu)}(\rho):=\frac{\Gamma(\mu)e^{i\pi(\mu-\nu+1/2)}(n+\mu)}{\sqrt{\pi}\,\Gamma(\nu)\rho^{\mu+1/2}(1-\rho^2)^{\nu-\mu-1/2}}Q_{n+\mu-1/2}^{\nu-\mu-1/2}\left(\frac{1+\rho^2}{2\rho}\right),$$

where Q_ν^μ is the associated Legendre function of the second kind [12, Chapter 14]. It is easy to demonstrate that $f_n^{(\nu,\nu)}(\rho)=\rho^n$. We have also successfully applied this technique to an extension of (1) expanded in Jacobi polynomials using a connection relation with two free parameters in Cohl [4, Theorem 1]. In this case the coefficients of the expansion are given in terms of Jacobi functions of the second kind. Applying this technique with connection relations with more than one free parameter is therefore possible, but it is more intricate and involves rearrangement and summation of three or more increasingly complicated sums. The goal of this paper is to demonstrate the effectiveness of the series rearrangement technique using connection relations with one free parameter by applying it to some of the most fundamental generating functions for hypergeometric orthogonal polynomials.

Unless otherwise stated, the domains of convergence given in this paper are those of the original generating function and/or its corresponding definite integral. In this paper, we only justify summation interchanges for a few of the theorems we present. For the interchange justifications we have given, we give all the details. However, for the sake of brevity, we leave justification for the remaining interchanges to the reader.

Here we will make a short review of the special functions which are used in this paper. The generalized hypergeometric function ${}_pF_q:\mathbf{C}^p\times(\mathbf{C}\setminus-\mathbf{N}_0)^q\times\{z\in\mathbf{C}:|z|<1\}\to\mathbf{C}$ (see Chapter 16 in Olver et al. [12]) is defined as

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right):=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{n!}.$$

When $p=2$, $q=1$, this is the special case referred to as the Gauss hypergeometric function ${}_2F_1:\mathbf{C}^2\times(\mathbf{C}\setminus-\mathbf{N}_0)\times\{z\in\mathbf{C}:|z|<1\}\to\mathbf{C}$ (see Chapter 15 of Olver et al. [12]). When $p=1$, $q=1$ this is Kummer's confluent hypergeometric function of the first kind $M:\mathbf{C}\times(\mathbf{C}\setminus-\mathbf{N}_0)\times\mathbf{C}\to\mathbf{C}$ (see Chapter 13 in Olver et al. [12]), namely

$$M(a,b,z):=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}={}_1F_1\left(\begin{matrix}a\\ b\end{matrix};z\right).$$

When $p=0$, $q=1$, this is related to the Bessel function of the first kind (see Chapter 10 in Olver et al. [12]) $J_\nu:\mathbf{C}\setminus\{0\}\to\mathbf{C}$, for $\nu\in\mathbf{C}$, defined by

$$J_\nu(z):=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\begin{matrix}-\\ \nu+1\end{matrix};\frac{-z^2}{4}\right). \tag{4}$$

The case of $p=0$, $q=1$ is also related to the modified Bessel function of the first kind (see Chapter 10 in Olver et al. [12]) $I_\nu:\mathbf{C}\setminus(-\infty,0]\to\mathbf{C}$, for $\nu\in\mathbf{C}$, defined by

$$I_\nu(z):=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\begin{matrix}-\\ \nu+1\end{matrix};\frac{z^2}{4}\right).$$

When $p=1$, $q=0$, this is the binomial expansion (see for instance Olver et al. [12, (15.4.6)]), namely

$${}_1F_0\left(\begin{matrix}\alpha\\ -\end{matrix};z\right)=(1-z)^{-\alpha}. \tag{5}$$

In these sums, the Pochhammer symbol (rising factorial) $(\cdot)_n:\mathbf{C}\to\mathbf{C}$ [12, (5.2.4)] is defined by

$$(z)_n:=\prod_{i=1}^{n}(z+i-1),$$

where $n\in\mathbf{N}_0$. Also, when $z\notin-\mathbf{N}_0$ we have (see [12, (5.2.5)])

$$(z)_n=\frac{\Gamma(z+n)}{\Gamma(z)}, \tag{6}$$

where $\Gamma:\mathbf{C}\setminus-\mathbf{N}_0\to\mathbf{C}$ is the gamma function (see Chapter 5 in Olver et al. [12]).

Throughout this paper we rely on the following definitions. For $a_1,a_2,a_3,\ldots\in\mathbf{C}$, if $i,j\in\mathbf{Z}$ and $j<i$ then $\sum_{n=i}^{j}a_n=0$ and $\prod_{n=i}^{j}a_n=1$. The set of natural numbers is given by $\mathbf{N}:=\{1,2,3,\ldots\}$, the set $\mathbf{N}_0:=\{0,1,2,\ldots\}=\mathbf{N}\cup\{0\}$, and the set $\mathbf{Z}:=\{0,\pm1,\pm2,\ldots\}$. The set $\mathbf{R}$ represents the real numbers.

2. Expansions in Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)} : \mathbf{C} \to \mathbf{C}$ can be defined in terms of a terminating Gauss hypergeometric series as follows (Olver et al. [12, (18.5.7)])

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-z}{2}\right),$$

for $n \in \mathbf{N}_0$, and $\alpha, \beta > -1$ such that if $\alpha, \beta \in (-1, 0)$ then $\alpha+\beta+1 \neq 0$. The orthogonality relation for Jacobi polynomials can be found in Olver et al. [12, (18.2.1), (18.2.5), Table 18.3.1]

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}\delta_{m,n}. \tag{7}$$

A connection relation with one free parameter for Jacobi polynomials can be found in Olver et al. [12, (18.18.14)], namely

$$P_n^{(\alpha,\beta)}(x) = \frac{(\beta+1)_n}{(\gamma+\beta+1)(\gamma+\beta+2)_n} \times \sum_{k=0}^{n} \frac{(\gamma+\beta+2k+1)(\gamma+\beta+1)_k\,(n+\beta+\alpha+1)_k(\alpha-\gamma)_{n-k}}{(\beta+1)_k\,(n+\gamma+\beta+2)_k(n-k)!}P_k^{(\gamma,\beta)}(x). \tag{8}$$

In the remainder of the paper, we will use the following global notation $\mathrm{R} := \sqrt{1+\rho^2-2\rho x}$.

Theorem 1. *Let $\alpha \in \mathbf{C}$, $\beta, \gamma > -1$ such that if $\beta, \gamma \in (-1, 0)$ then $\beta+\gamma+1 \neq 0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. Then*

$$\frac{2^{\alpha+\beta}}{\mathrm{R}\,(1+\mathrm{R}-\rho)^\alpha\,(1+\mathrm{R}+\rho)^\beta} = \frac{1}{\gamma+\beta+1}\sum_{k=0}^{\infty}\frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k\left(\frac{\alpha+\beta+1}{2}\right)_k\left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+\beta+1)_k\left(\frac{\gamma+\beta+2}{2}\right)_k\left(\frac{\gamma+\beta+3}{2}\right)_k} \times {}_3F_2\left(\begin{matrix}\beta+k+1, \alpha+\beta+2k+1, \alpha-\gamma\\ \alpha+\beta+k+1, \gamma+\beta+2k+2\end{matrix}; \rho\right)\rho^k P_k^{(\gamma,\beta)}(x). \tag{9}$$

Proof. Olver et al. [12, (18.12.1)] give the generating for Jacobi polynomials, namely

$$\frac{2^{\alpha+\beta}}{\mathrm{R}\,(1+\mathrm{R}-\rho)^\alpha\,(1+\mathrm{R}+\rho)^\beta} = \sum_{n=0}^{\infty}\rho^n P_n^{(\alpha,\beta)}(x). \tag{10}$$

This generating function is special in that it is the only known algebraic generating function for Jacobi polynomials (see [9, p. 90]). Using the Jacobi connection relation (8) in (10) produces a double sum. In order to justify reversing the order of the double summation expression we show that

$$\sum_{n=0}^{\infty}|c_n|\sum_{k=0}^{n}|a_{nk}|\left|P_k^{(\alpha,\beta)}(x)\right| < \infty, \tag{11}$$

where $c_n = \rho^n$ and a_{nk} are the connection coefficients satisfying

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} a_{nk}P_k^{(\gamma,\beta)}(x).$$

We assume that $\alpha, \beta, \gamma > -1$, $x \in [-1, 1]$, and $|\rho| < 1$. It follows from [15, Theorem 7.32.1] that

$$\max_{x\in[-1,1]}\left|P_n^{(\alpha,\beta)}(x)\right| \le K_1(1+n)^\sigma, \tag{12}$$

where K_1 and σ are positive constants. In order to estimate a_{nk} we use

$$a_{nk} = \frac{\int_{-1}^{1}(1-x)^\gamma(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_k^{(\gamma,\beta)}(x)\,dx}{\int_{-1}^{1}(1-x)^\gamma(1+x)^\beta\{P_k^{(\gamma,\beta)}(x)\}^2\,dx}.$$

Using (12) we have

$$\left|\int_{-1}^{1}(1-x)^{\gamma}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_k^{(\gamma,\beta)}(x)\,dx\right| \le K_2(1+k)^{\sigma}(1+n)^{\sigma}.$$

Using (7), we get

$$\left|\int_{-1}^{1}(1-x)^{\gamma}(1+x)^{\beta}\{P_k^{(\gamma,\beta)}(x)\}^2\,dx\right| \ge \frac{K_3}{1+k}$$

where $K_3 > 0$. Therefore,

$$|a_{nk}| \le K_4(1+k)^{\sigma+1}(1+n)^{\sigma}.$$

Finally, we show (11):

$$\sum_{n=0}^{\infty}|c_n|\sum_{k=0}^{n}|a_{nk}|\left|P_k^{(\alpha,\beta)}(x)\right| \le K_5\sum_{n=0}^{\infty}|\rho|^n\sum_{k=0}^{n}(1+k)^{2\sigma+1}(1+n)^{\sigma} \le K_5\sum_{n=0}^{\infty}|\rho|^n(1+n)^{3\sigma+2} < \infty,$$

because $|\rho| < 1$. Reversing the order of the summation and shifting the n-index by k with simplification, and by analytic continuation in α, we produce the generalization (9). □

Theorem 2. *Let $\alpha \in \mathbf{C}$, $\beta, \gamma > -1$ such that if $\beta, \gamma \in (-1, 0)$ then $\beta + \gamma + 1 \neq 0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in (-1, 1)$. Then*

$$\left(\frac{2}{(1-x)\rho}\right)^{\alpha/2}\left(\frac{2}{(1+x)\rho}\right)^{\beta/2} J_\alpha\left(\sqrt{2(1-x)\rho}\right) I_\beta\left(\sqrt{2(1+x)\rho}\right)$$
$$= \frac{1}{(\gamma+\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k\left(\frac{\alpha+\beta+1}{2}\right)_k\left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+1)_k(\beta+1)_k(\alpha+\beta+1)_k\left(\frac{\gamma+\beta+2}{2}\right)_k\left(\frac{\gamma+\beta+3}{2}\right)_k}$$
$$\times {}_2F_3\left(\begin{matrix}2k+\alpha+\beta+1, \alpha-\gamma\\ \alpha+\beta+k+1, \gamma+\beta+2k+2, \alpha+k+1\end{matrix}; \rho\right)\rho^k P_k^{(\gamma,\beta)}(x). \tag{13}$$

Proof. Olver et al. [12, (18.12.2)] give a generating function for Jacobi polynomials, namely

$$\left(\frac{2}{(1-x)\rho}\right)^{\alpha/2}\left(\frac{2}{(1+x)\rho}\right)^{\beta/2} J_\alpha\left(\sqrt{2(1-x)\rho}\right) I_\beta\left(\sqrt{2(1+x)\rho}\right)$$
$$= \sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}\rho^n P_n^{(\alpha,\beta)}(x). \tag{14}$$

Using the connection relation for Jacobi polynomials (8) in (14) produces a double sum. Reversing the order of the summation and shifting the n-index by k with simplification produces this generalization for a generating function of Jacobi polynomials. □

Definition 3. A *companion identity* is one which is produced by applying the map $x \mapsto -x$ to an expansion over Jacobi polynomials or in terms of those orthogonal polynomials which can be obtained as limiting cases of Jacobi polynomials (i.e., Gegenbauer, Chebyshev, and Legendre polynomials) with argument x in conjunction with the parity relations for those orthogonal polynomials.

By starting with (9) and (13), applying the parity relation for Jacobi polynomials (see for instance Olver et al. [12, Table 18.6.1])

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x),$$

and mapping $\rho \mapsto -\rho$, one obtains the corresponding companion identities. Although for (13), one must substitute $-1 = e^{\pm i\pi}$, and use Olver et al. [12, (10.27.6)]. Therefore Theorems 1 and 2 are valid when the left-hand sides remain the same, and on the right-hand sides $\alpha, \beta \mapsto \beta, \alpha$, the arguments of the ${}_3F_2$ and ${}_2F_3$ are replaced by $-\rho$, and the order of the Jacobi polynomials become (α, γ).

Theorem 4. *Let $\alpha \in \mathbf{C}$, $\beta, \gamma > -1$ such that if $\beta, \gamma \in (-1, 0)$ then $\beta + \gamma + 1 \neq 0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1, 0]$, $x \in [-1, 1]$. Then*

$$\frac{(1+x)^{-\beta/2}}{\mathrm{R}^{\alpha+1}} P_{\alpha}^{-\beta}\left(\frac{1+\rho}{\mathrm{R}}\right) = \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2}\Gamma(\beta+1)(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}} \times \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k(\alpha+\beta+1)_{2k}}{(\beta+1)_k} \times P_{\gamma-\alpha}^{-\gamma-\beta-2k-1}\left(\frac{1+\rho}{1-\rho}\right) P_k^{(\gamma,\beta)}(x). \tag{15}$$

Proof. Olver et al. [12, (18.12.3)] give a generating function for Jacobi polynomials, namely

$$(1+\rho)^{-\alpha-\beta-1} {}_2F_1\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2) \\ \beta+1\end{matrix}; \frac{2(1+x)\rho}{(1+\rho)^2}\right) = \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} \rho^n P_n^{(\alpha,\beta)}(x). \tag{16}$$

Using the connection relation for Jacobi polynomials (8) in (16) produces a double sum on the right-hand side of the equation. Reversing the order of the summation and shifting the n-index by k with simplification gives a Gauss hypergeometric function as the coefficient of the expansion. The resulting expansion formula is

$$(1+\rho)^{-\alpha-\beta-1} {}_2F_1\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2) \\ \beta+1\end{matrix}; \frac{2(1+x)\rho}{(1+\rho)^2}\right)$$
$$= \frac{1}{\gamma+\beta+1} \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k \left(\frac{\alpha+\beta+1}{2}\right)_k \left(\frac{\alpha+\beta+2}{2}\right)_k}{(\beta+1)_k \left(\frac{\gamma+\beta+2}{2}\right)_k \left(\frac{\gamma+\beta+3}{2}\right)_k} \times {}_2F_1\left(\begin{matrix}\alpha+\beta+1+2k, \alpha-\gamma \\ \gamma+\beta+2+2k\end{matrix}; \rho\right) \rho^k P_k^{(\gamma,\beta)}(x). \tag{17}$$

The Gauss hypergeometric function coefficient is realized to be an associated Legendre function of the first kind. The associated Legendre function of the first kind $P_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$ (see Chapter 14 in Olver et al. [12]) can be defined in terms of the Gauss hypergeometric function as follows (Olver et al. [12, (14.3.6), (15.2.2), Section 14.21(i)])

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(\begin{matrix}-\nu, \nu+1 \\ 1-\mu\end{matrix}; \frac{1-z}{2}\right), \tag{18}$$

for $|1-z| < 2$. Using a relation for the Gauss hypergeometric function from Olver et al. [12, (15.9.19)], namely

$${}_2F_1\left(\begin{matrix}a, b \\ a-b+1\end{matrix}; z\right) = \frac{z^{(b-a)/2}\,\Gamma(a-b+1)}{(1-z)^b} P_{-b}^{b-a}\left(\frac{1+z}{1-z}\right), \tag{19}$$

for $z \in \mathbf{C} \setminus \{(-\infty, 0] \cup (1, \infty)\}$, the Gauss hypergeometric function coefficient of the expansion can be expressed as an associated Legendre function of the first kind. The Gauss hypergeometric function on the left-hand side of (17) can also be expressed in terms of the associated Legendre function of the first kind using Magnus et al. [11, p. 157, entry 11], namely

$$P_\nu^\mu(z) = \frac{2^\mu z^{\nu+\mu}}{\Gamma(1-\mu)(z^2-1)^{\mu/2}} {}_2F_1\left(\begin{matrix}\frac{-\nu-\mu}{2}, \frac{-\nu-\mu+1}{2} \\ 1-\mu\end{matrix}; 1-\frac{1}{z^2}\right),$$

where $\Re z > 0$. This completes the proof. $\square$

Corollary 5. *Let $\beta \in \mathbf{C}$, $\alpha, \gamma > -1$ such that if $\alpha, \gamma \in (-1, 0)$ then $\alpha + \gamma + 1 \neq 0$, $\rho \in (0, 1)$, $x \in [-1, 1]$. Then*

$$\frac{(1-x)^{-\alpha/2}}{\mathrm{R}^{\beta+1}} \mathrm{P}_{\beta}^{-\alpha}\left(\frac{1-\rho}{\mathrm{R}}\right) = \frac{\Gamma(\gamma+\alpha+1)}{2^{\alpha/2}\Gamma(\alpha+1)(1+\rho)^{\beta-\gamma}\rho^{(\gamma+1)/2}} \times \sum_{k=0}^{\infty} \frac{(2k+\gamma+\alpha+1)(\gamma+\alpha+1)_k(\alpha+\beta+1)_{2k}}{(\alpha+1)_k} \times \mathrm{P}_{\gamma-\beta}^{-\gamma-\alpha-2k-1}\left(\frac{1-\rho}{1+\rho}\right) P_k^{(\alpha,\gamma)}(x). \tag{20}$$

Proof. Applying the parity relation for Jacobi polynomials to (17) and mapping $\rho \mapsto -\rho$ produces its companion identity. The Gauss hypergeometric functions appearing in this expression are Ferrers functions of the first kind (often referred to as the associated Legendre function of the first kind on the cut). The Ferrers function of the first kind $\mathrm{P}_\nu^\mu : (-1, 1) \to \mathbf{C}$ can be defined in terms of the Gauss hypergeometric function as follows (Olver et al. [12, (14.3.1)])

$$\mathrm{P}_\nu^\mu(x) := \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(\begin{matrix}-\nu, \nu+1\\ 1-\mu\end{matrix}; \frac{1-x}{2}\right), \tag{21}$$

for $x \in (-1, 1)$. The Gauss hypergeometric function coefficient of the expansion is seen to be a Ferrers function of the first kind by starting with (21) and using the linear transformation for the Gauss hypergeometric function [12, (15.8.1)]. This derives a Gauss hypergeometric function representation of the Ferrers function of the first kind, namely

$${}_2F_1\left(\begin{matrix}a, b\\ a-b+1\end{matrix}; -x\right) = \frac{x^{(b-a)/2}\,\Gamma(a-b+1)}{(1+x)^b}\mathrm{P}_{-b}^{b-a}\left(\frac{1-x}{1+x}\right),$$

for $x \in (0, 1)$. The Gauss hypergeometric function on the left-hand side of the companion identity for (17) is shown to be a Ferrers function of the first kind through Magnus et al. [11, p. 167], namely

$$\mathrm{P}_\nu^\mu(x) = \frac{2^\mu x^{\nu+\mu}}{\Gamma(1-\mu)(1-x^2)^{\mu/2}}\,{}_2F_1\left(\begin{matrix}\frac{-\nu-\mu}{2}, \frac{-\nu-\mu+1}{2}\\ 1-\mu\end{matrix}; 1-\frac{1}{x^2}\right),$$

for $x \in (0, 1)$. This completes the proof. □

The above two theorems are interesting results, as they are general cases for other generalizations found from related generating functions. For instance, by applying the connection relation for Jacobi polynomials (8) to an important extension of the generating function (16) (with the parity relation for Jacobi polynomials applied) given by Ismail [9, (4.3.2)], namely

$$\frac{1+\rho}{(1-\rho)^{\alpha+\beta+2}}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}\\ \alpha+1\end{matrix}; \frac{2\rho(x-1)}{(1-\rho)^2}\right) = \sum_{n=0}^\infty \frac{(\alpha+\beta+1+2n)(\alpha+\beta+1)_n}{(\alpha+\beta+1)(\alpha+1)_n}\rho^n P_n^{(\alpha,\beta)}(x), \tag{22}$$

produces a generalization that is equivalent to mapping $\alpha \mapsto \alpha+1$ in Theorem 4.

It is interesting that trying to generalize (22) using the connection relation (8), does not produce a new generalized formula, since (15) is its generalization—Ismail proves (22) by multiplying (16) by $\rho^{(\alpha+\beta+1)/2}$, after the companion identity is applied, and then differentiating by ρ. Ismail also mentions that (22) (and therefore Theorem 4 and Corollary 5) are closely connected to the Poisson kernel of $\left\{P_n^{(\alpha,\beta)}(x)\right\}$. One can also see that these expansions are related to the translation operator associated with Jacobi polynomials by mapping $\alpha \mapsto \alpha+1$ in Theorem 4.

Theorem 4 and Corollary 5 are also generalizations of the expansion (see Cohl and MacKenzie [6])

$$\begin{aligned}\frac{(1+x)^{-\beta/2}}{\mathrm{R}^{\alpha+m+1}}P_{\alpha+m}^{-\beta}\left(\frac{1+\rho}{\mathrm{R}}\right) &= \frac{\rho^{-(\alpha+1)/2}}{2^{\beta/2}(1-\rho)^m}\sum_{n=0}^\infty \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{\Gamma(\beta+n+1)}\\ &\quad\times P_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1+\rho}{1-\rho}\right)P_n^{(\alpha,\beta)}(x),\end{aligned} \tag{23}$$

found by mapping $\alpha, \gamma \mapsto \alpha+m, \alpha$ for $m \in \mathbf{N}_0$ in Theorem 4; and its companion identity (see Cohl and MacKenzie [6]),

$$\begin{aligned}\frac{(1-x)^{-\alpha/2}}{\mathrm{R}^{\beta+m+1}}\mathrm{P}_{\beta+m}^{-\alpha}\left(\frac{1-\rho}{\mathrm{R}}\right) &= \frac{\rho^{-(\beta+1)/2}}{2^{\alpha/2}(1+\rho)^m}\sum_{n=0}^\infty \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{\Gamma(\alpha+n+1)}\\ &\quad\times \mathrm{P}_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1-\rho}{1+\rho}\right)P_n^{(\alpha,\beta)}(x),\end{aligned} \tag{24}$$

found by mapping $\beta, \gamma \mapsto \beta+m, \beta$ for $m \in \mathbf{N}_0$ in Corollary 5. The expansions (23), (24) are produced using the definition of the Gauss hypergeometric function on the left hand side of (22) and the expansion of $(1+x)^n$ in terms of Jacobi polynomials (see Cohl and MacKenzie [6, (7), (13)]). Interestingly, the expansions (23) and (24) are also related to the generalized translation operator, but with a more general translation that can be seen with the $\alpha \mapsto \alpha+m$ or $\beta \mapsto \beta+m$.

3. Expansions in Gegenbauer polynomials

The Gegenbauer polynomials $C_n^\mu : \mathbf{C} \to \mathbf{C}$ can be defined in terms of the terminating Gauss hypergeometric series as follows (Olver et al. [12, (18.5.9)])

$$C_n^\mu(z) := \frac{(2\mu)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+2\mu\\ \mu+\frac{1}{2}\end{matrix}; \frac{1-z}{2}\right),$$

for $n \in \mathbf{N}_0$ and $\mu \in (-1/2, \infty) \setminus \{0\}$. The orthogonality relation for Gegenbauer polynomials can be found in Olver et al. [12, (18.2.1), (18.2.5), Table 18.3.1] for $m, n \in \mathbf{N}_0$, namely

$$\int_{-1}^{1} C_m^\mu(x) C_n^\mu(x)(1-x^2)^{\mu-1/2} dx = \frac{\pi 2^{1-2\mu}\Gamma(2\mu+n)}{(n+\mu)\Gamma^2(\mu) n!}\delta_{m,n}. \tag{25}$$

Theorem 6. *Let* $\lambda, \mu \in \mathbf{C}$, $\nu \in (-1/2, \infty) \setminus \{0\}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. *Then*

$$(1-x^2)^{1/4-\mu/2} \mathrm{P}^{1/2-\mu}_{\lambda+\mu-1/2}(\mathrm{R}+\rho)\, \mathrm{P}^{1/2-\mu}_{\lambda+\mu-1/2}(\mathrm{R}-\rho) = \frac{(\rho/2)^{\mu-1/2}}{\nu\,\Gamma^2(\mu+1/2)} \sum_{n=0}^{\infty} \frac{(\nu+n)(-\lambda)_n\,(2\mu+\lambda)_n\,(\mu)_n}{(2\mu)_n\,(\mu+1/2)_n\,(\nu+1)_n}$$
$$\times {}_6F_5\left(\begin{matrix} \frac{-\lambda+n}{2}, \frac{-\lambda+n+1}{2}, \frac{2\mu+\lambda+n}{2}, \frac{2\mu+\lambda+n+1}{2}, \mu+n, \mu-\nu \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+\frac{1}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}, \nu+1+n \end{matrix}; \rho^2\right) \rho^n C_n^\nu(x). \tag{26}$$

Proof. In Koekoek, Lesky and Swarttouw [10, (9.8.32)], there is the following generating function for Gegenbauer polynomials

$${}_2F_1\left(\begin{matrix}\lambda, 2\mu-\lambda \\ \mu+\frac{1}{2}\end{matrix}; \frac{1-\rho-\mathrm{R}}{2}\right) {}_2F_1\left(\begin{matrix}\lambda, 2\mu-\lambda \\ \mu+\frac{1}{2}\end{matrix}; \frac{1+\rho-\mathrm{R}}{2}\right) = \sum_{n=0}^{\infty} \frac{(\lambda)_n\,(2\mu-\lambda)_n}{(2\mu)_n\,\left(\mu+\frac{1}{2}\right)_n}\rho^n C_n^\mu(x).$$

These Gauss hypergeometric functions can be re-written in terms of associated Legendre and Ferrers functions of the first kind. The first Gauss hypergeometric function can be written in terms the Ferrers function of the first kind using Abramowitz and Stegun [1, (15.4.19)], namely

$${}_2F_1\left(\begin{matrix}a, b \\ \frac{a+b+1}{2}\end{matrix}; x\right) = \Gamma\left(\frac{a+b+1}{2}\right)(x(1-x))^{(1-a-b)/4}\, \mathrm{P}^{(1-a-b)/2}_{(a-b-1)/2}(1-2x),$$

for $x \in (0, 1)$, with $a = \lambda$, $b = 2\mu - \lambda$. The second Gauss hypergeometric function can be written in terms of the associated Legendre function of the first kind using [12, (14.3.6)] and Euler's transformation [12, (15.8.1)]. The substitutions yield

$$(1-x^2)^{1/4-\mu/2} P^{1/2-\mu}_{\lambda+\mu-1/2}(\mathrm{R}+\rho) \mathrm{P}^{1/2-\mu}_{\lambda+\mu-1/2}(\mathrm{R}-\rho) = \frac{(\rho/2)^{\mu-1/2}}{\Gamma^2(\mu+1/2)} \sum_{n=0}^{\infty} \frac{(-\lambda)_n(2\mu+\lambda)_n}{(2\mu)_n(\mu+1/2)_n}\rho^n C_n^\mu(x). \tag{27}$$

Using the connection relation for Gegenbauer polynomials (2) on the generating function (27) produces a double sum. Reversing the order of the summation and shifting the n-index by $2k$ with simplification completes the proof. □

Associated Legendre and Ferrers functions of the first kind with special values of the degree and order reduce to Gegenbauer polynomials. For instance, if $n \in \mathbf{N}_0$, then through [12, (14.3.22)]

$$P^{1/2-\mu}_{n+\mu-1/2}(z) = \frac{2^{\mu-1/2}\Gamma(\mu) n!}{\sqrt{\pi}\,\Gamma(2\mu+n)}(z^2-1)^{\mu/2-1/4}\, C_n^\mu(z),$$

and from [12, (14.3.21)], one has

$$\mathrm{P}^{1/2-\mu}_{n+\mu-1/2}(x) = \frac{2^{\mu-1/2}\Gamma(\mu) n!}{\sqrt{\pi}\,\Gamma(2\mu+n)}(1-x^2)^{\mu/2-1/4}\, C_n^\mu(x).$$

From (27) using the above expressions, we have the following finite-summation generating function expression with $m \in \mathbf{N}_0$,

$$C_m^\mu(\mathrm{R}+\rho) C_m^\mu(\mathrm{R}-\rho) = \frac{(2\mu)_m^2}{(m!)^2}\sum_{n=0}^{m} \frac{(-m)_n(2\mu+m)_n}{(2\mu)_n\left(\mu+\frac{1}{2}\right)_n}\rho^n C_n^\mu(x), \tag{28}$$

and from the generalized result (26) we have

$$C_m^\mu(\mathrm{R}+\rho) C_m^\mu(\mathrm{R}-\rho) = \frac{(2\mu)_m^2}{\nu(m!)^2}\sum_{n=0}^{m} \frac{(\nu+n)\,(-m)_n\,(2\mu+m)_n\,(\mu)_n}{(2\mu)_n\,(\mu+1/2)_n\,(\nu+1)_n}$$
$$\times {}_6F_5\left(\begin{matrix} \frac{-m+n}{2}, \frac{-m+n+1}{2}, \frac{2\mu+m+n}{2}, \frac{2\mu+m+n+1}{2}, \mu-\nu, \mu+n \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+\frac{1}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}, \nu+1+n \end{matrix}; \rho^2\right) \rho^n C_n^\nu(x),$$

which reduces to (28) when $\nu = \mu$.

Consider the generating function for Gegenbauer polynomials, Olver et al. [12, (18.12.5)]

$$\frac{1-\rho x}{(1+\rho^2-2\rho x)^{\nu+1}} = \frac{1}{2\nu}\sum_{n=0}^{\infty}(n+2\nu)\rho^n C_n^\nu(x), \tag{29}$$

and the generating function

$$\frac{x-\rho}{(1+\rho^2-2\rho x)^{\nu+1}} = \frac{1}{2\nu\rho}\sum_{n=0}^{\infty} n\rho^n C_n^\nu(x), \tag{30}$$

which follows from (29) using (1). The technique of this paper can also be applied to generalize (29) and (30). However, note that

$$\frac{1-\rho x}{(1+\rho^2-2\rho x)^{\nu+1}} = \frac{1-\rho^2}{2}\frac{1}{(1+\rho^2-2\rho x)^{\nu+1}} + \frac{1}{2}\frac{1}{(1+\rho^2-2\rho x)^{\nu}},$$
$$\frac{x-\rho}{(1+\rho^2-2\rho x)^{\nu+1}} = \frac{1-\rho^2}{2\rho}\frac{1}{(1+\rho^2-2\rho x)^{\nu+1}} - \frac{1}{2\rho}\frac{1}{(1+\rho^2-2\rho x)^{\nu}},$$

so it is easier to use (3) on the right-hand sides.

The Gegenbauer polynomials can be defined as symmetric Jacobi polynomials, namely

$$C_n^\nu(x) = \frac{(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n} P_n^{(\nu-1/2,\nu-1/2)}(x). \tag{31}$$

Therefore the expansions given in the section on Jacobi polynomials can also be written as expansions in Gegenbauer polynomials by using symmetric parameters. Furthermore, these expansions can also be written as expansions over Chebyshev polynomials of the second kind and Legendre polynomials using $U_n(z) = C_n^1(z), P_n(z) = C_n^{1/2}(z)$, for $n \in \mathbf{N}_0$. One may also take the limit of an expansion in Gegenbauer polynomials as $\mu \to 0$. This limit may be well defined with the interpretation of obtaining Chebyshev polynomials of the first kind through Andrews et al. [2, (6.4.13)], namely

$$T_n(z) = \frac{1}{\epsilon_n}\lim_{\mu\to 0}\frac{n+\mu}{\mu}C_n^\mu(z), \tag{32}$$

where the Neumann factor $\epsilon_n \in \{1, 2\}$, is defined by $\epsilon_n := 2 - \delta_{n,0}$, commonly seen in Fourier cosine series. We can, for example, derive the following corollaries.

Corollary 7. *Let $\alpha \in \mathbf{C}$, $\gamma \in (-1/2, \infty) \setminus \{0\}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. Then*

$$\frac{2^{\alpha+\gamma-1}}{\mathrm{R}\,(1+\mathrm{R}-\rho)^{\alpha-1/2}\,(1+\mathrm{R}+\rho)^{\gamma-1/2}} = \frac{1}{\gamma}\sum_{k=0}^{\infty}\frac{(k+\gamma)\left(\frac{\alpha+\gamma}{2}\right)_k\left(\frac{\alpha+\gamma+1}{2}\right)_k}{(\alpha+\gamma)_k\,(\gamma+1)_k}$$
$$\times\,{}_3F_2\left(\begin{matrix}\gamma+k+\frac{1}{2}, \alpha+\gamma+2k, \alpha-\gamma\\ \alpha+\gamma+k, 2\gamma+2k+1\end{matrix}; \rho\right)\rho^k C_k^\gamma(x). \tag{33}$$

Proof. Using (9), mapping $\alpha \mapsto \alpha - 1/2$ and $\beta, \gamma \mapsto \gamma - 1/2$, and using (31) completes the proof. □

Corollary 8. *Let $\alpha \in \mathbf{C}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. Then*

$$\frac{(1+\mathrm{R}+\rho)^{1/2}}{\mathrm{R}(1+\mathrm{R}-\rho)^{\alpha-1/2}} = 2^{1-\alpha}\sum_{k=0}^{\infty}\epsilon_k\frac{\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha+1}{2}\right)_k}{(\alpha)_k\,k!}\,{}_3F_2\left(\begin{matrix}k+\frac{1}{2}, \alpha+2k, \alpha\\ 2k+1, \alpha+k\end{matrix}; \rho\right)\rho^k T_k(x). \tag{34}$$

Proof. Taking the limit as $\gamma \to 0$ of (33) and using (32) completes the proof. □

4. Expansions in Laguerre polynomials

The Laguerre polynomials $L_n^\alpha : \mathbf{C} \to \mathbf{C}$ can be defined in terms of Kummer's confluent hypergeometric function of the first kind as follows (Olver et al. [12, (18.5.12)])

$$L_n^\alpha(z) := \frac{(\alpha+1)_n}{n!}M(-n, \alpha+1, z),$$

for $n \in \mathbf{N}_0$, and $\alpha > -1$. The Laguerre function $L_\nu^\alpha : \mathbf{C} \to \mathbf{C}$, which generalizes the Laguerre polynomials is defined as follows (Erdélyi et al. [7, (6.9.2.37), this equation is stated incorrectly therein]) for $\nu, \alpha \in \mathbf{C}$,

$$L_\nu^\alpha(z) := \frac{\Gamma(1+\nu+\alpha)}{\Gamma(\nu+1)\Gamma(\alpha+1)} M(-\nu, \alpha+1, z). \tag{35}$$

The orthogonality relation for Laguerre polynomials can be found in Olver et al. [12, (18.2.1), (18.2.5), Table 18.3.1]

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}. \tag{36}$$

The connection relation for Laguerre polynomials, given by Olver et al. [12, (18.18.18)] (see also Ruiz and Dehesa [13]), is

$$L_n^\alpha(x) = \sum_{k=0}^n \frac{(\alpha-\beta)_{n-k}}{(n-k)!} L_k^\beta(x). \tag{37}$$

Theorem 9. *Let $\alpha, \beta \in \mathbf{R}$, $x > 0$, $\rho \in \mathbf{C}$. Then*

$$x^{-\alpha/2} J_\alpha\left(2\sqrt{x\rho}\right) = \rho^{\alpha/2} e^{-\rho} \sum_{k=0}^\infty \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1+k)} L_{\beta-\alpha}^{\alpha+k}(\rho)\rho^k L_k^\beta(x). \tag{38}$$

Proof. Olver et al. [12, (18.12.14)] give a generating function for Laguerre polynomials, namely

$$x^{-\alpha/2} J_\alpha(2\sqrt{x\rho}) = \rho^{\alpha/2} e^{-\rho} \sum_{n=0}^\infty \frac{\rho^n}{\Gamma(\alpha+1+n)} L_n^\alpha(x),$$

where J_α is the Bessel function of the first kind (4). Using the Laguerre connection relation (37) to replace the Laguerre polynomial in the generating function produces a double sum. In order to justify reversing the resulting order of summation, we demonstrate that

$$\sum_{n=0}^\infty |c_n| \sum_{k=0}^n |a_{nk}| \left|L_k^\beta(x)\right| < \infty, \tag{39}$$

where

$$c_n = \frac{\rho^n}{\Gamma(\alpha+1+n)}$$

and

$$a_{nk} = \frac{(\alpha-\beta)_{n-k}}{(n-k)!}. \tag{40}$$

We assume that $\alpha, \beta \in \mathbf{R}$, $\rho \in \mathbf{C}$ and $x > 0$. It is known [15, Theorem 8.22.1] that

$$\left|L_n^\alpha(x)\right| \le K_1(1+n)^{\sigma_1}, \tag{41}$$

where K_1, $\sigma_1 = \frac{\alpha}{2} - \frac{1}{4}$ are constants independent of n (but depend on x and α). We also have

$$|a_{nk}| \le (1+n-k)^{\sigma_2} \le (1+n)^{\sigma_2}, \tag{42}$$

where $\sigma_2 = |\alpha-\beta|$. Therefore,

$$\sum_{n=0}^\infty |c_n| \sum_{k=0}^n |a_{nk}| \left|L_k^\beta(x)\right| \le K_1 \sum_{n=0}^\infty \frac{|\rho|^n}{\Gamma(\alpha+1+n)} (1+n)^{\sigma_1+\sigma_2+1} < \infty.$$

Reversing the order of summation and shifting the n-index by k yields

$$x^{-\alpha/2} J_\alpha(2\sqrt{x\rho}) = \rho^{\alpha/2} e^{-\rho} \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{(\alpha-\beta)_n\, \rho^{n+k}}{\Gamma(\alpha+1+n+k)\, n!} L_k^\beta(x).$$

Using (6) produces a Kummer's confluent hypergeometric function of the first kind as the coefficient of the expansion. Using the definition of Laguerre functions (35) to replace the confluent hypergeometric function completes the proof. □

Consider the generating function (Srivastava and Manocha [14, p. 209])

$$e^{-x\rho} = \frac{1}{(1+\rho)^{\alpha}} \sum_{n=0}^{\infty} \rho^n L_n^{\alpha-n}(x), \tag{43}$$

for $\alpha \in \mathbf{C}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x > 0$. Using the connection relation for Laguerre polynomials (37) in the generating function (43), yields a double sum. Reversing the order of the summation and shifting the n-index by k produces

$$e^{-x\rho} = \frac{1}{(1+\rho)^{\alpha}} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha-n-k-\beta)_n}{n!} \rho^{n+k} L_k^{\beta}(x).$$

Using (5), (6), and substituting $z = \rho/(1+\rho)$ yields the known generating function for Laguerre polynomials Olver et al. [12, (18.12.13)], namely

$$\exp\left(\frac{x\rho}{\rho-1}\right) = (1-\rho)^{\beta+1} \sum_{n=0}^{\infty} \rho^n L_n^{\beta}(x), \tag{44}$$

for $\beta \in \mathbf{C}$. Note that using the connection relation for Laguerre polynomials (37) on (44) leaves this generating function invariant.

Theorem 10. *Let $\lambda \in \mathbf{C}$, $\alpha \in \mathbf{C} \setminus -\mathbf{N}$, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x > 0$. Then*

$$M\left(\lambda, \alpha+1, \frac{x\rho}{\rho-1}\right) = (1-\rho)^{\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{(\alpha+1)_k} \, {}_2F_1\left(\begin{matrix}\lambda+k, \alpha-\beta \\ \alpha+1+k\end{matrix}; \rho\right) \rho^k L_k^{\beta}(x).$$

Proof. On p. 132 of Srivastava and Manocha [14] there is a generating function for Laguerre polynomials, namely

$$M\left(\lambda, \alpha+1, \frac{x\rho}{\rho-1}\right) = (1-\rho)^{\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n \, \rho^n}{(\alpha+1)_n} L_n^{\alpha}(x).$$

Using the connection relation for Laguerre polynomials (37) we obtain a double summation. In order to justify reversing the resulting order of summation, we demonstrate (39), where

$$c_n = \frac{(\lambda)_n \rho^n}{(\alpha+1)_n}$$

and a_{nk} is given in (40). We assume that $\alpha \in \mathbf{C} \setminus -\mathbf{N}$, $\beta > -1$, $|\rho| < 1$ and $x > 0$. Given (41), (42), then

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^{n} |a_{nk}| \left|L_k^{\beta}(x)\right| \le K_1 \sum_{n=0}^{\infty} \frac{|(\lambda)_n| \, |\rho|^n}{|(\alpha+1)_n|} (1+n)^{\sigma_1+\sigma_2+1} \le K_3 \sum_{n=0}^{\infty} |\rho|^n (1+n)^{\sigma_1+\sigma_2+\lambda-\alpha} < \infty,$$

for some $K_3 \in \mathbf{R}$. Reversing the order of the summation and shifting the n-index by k produces

$$M\left(\lambda, \alpha+1, \frac{x\rho}{\rho-1}\right) = (1-\rho)^{\lambda} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{n+k} \, (\alpha-\beta)_n}{(\alpha+1)_{n+k} \, n!} \rho^{n+k} L_k^{\beta}(x).$$

Then, using (6) with simplification completes the proof. □

5. Expansions in Wilson polynomials

The Wilson polynomials $W_n\left(x^2; a, b, c, d\right)$, originally introduced in Wilson [16], can be defined in terms of a terminating generalized hypergeometric series as follows (Olver et al. [12, (18.26.1)])

$$W_n(x^2; a, b, c, d) := (a+b)_n (a+c)_n (a+d)_n \, {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d\end{matrix}; 1\right).$$

These polynomials are perhaps the most general hypergeometric orthogonal polynomials in existence being at the very top of the Askey scheme which classifies these orthogonal polynomials (see for instance [12, Figure 18.21.1]). The orthogonality relation for Wilson polynomials can be found in [10, Section 9.1], namely

$$\int_0^{\infty} \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 W_m\left(x^2; a, b, c, d\right) W_n\left(x^2; a, b, c, d\right) dx$$
$$= \frac{2\pi \, n! \, \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)}{(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)} \delta_{m,n}$$

where $\Re a, \Re b, \Re c, \Re d > 0$, and non-real parameters occurring in conjugate pairs. A connection relation with one free parameter for the Wilson polynomials is given by [13, equation just below (15)], namely

$$W_n\left(x^2;a,b,c,d\right) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} W_k\left(x^2;a,b,c,h\right) \times \frac{(n+a+b+c+d-1)_k\,(d-h)_{n-k}\,(k+a+b)_{n-k}\,(k+a+c)_{n-k}\,(k+b+c)_{n-k}}{(k+a+b+c+h-1)_k\,(2k+a+b+c+h)_{n-k}}. \tag{45}$$

In this section, we give a generalization of a generating function for Wilson polynomials. This example is intended to be illustrative. In Koekoek, Lesky and Swarttouw [10] for instance there are four separate generating functions given for the Wilson polynomials. The technique applied in the proof of the theorem presented in this section, can be easily applied to the rest of the generating functions for Wilson polynomials in [10]. Generalizations of these generating functions (and their corresponding definite integrals) can be extended by a well-established limiting procedure (see [10, Chapter 9]) to the continuous dual Hahn, continuous Hahn, Meixner–Pollaczek, pseudo Jacobi, Jacobi, Laguerre and Hermite polynomials.

Theorem 11. *Let $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in (0,\infty)$, $\Re a, \Re b, \Re c, \Re d, \Re h > 0$ and non-real parameters a, b, c, d, h occurring in conjugate pairs. Then*

$$ {}_2F_1\left(\begin{matrix} a+ix,\ b+ix \\ a+b \end{matrix};\rho\right) {}_2F_1\left(\begin{matrix} c-ix,\ d-ix \\ c+d \end{matrix};\rho\right) = \sum_{k=0}^{\infty} \frac{(k+a+b+c+d-1)_k}{(k+a+b+c+h-1)_k\,(a+b)_k\,(c+d)_k\,k!} \times {}_4F_3\left(\begin{matrix} d-h,\ 2k+a+b+c+d-1,\ k+a+c,\ k+b+c \\ k+a+b+c+d-1,\ 2k+a+b+c+h,\ k+c+d \end{matrix};\rho\right)\rho^k\, W_k\left(x^2;a,b,c,h\right). \tag{46}$$

Proof. Koekoek et al. [10, (1.1.12)] give a generating function for Wilson polynomials, namely

$$ {}_2F_1\left(\begin{matrix} a+ix,\ b+ix \\ a+b \end{matrix};\rho\right) {}_2F_1\left(\begin{matrix} c-ix,\ d-ix \\ c+d \end{matrix};\rho\right) = \sum_{n=0}^{\infty} \frac{\rho^n\, W_n\left(x^2;a,b,c,d\right)}{(a+b)_n\,(c+d)_n\,n!}.$$

Using the connection relation for Wilson polynomials (45) in the above generating function produces a double sum. In order to justify reversing the summation symbols we show that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^{n} |a_{nk}| \left|W_k(x^2;a,b,c,h)\right| < \infty,$$

where

$$c_n = \frac{\rho^n}{(a+b)_n(c+d)_n n!},$$

and a_{nk} are the connection coefficients satisfying

$$W_n(x^2;a,b,c,d) = \sum_{k=0}^{n} a_{nk} W_k(x^2;a,b,c,h).$$

We assume that a, b, c, d and a, b, c, h are positive except for complex conjugate pairs with positive real parts, and $x > 0$. It follows from [17, bottom of p. 59] that

$$\left|W_n(x^2;a,b,c,d)\right| \le K_1 (n!)^3 (1+n)^{\sigma_1}, \tag{47}$$

where K_1 and σ_1 are positive constants independent of n.

We will need the following lemma.

Lemma 12. *Let $j \in \mathbf{N}$, $k, n \in \mathbf{N}_0$, $z \in \mathbf{C}$, $\Re u > 0$, $w > -1$, $v \ge 0$, $x > 0$. Then*

$$|(u)_j| \ge (\Re u)(j-1)!, \tag{48}$$

$$\frac{(v)_n}{n!} \le (1+n)^v, \tag{49}$$

$$(n+w)_k \le \max\{1, 2^w\} \frac{(n+k)!}{n!}, \quad (k \le n), \tag{50}$$

$$|(k+z)_{n-k}| \le (1+n)^{|z|} \frac{n!}{k!}, \quad (k \le n), \tag{51}$$

$$(k+x-1)_k \geq \min\left\{\frac{x}{2}, \frac{1}{6}\right\}\frac{(2k)!}{k!}, \tag{52}$$

$$(2k+x)_{n-k} \geq \min\{x, 1\}\frac{1}{1+n}\frac{(n+k)!}{(2k)!}, \quad (k \leq n). \tag{53}$$

Proof. Let us consider

$$|(u)_j| = |u|\,|u+1|\dots|u+j-1| \geq \Re u(\Re u+1)\dots(\Re u+j-1) \geq (\Re u)(j-1)!.$$

This completes the proof of (48). Choose $m \in \mathbf{N}_0$ such that $m \leq v \leq m+1$. Then

$$\frac{(v)_n}{n!} \leq \frac{(m+1)_n}{n!} = \frac{(n+1)(n+2)\dots(n+m)}{m!} = (1+n)\left(1+\frac{n}{2}\right)\dots\left(1+\frac{n}{m}\right) \leq (1+n)^m \leq (1+n)^v.$$

This completes the proof of (49). If $-1 < w \leq 1$ then

$$n!(n+w)_k \leq n!(n+1)(n+2)\dots(n+k) = (n+k)!.$$

If $m \leq w \leq m+1$ with $m \in \mathbf{N}$ then

$$n!(n+w)_k \leq (n+k)!\frac{n+k+1}{n+1}\frac{n+k+2}{n+2}\dots\frac{n+m+k}{n+m} \leq 2^m(n+k)! \leq 2^w(n+k)!$$

This completes the proof of (50). If $|z| \leq 1$ then

$$|(k+z)_{n-k}| \leq (k+1)(k+2)\dots n = \frac{n!}{k!}.$$

If $|z| > 1$ then, using (49),

$$k!|(k+z)_{n-k}| \leq |z|(|z|+1)\dots(|z|+n-1) = (|z|)_n \leq n!(1+n)^{|z|}.$$

This completes the proof of (51). Let $k \geq 2$. Then

$$(k+x-1)_k \geq (k-1)k\dots(2k-2) = \frac{k(k-1)}{2k(2k-1)}\frac{(2k)!}{k!} \geq \frac{1}{6}\frac{(2k)!}{k!}.$$

The cases $k = 0, 1$ can be verified directly. This completes the proof of (52). Let $k \geq 1$. Then

$$(2k+x)_{n-k} \geq (2k)_{n-k} = \frac{(n+k)!}{(2k)!}\frac{2k}{n+k} \geq \frac{1}{1+n}\frac{(n+k)!}{k!}.$$

The case $k = 0$ can be verified separately. This completes the proof of (53). □

Using (48), we obtain, for $n \in \mathbf{N}$,

$$|c_n| = \frac{|\rho|^n}{|(a+b)_n|\,|(c+d)_n|n!} \leq \frac{|\rho^n|}{\Re(a+b)\Re(c+d)(n-1)!^2 n!} = \frac{1}{\Re(a+b)\Re(c+d)}\frac{n^2|\rho|^n}{(n!)^3}.$$

Therefore, we obtain, for all $n \in \mathbf{N}_0$,

$$|c_n| \leq K_2(1+n)^2\frac{|\rho|^n}{(n!)^3}, \tag{54}$$

where

$$K_2 = \max\left\{1, \frac{1}{\Re(a+b)\Re(c+d)}\right\}.$$

Using (49), we find

$$\left|\frac{(d-h)_{n-k}}{(n-k)!}\right| \leq (1+n)^{|d-h|}. \tag{55}$$

Using (50), we obtain

$$|n!(n+a+b+c+d-1)_k| \leq K_3(n+k)!. \tag{56}$$

From (51), we find

$$|(k+a+b)_{n-k}| \leq (1+n)^{\sigma_4}\frac{n!}{k!}, \tag{57}$$

and similar estimates with $a+c$ and $b+c$ in place of $a+b$. Using (52), we obtain

$$|(k+a+b+c+h-1)_k| \geq K_4 \frac{(2k)!}{k!}, \tag{58}$$

where $K_4 > 0$. Using (53), we obtain

$$|(2k+a+b+c+h)_{n-k}| \geq \frac{K_5}{1+n} \frac{(n+k)!}{(2k)!}, \tag{59}$$

where $K_5 > 0$.

Combining (55)–(59), we find

$$|a_{nk}| \leq K_6 (1+n)^{\sigma_6} \left(\frac{n!}{k!}\right)^3. \tag{60}$$

Now (47), (54), (60) give

$$\begin{aligned}\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^{n} |a_{nk}| \left|W_k(x^2; a, b, c, h)\right| &\leq K_1 K_2 K_6 \sum_{n=0}^{\infty} |\rho|^n (1+n)^2 \sum_{k=0}^{n} (1+n)^{\sigma_1+\sigma_6} \\ &= K_1 K_2 K_6 \sum_{n=0}^{\infty} |\rho|^n (1+n)^{\sigma_1+\sigma_6+3} < \infty\end{aligned}$$

since $|\rho| < 1$. Reversing the order of the summation and shifting the n-index by k produces the generalized expansion (46).

This completes the proof of Theorem 11. □

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Appendix. Definite integrals

As a consequence of the series expansions given above, one may generate corresponding definite integrals (in a one-step procedure) as an application of the orthogonality relation for these hypergeometric orthogonal polynomials. Integrals of such sort are always of interest since they are very likely to find applications in applied mathematics and theoretical physics.

Corollary 13. *Let $k \in \mathbf{N}_0$, $\alpha \in \mathbf{C}$, $\beta, \gamma > -1$ such that if $\beta, \gamma \in (-1, 0)$ then $\beta + \gamma + 1 \neq 0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$. Then*

$$\begin{aligned}\int_{-1}^{1} \frac{(1-x)^{\gamma}(1+x)^{\beta}}{\mathrm{R}\,(1+\mathrm{R}-\rho)^{\alpha}\,(1+\mathrm{R}+\rho)^{\beta}} P_k^{(\gamma,\beta)}(x)dx &= \frac{2^{1+\gamma-\alpha}\Gamma(\gamma+k+1)\Gamma(\beta+k+1)\left(\frac{\alpha+\beta+1}{2}\right)_k\left(\frac{\alpha+\beta+2}{2}\right)_k}{\Gamma(\gamma+\beta+2)(\alpha+\beta+1)_k\left(\frac{\gamma+\beta+2}{2}\right)_k\left(\frac{\gamma+\beta+3}{2}\right)_k k!} \\ &\quad \times {}_3F_2\left(\begin{matrix}\beta+k+1, \alpha+\beta+2k+1, \alpha-\gamma \\ \alpha+\beta+k+1, \gamma+\beta+2k+2\end{matrix}; \rho\right)\rho^k.\end{aligned}$$

Proof. Multiplying both sides of (9) by $P_n^{(\gamma,\beta)}(x)(1-x)^{\gamma}(1+x)^{\beta}$ and integrating from -1 to 1 using the orthogonality relation for Jacobi polynomials (7) with simplification completes the proof. □

Corollary 14. *Let $k \in \mathbf{N}_0$, $\alpha \in \mathbf{C}$, $\beta, \gamma > -1$ such that if $\beta, \gamma \in (-1, 0)$ then $\beta + \gamma + 1 \neq 0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$. Then*

$$\begin{aligned}&\int_{-1}^{1} (1-x)^{\gamma-\alpha/2}(1+x)^{\beta/2} J_{\alpha}\left(\sqrt{2(1-x)\rho}\right) I_{\beta}\left(\sqrt{2(1+x)\rho}\right) P_k^{(\gamma,\beta)}(x)dx \\ &= \frac{2^{\gamma+\beta/2-\alpha/2+1}\,\Gamma(\gamma+k+1)\left(\frac{\alpha+\beta+1}{2}\right)_k\left(\frac{\alpha+\beta+2}{2}\right)_k}{\Gamma(\gamma+\beta+2)\,\Gamma(\alpha+k+1)\,(\alpha+\beta+1)_k\left(\frac{\gamma+\beta+2}{2}\right)_k\left(\frac{\gamma+\beta+3}{2}\right)_k k!} \\ &\quad \times {}_2F_3\left(\begin{matrix}2k+\alpha+\beta+1, \alpha-\gamma \\ \alpha+\beta+k+1, \gamma+\beta+2k+2, \alpha+1+k\end{matrix}; \rho\right)\rho^{\alpha/2+\beta/2+k}.\end{aligned}$$

Proof. Same as the proof of Corollary 13, except apply to both sides of (13). □

Corollary 15. *Let $\beta \in \mathbf{C}, \alpha, \gamma > -1$ such that if $\alpha, \gamma \in (-1, 0)$ then $\alpha + \gamma + 1 \neq 0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1, 0]$. Then*

$$\int_{-1}^{1} \frac{(1+x)^{\beta/2}(1-x)^{\gamma}}{\mathrm{R}^{\alpha+1}} P_{\alpha}^{-\beta}\left(\frac{1+\rho}{\mathrm{R}}\right) P_k^{(\gamma,\beta)}(x)dx = \frac{2^{\gamma+\beta/2+1}\Gamma(\gamma+k+1)(\alpha+\beta+1)_{2k}}{(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}k!} P_{\gamma-\alpha}^{-\gamma-\beta-2k-1}\left(\frac{1+\rho}{1-\rho}\right).$$

Proof. Same as the proof of Corollary 13, except apply to both sides of (15). □

Corollary 16. *Let $\beta \in \mathbf{C}, \alpha, \gamma > -1$ such that if $\alpha, \gamma \in (-1, 0)$ then $\alpha + \gamma + 1 \neq 0$, $\rho \in (0, 1)$. Then*

$$\int_{-1}^{1} \frac{(1-x)^{\alpha/2}(1+x)^{\gamma}}{\mathrm{R}^{\beta+1}} \mathrm{P}_{\beta}^{-\alpha}\left(\frac{1-\rho}{\mathrm{R}}\right) P_k^{(\alpha,\gamma)}(x)dx = \frac{2^{\gamma+\alpha/2+1}\Gamma(\gamma+k+1)(\alpha+\beta+1)_{2k}}{(1+\rho)^{\beta-\gamma}\rho^{(\gamma+1)/2}k!} P_{\gamma-\beta}^{-\gamma-\alpha-2k-1}\left(\frac{1-\rho}{1+\rho}\right).$$

Proof. Same as the proof of Corollary 13, except apply to both sides of (20). □

Corollary 17. *Let $n \in \mathbf{N}_0, \alpha, \mu \in \mathbf{C}, \nu \in (-1/2, \infty) \setminus \{0\}, \rho \in (0, 1)$. Then*

$$\begin{aligned}
&\int_{-1}^{1} \left(1-x^2\right)^{\nu-\mu/2-1/4} P_{\mu-\alpha-1/2}^{1/2-\mu}(\mathrm{R}+\rho)\, \mathrm{P}_{\mu-\alpha-1/2}^{1/2-\mu}(\mathrm{R}-\rho)\, C_n^{\nu}(x)dx \\
&= \frac{\sqrt{\pi}2^{1/2-\mu}(2\nu)_n(\alpha)_n(2\mu-\alpha)_n(\mu)_n\Gamma\left(\frac12+\nu\right)}{(2\mu)_n\Gamma(\frac12+\mu+n)\Gamma(1+\nu+n)\Gamma(\frac12+\mu)n!}\rho^{n+\mu-1/2} \\
&\quad \times {}_6F_5\left(\begin{matrix}\frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}, \frac{2\mu-\alpha+n}{2}, \frac{2\mu-\alpha+n+1}{2}, \mu+n, \mu-\nu \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+\frac12}{2}, \frac{\mu+n+\frac32}{2}, 1+\nu+n\end{matrix}; \rho^2\right).
\end{aligned}$$

Proof. Multiplying both sides of (26) by $C_n^{\nu}(x)(1-x^2)^{\nu-1/2}$ and integrating from -1 to 1 using the orthogonality relation for Gegenbauer polynomials (25) with simplification completes the proof. □

Corollary 18. *Let $k \in \mathbf{N}_0, \alpha \in \mathbf{C}, \gamma \in (-1/2, \infty) \setminus \{0\}, \rho \in \{z \in \mathbf{C} : |z| < 1\}$. Then*

$$\begin{aligned}
&\int_{-1}^{1} \frac{(1-x^2)^{\gamma-1/2}}{\mathrm{R}\,(1+\mathrm{R}-\rho)^{\alpha-1/2}(1+\mathrm{R}+\rho)^{\gamma-1/2}} C_k^{\gamma}(x)dx \\
&= \frac{\sqrt{\pi}\,2^{1-\gamma-\alpha}\Gamma(\gamma+1/2)\left(\frac{\alpha+\gamma}{2}\right)_k\left(\frac{\alpha+\gamma+1}{2}\right)_k(2\gamma)_k}{\Gamma(\gamma+k+1)(\alpha+\gamma)_k\,k!}\,{}_3F_2\left(\begin{matrix}\gamma+k+\frac12, \alpha+\gamma+2k, \alpha-\gamma \\ \alpha+\gamma+k, 2\gamma+2k+1\end{matrix}; \rho\right)\rho^k.
\end{aligned}$$

Proof. Same as in the proof of Corollary 17, except apply to both sides of (33). □

Corollary 19. *Let $k \in \mathbf{N}_0, \alpha \in \mathbf{C}, \rho \in \{z \in \mathbf{C} : |z| < 1\}$. Then*

$$\int_{-1}^{1} \frac{(1+\mathrm{R}+\rho)^{1/2}}{\mathrm{R}(1+\mathrm{R}-\rho)^{\alpha-1/2}(1-x^2)^{1/2}} T_k(x)dx = \frac{\pi\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha+1}{2}\right)_k}{2^{\alpha-1}(\alpha)_k\,k!}\,{}_3F_2\left(\begin{matrix}k+\frac12, \alpha+2k, \alpha \\ 2k+1, \alpha+k\end{matrix}; \rho\right)\rho^k.$$

Proof. Multiplying both sides of (34) by $T_k(x)(1-x^2)^{-1/2}$ and integrating from -1 to 1 using the orthogonality relation for Chebyshev polynomials of the first kind, Olver et al. [12, (18.2.1), (18.2.5), Table 18.3.1]

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}dx = \frac{\pi}{\epsilon_n}\delta_{m,n},$$

with simplification completes the proof. □

Corollary 20. *Let $k \in \mathbf{N}_0, \alpha, \beta \in \mathbf{R}, \rho \in \mathbf{C} \setminus \{0\}$. Then*

$$\int_0^{\infty} x^{\beta-\alpha/2}e^{-x}J_{\alpha}(2\sqrt{\rho x})L_k^{\beta}(x)dx = \Gamma(\beta-\alpha+1)\frac{e^{-\rho}\rho^{k+\alpha/2}}{k!}L_{\beta-\alpha}^{\alpha+k}(\rho).$$

Proof. Multiplying both sides of (38) by $x^\beta e^{-x} L_{k'}^\beta(x)$ for $k' \in \mathbf{N}_0$, integrating over $(0, \infty)$ and using the orthogonality relation for Laguerre polynomials (36) completes the proof. □

Applying the process of Corollary 20 to both sides of (44) produces the definite integral for Laguerre polynomials

$$\int_0^\infty x^\beta \exp\left(\frac{x}{\rho - 1}\right) L_n^\beta(x) dx = \frac{\Gamma(n + \beta + 1)(1 - \rho)^{\beta+1}}{n!} \rho^n, \tag{61}$$

which is a specific case of the definite integral given by Gradshteyn and Ryzhik [8, (7.414.8)]. This is not surprising since (61) was found using the generating function for Laguerre polynomials.

Corollary 21. *Let $k \in \mathbf{N}_0$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $\Re a, \Re b, \Re c, \Re d, \Re h > 0$ and non-real parameters occurring in conjugate pairs. Then*

$$\int_0^\infty {}_2F_1\left(\begin{matrix} a + ix,\ b + ix \\ a + b \end{matrix}; \rho\right) {}_2F_1\left(\begin{matrix} c - ix,\ d - ix \\ c + d \end{matrix}; \rho\right) W_k\left(x^2; a, b, c, h\right) w(x)\, dx$$
$$= \frac{2\pi \Gamma(a + b)\Gamma(k + a + c)\Gamma(k + a + h)\Gamma(k + b + c)\Gamma(k + b + h)\Gamma(k + c + h)}{(c + d)_k \Gamma(2k + a + b + c + h) \{(k + a + b + c + d - 1)_k\}^{-1}}$$
$$\times {}_4F_3\left(\begin{matrix} d - h, 2k + a + b + c + d - 1, k + a + c, k + b + c \\ k + a + b + c + d - 1, 2k + a + b + c + h, k + c + d \end{matrix}; \rho\right) \rho^k$$

where $w : (0, \infty) \to \mathbf{R}$ is defined by

$$w(x) := \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(h + ix)}{\Gamma(2ix)} \right|^2.$$

Proof. Multiplying both sides of (46) by $W_{k'}(x^2; a, b, c, h)w(x)$, for $k' \in \mathbf{N}_0$, integrating over $x \in (0, \infty)$ and using the orthogonality relation for Wilson polynomials (45) completes the proof. □

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