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Expansions for a fundamental solution of Laplace's equation on $\mathbb{R}^3$ in 5-cyclidic harmonics

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We derive eigenfunction expansions for a fundamental solution of Laplace's equation in three-dimensional Euclidean space in 5-cyclidic coordinates. There are three such expansions in terms of internal and external 5-cyclidic harmonics of first, second and third kind. The internal and external 5-cyclidic harmonics are expressed by solutions of a Fuchsian differential equation with five regular singular points.

Keywords: Laplace equation; fundamental solution; general spectral theory; eigenfunction expansions; asymmetric cyclidic coordinate system.

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1. Introduction

Expansions for a fundamental solution of Laplace's equation on $\mathbb{R}^3$ in terms of solutions found by the method of separation of variables in a suitable curvilinear coordinate system are known for a long time. For example, when we choose spherical coordinates, we obtain the well-known expansion [25]

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}=\sum_{\ell=0}^{\infty}\frac{r^\ell}{(r')^{\ell+1}}\sum_{m=-\ell}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\mathsf{P}_\ell^m(\cos\theta)\mathsf{P}_\ell^m(\cos\theta')e^{im(\phi-\phi')}, \tag{1.1}$$

where $\|\mathbf{r}\|<\|\mathbf{r}'\|$ ($\|\mathbf{r}\|$ denotes the Euclidean norm of $\mathbf{r}\in\mathbb{R}^3$), and r,θ,ϕ, r',θ',ϕ' are the spherical coordinates of $\mathbf{r}$ and $\mathbf{r}'$, respectively. The expansion (1.1) contains the Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut) P_ℓ^m [26, (14.3.1)]. We may write expansion (1.1) in the more concise form

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} G_\ell^m(\mathbf{r})\overline{H_\ell^m(\mathbf{r}')}, \tag{1.2}$$

where $G_\ell^m : \mathbb{R}^3 \to \mathbb{C}$ is the internal spherical harmonic

$$G_\ell^m(\mathbf{r}) := \left(\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} r^\ell \mathsf{P}_\ell^m(\cos\theta)e^{im\phi}, \tag{1.3}$$

and $H_\ell^m : \mathbb{R}^3\backslash\{(0,0,0)\} \to \mathbb{C}$ is the external spherical harmonic

$$H_\ell^m(\mathbf{r}') := \left(\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} (r')^{-\ell-1}\mathsf{P}_\ell^m(\cos\theta')e^{im\phi'}. \tag{1.4}$$

In this paper we derive expansions analogous to (1.2) for the 5-cyclidic coordinate system [24, (6.24)] in place of spherical coordinates. The coordinate surfaces of 5-cyclidic coordinates are triply orthogonal confocal cyclides. There are three kinds of internal and external 5-cyclidic harmonics, one for each family of coordinate surfaces, and three corresponding expansions. The authors already introduced internal 5-cyclidic harmonics in [13]. As far as we know, the definition of external 5-cyclidic harmonics and the expansions analogous to (1.2) are given in this paper for the first time. We also derive some needed additional properties of internal 5-cyclidic harmonics. In the definitions of internal and external spherical harmonics (1.3), (1.4) there appear only the associated Legendre functions apart from elementary functions. In the case of 5-cyclidic coordinates the definition of internal and external harmonics requires solutions of a Fuchsian differential equation with five regular singularities. The particular solutions of interest are eigenfunctions of two-parameter Sturm–Liouville eigenvalue problems; see [13].

In Bôcher's 1891 dissertation, *Ueber die Reihenentwickelungen der Potentialtheorie* [4], it was shown that the Laplace equation on $\mathbb{R}^3$ can be solved using separation of variables in seventeen conformally distinct quadric and cyclidic coordinate systems. The latter have been introduced by young Darboux in his master's thesis of 1864 [14]. These early ideas were improved in his doctoral thesis of 1866 [15]. The Helmholtz equation on $\mathbb{R}^3$ admits simply separable solutions in the same eleven quadric coordinate systems that the Laplace equation admits separable solutions [17]. The Laplace equation also admits R-separable solutions in an additional six conformally distinct coordinate systems [24, Table 17, p. 210]. Unlike the Laplace equation, the Helmholtz equation does not admit solutions via R-separation of variables. The appearance of R-separation is intrinsic to the existence of conformal symmetries for a linear partial differential equation (see [5]), i.e. dilatations, special conformal transformations, inversions and reflections. The theory of separation of variables from a Lie group theoretic viewpoint has been treated in [24]. In Miller's book [24, Table 17, System 12], separation of variables for the Laplace equation on $\mathbb{R}^3$ was treated and the general asymmetric R-separable 5-cyclidic coordinate system was introduced. In regard to this coordinate system, and the corresponding separable harmonic solutions, Miller indicates that "Very little is known about the solutions."

To the authors' knowledge, eigenfunction expansions for the fundamental solution (the $1/r$ potential) have been obtained for the following coordinate systems. See [12, 19, 22, 25] for expansions in spherical, circular/parabolic/elliptic cylinder, oblate/prolate spheroidal, parabolic, bi-spherical and toroidal coordinates. The expansion in confocal ellipsoidal coordinates is treated in [3, 18]. This paper is a stepping-stone for derivations of eigenfunction expansions for a fundamental solution of Laplace's equation in coordinate systems where these expansions are not known such as paraboloidal, flat-ring cyclide, flat-disk cyclidic, bi-cyclide, cap-cyclide and 3-cyclide (see [24, Table 17, System 13]) coordinates. For results on the geometry of cyclides, we refer to [27].

These eigenfunction expansions are often connected with other results such as:

(a) integral identities [28, Secs. 13.2 and 3.21] (e.g. circular cylindrical and parabolic coordinates);
(b) addition theorems [28, Secs. 11.1 and 11.3]; [29] (e.g. Neumann's addition theorem, and Graf's generalization of Neumann's addition theorem in circular cylindrical coordinates and the addition theorem for spherical harmonics in spherical coordinates);
(c) generating functions for orthogonal polynomials [26, (18.12.4)] (e.g. the generating function for Legendre polynomials in spherical coordinates); and
(d) special function expansion identities [8, (3.11)] (e.g. Heine's reciprocal square root identity in circular cylindrical coordinates).

In this setting, one may perform eigenfunction expansions for a fundamental solution of Laplace's equation in alternative separable coordinate systems to obtain new special function summation and integration identities which often have interesting geometrical interpretations (see for instance [6, 10, 11]). Eigenfunction expansions for fundamental solutions of elliptic partial differential equations have been extended to more general separable linear partial differential equations [7] and to partial differential equations on Riemannian manifolds of constant curvature [9].

The outline of this paper is as follows. The 5-cyclidic coordinate system s_1, s_2, s_3 is discussed in Sec. 2. In Sec. 3, we consider internal and external 5-cyclidic harmonics of the second kind which are related to the coordinate surfaces $s_2 = \text{const.}$ We start with functions of the second kind because they are slightly easier to treat than the harmonics of the first and third kind related to the coordinate surfaces $s_1 = \text{const}$, $s_3 = \text{const}$, respectively. In Sec. 4, as one of our main results, we obtain the expansion of the fundamental solution of Laplace's equation in terms of internal and external 5-cyclidic harmonics of the second kind. The proof is based on (a) an integral representation of the external harmonics in terms of internal harmonics given in Sec. 4, and (b) the completeness property of internal harmonics obtained in [13]. In Secs. 5 and 6, we treat 5-cyclidic harmonics of the first kind. In Secs. 7 and 8, we treat 5-cyclidic harmonics of the third kind.

2. 5-Cyclidic Coordinates

We work on $\mathbb{R}^3$ with Cartesian coordinates x, y, z, and we use the notations $\mathbf{r} = (x, y, z)$ and $\|\mathbf{r}\| = (x^2 + y^2 + z^2)^{1/2}$. Fix $a_0 < a_1 < a_2 < a_3$. The 5-cyclidic coordinates of a point $\mathbf{r} \in \mathbb{R}^3$ are the solutions $s = s_1, s_2, s_3$ of the equation

$$\frac{(\|\mathbf{r}\|^2 - 1)^2}{s - a_0} + \frac{4x^2}{s - a_1} + \frac{4y^2}{s - a_2} + \frac{4z^2}{s - a_3} = 0 \tag{2.1}$$

(strictly speaking, this equation is multiplied by the common denominator of the left-hand side), where

$$a_0 \le s_1 \le a_1 \le s_2 \le a_2 \le s_3 \le a_3;$$

see [13, Sec. 4]. On the set

$$R := \{\mathbf{r} : x, y, z > 0,\ \|\mathbf{r}\| < 1\}, \tag{2.2}$$

the map $(x, y, z) \in R \mapsto (s_1, s_2, s_3) \in (a_0, a_1) \times (a_1, a_2) \times (a_2, a_3)$ is bijective. The inverse map is given by

$$x = \frac{x_1}{1 + x_0}, \quad y = \frac{x_2}{1 + x_0}, \quad z = \frac{x_3}{1 + x_0}, \tag{2.3}$$

where

$$x_j^2 = \frac{\displaystyle\prod_{i=1}^{3}(s_i - a_j)}{\displaystyle\prod_{j \ne i = 0}^{3}(a_i - a_j)}, \quad x_j > 0. \tag{2.4}$$

We note that each s_i is a continuous function on $\mathbb{R}^3$. Of particular interest are the sets

$$A_1 := \{\mathbf{r} : s_1 = s_2\} = \{(0, y, z) : g_1(y, z) = 0\},$$
$$A_2 := \{\mathbf{r} : s_2 = s_3\} = \{(x, 0, z) : g_2(x, z) = 0\},$$

where

$$g_1(y, z) := \frac{(y^2 + z^2 - 1)^2}{a_1 - a_0} + \frac{4y^2}{a_1 - a_2} + \frac{4z^2}{a_1 - a_3},$$
$$g_2(x, z) := \frac{(x^2 + z^2 - 1)^2}{a_2 - a_0} + \frac{4x^2}{a_2 - a_1} + \frac{4z^2}{a_2 - a_3}.$$

Each set A_1, A_2 consists of two closed curves; see Figs. 1 and 2. The function s_1 is (real-)analytic on $\mathbb{R}^3 \backslash A_1$, s_2 is analytic on $\mathbb{R}^3 \backslash (A_1 \cup A_2)$, and s_3 is analytic on

$\mathbb{R}^3 \backslash A_2$. We will also encounter the sets

$$K_1 := \{\mathbf{r} : \|\mathbf{r}\| < 1, s_1 = a_1\} = \{(0,y,z) : y^2 + z^2 < 1, g_1(y,z) \geq 0\},$$

$$L_1 := \{\mathbf{r} : s_2 = a_1\} = \{(0,y,z) : g_1(y,z) \leq 0\},$$

$$M_1 := \{\mathbf{r} : \|\mathbf{r}\| > 1, s_1 = a_1\} = \{(0,y,z) : y^2 + z^2 > 1, g_1(y,z) \geq 0\},$$

$$K_2 := \{\mathbf{r} : z > 0, s_3 = a_2\} = \{(x,0,z) : z > 0, g_2(x,z) \leq 0\},$$

$$L_2 := \{\mathbf{r} : s_2 = a_2\} = \{(x,0,z) : g_2(x,z) \geq 0\},$$

$$M_2 := \{\mathbf{r} : z < 0, s_3 = a_2\} = \{(x,0,z) : z < 0, g_2(x,z) \leq 0\}.$$

The sets A_1, K_1, L_1, M_1 are subsets of the plane $x = 0$, and A_2, K_2, L_2, M_2 are subsets of the plane $y = 0$; see Figs. 1 and 2.

We denote the inversion at the unit sphere on $\mathbb{R}^3$ by

$$\sigma_0(\mathbf{r}) := \|\mathbf{r}\|^{-2}\mathbf{r}, \tag{2.5}$$

and the reflections at the coordinate planes by

$$\sigma_1(x,y,z) := (-x,y,z), \quad \sigma_2(x,y,z) := (x,-y,z), \quad \sigma_3(x,y,z) := (x,y,-z). \tag{2.6}$$

We note that the functions s_1, s_2, s_3 are invariant under σ_j, $j = 0,1,2,3$.

We define auxiliary functions $\chi_j : \mathbb{R}^3 \to \mathbb{R}$, $j = 0,1,2,3$, by

$$\chi_0(\mathbf{r}) := \operatorname{sgn}(1 - \|\mathbf{r}\|)(s_1 - a_0)^{1/2},$$

$$\chi_1(\mathbf{r}) := \operatorname{sgn}(x)((s_2 - a_1)(a_1 - s_1))^{1/2},$$

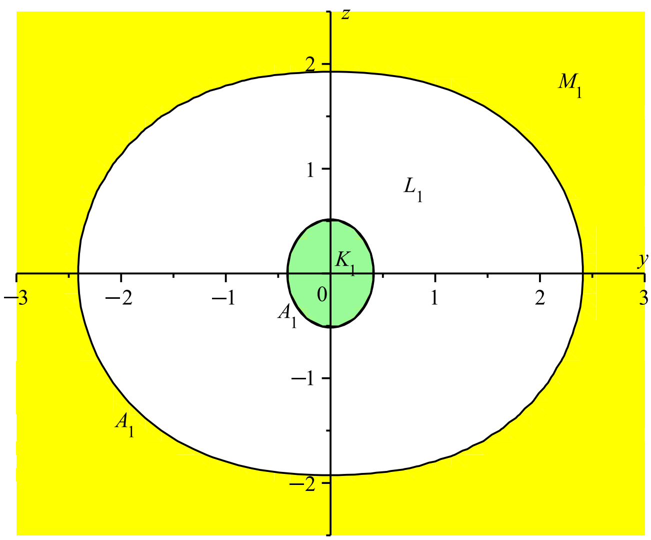

Fig. 1. Curves A_1 and regions K_1, L_1, M_1 for $a_j = j$.

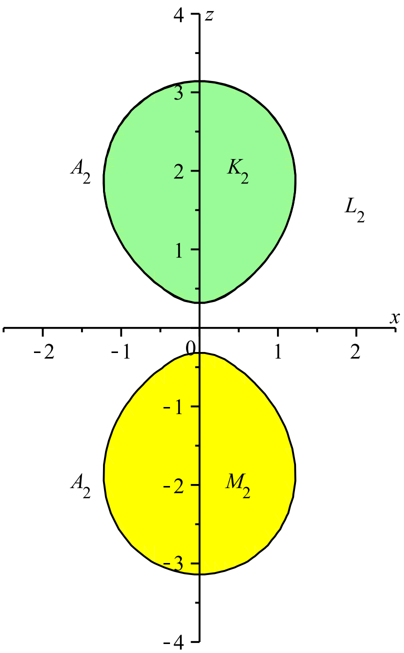

Fig. 2. Curves A_2 and regions K_2, L_2, M_2 for $a_j = j$.

$$\chi_2(\mathbf{r}) := \operatorname{sgn}(y)((s_3 - a_2)(a_2 - s_2))^{1/2},$$
$$\chi_3(\mathbf{r}) := \operatorname{sgn}(z)(a_3 - s_3)^{1/2}.$$

Lemma 2.1. *The functions* $\chi_j, j = 0, 1, 2, 3,$ *are continuous on* $\mathbb{R}^3$, χ_0, χ_2 *are analytic on* $\mathbb{R}^3 \backslash A_1$, *and* χ_1, χ_3 *are analytic on* $\mathbb{R}^3 \backslash A_2$. *Moreover,*

$$\chi_j \circ \sigma_i = \begin{cases} \chi_j & \text{if } i \neq j, \\ -\chi_j & \text{if } i = j. \end{cases} \tag{2.7}$$

Proof. Consider first χ_3. The function s_3 is continuous, and $s_3 = a_3$ if and only if $z = 0$. Therefore, χ_3 is continuous. In order to prove that χ_3 is analytic on $\mathbb{R}^3 \backslash A_2$, it is enough to show that χ_3 is analytic at every point of the plane $z = 0$. Let $\mathbf{r}_0 = (x_0, y_0, 0)$. There is $\epsilon \in (0, 1)$ such that $s_3 \neq a_2$ for $\mathbf{r} \in B_\epsilon(\mathbf{r}_0) = \{\mathbf{r} : \|\mathbf{r} - \mathbf{r}_0\| < \epsilon\}$. Then (2.1) with $s = s_3$ implies

$$a_3 - s_3 = \frac{4z^2}{f(\mathbf{r})} \quad \text{for } \mathbf{r} \in B_\epsilon(\mathbf{r}_0),$$

where

$$f(\mathbf{r}) := \frac{(\|\mathbf{r}\|^2 - 1)^2}{s_3 - a_0} + \frac{4x^2}{s_3 - a_1} + \frac{4y^2}{s_3 - a_2}$$

is positive and analytic on $B_\epsilon(\mathbf{r}_0)$. Therefore, we obtain

$$\chi_3(\mathbf{r}) = \frac{2z}{(f(\mathbf{r}))^{1/2}} \quad \text{for } \mathbf{r} \in B_\epsilon(\mathbf{r}_0),$$

and this shows that χ_3 is analytic at $\mathbf{r}_0$. χ_0 is treated similarly.

Consider next χ_2. The functions s_2, s_3 are continuous, and $(a_2 - s_2)(s_3 - a_2) = 0$ if and only if $y = 0$. Thus χ_2 is continuous. In order to prove that χ_2 is analytic on $\mathbb{R}^3 \backslash A_1$, it is enough to show that χ_2 is analytic at all points of the plane $y = 0$ which do not lie in A_1. Suppose first $\mathbf{r}_0 = (x_0, 0, z_0) \in (K_2 \cup M_2) \backslash A_2$. There is $\epsilon > 0$ such that $s_3 \neq a_3$ and $s_2 \neq a_2$ for $\mathbf{r} \in B_\epsilon(\mathbf{r}_0)$. Then, by (2.1) with $s = s_3$, we obtain

$$s_3 - a_2 = \frac{4y^2}{g(\mathbf{r})},$$

where

$$g(\mathbf{r}) := -\frac{(\|\mathbf{r}\|^2 - 1)^2}{s_3 - a_0} - \frac{4x^2}{s_3 - a_1} - \frac{4z^2}{s_3 - a_3}$$

is analytic on $B_\epsilon(\mathbf{r}_0)$. Since $g(\mathbf{r}_0) = -g_2(x_0, z_0) > 0, g$ is also positive on $B_\epsilon(\mathbf{r}_0)$ for sufficiently small $\epsilon > 0$. Then

$$\chi_2(\mathbf{r}) = (a_2 - s_2)^{1/2} \frac{2y}{(g(\mathbf{r}))^{1/2}} \quad \text{for } \mathbf{r} \in B_\epsilon(\mathbf{r}_0).$$

This shows that χ_2 is analytic at $\mathbf{r}_0$ provided that $\mathbf{r}_0 \notin A_1$. In a similar way, by using (2.1) with $s = s_2$, we show that χ_2 is analytic at all points $\mathbf{r}_0 \in L_2 \backslash A_2$. Finally, by subtracting equations (2.1) with $s = s_2, s_3$ from each other, we show that χ_2 is analytic at all points $\mathbf{r}_0 \in A_2$. χ_1 is treated similarly.

The symmetries (2.7) follow from the definition of χ_j. □

Solving the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2.8}$$

by the method of separation of variables, we find solutions

$$u(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2} w_1(s_1) w_2(s_2) w_3(s_3), \quad s_i \in (a_{i-1}, a_i). \tag{2.9}$$

Each function $w = w_1, w_2, w_3$ satisfies the Fuchsian equation

$$\prod_{j=0}^{3}(s - a_j)\left[w'' + \frac{1}{2}\sum_{j=0}^{3}\frac{1}{s - a_j}w'\right] + \left(\frac{3}{16}s^2 + \lambda_1 s + \lambda_2\right)w = 0, \tag{2.10}$$

where λ_1, λ_2 are separation constants; see [13]. This equation has five regular singularities at $a_0, a_1, a_2, a_3, \infty$. The exponents at each finite singularity are 0 or $\frac{1}{2}$.

The function $u(\mathbf{r})$ defined in (2.9) is harmonic for all choices of solutions w_i to (2.10). However, it is harmonic only in the open set obtained from $\mathbb{R}^3$ by removing the coordinate planes $x = 0, y = 0, z = 0$ and the unit sphere $\|\mathbf{r}\| = 1$. In order to obtain globally defined harmonic functions we have to select the Frobenius solutions w at the finite singularities, that is, solutions that are either analytic at a_j or of the form $(s-a_j)^{1/2}g(s)$ with $g(s)$ analytic at $s=a_j$. It is impossible to choose the parameters λ_1, λ_2 in such a way that each solution $w_i, i=1,2,3$, is a nontrivial Frobenius solution belonging to either one of the exponents 0 or $\frac{1}{2}$ at both end points a_{i-1}, a_i. If this were possible (2.9) would define a function which is harmonic in the whole space $\mathbb{R}^3$ (as we see later) and converges to 0 as $\|\mathbf{r}\| \to \infty$. But such a function would have to be identically zero. However, as shown in [13], we can determine special values of λ_1, λ_2 (eigenvalues) such that two solutions (either (1) w_2, w_3, or (2) w_1, w_3, or (3) w_1, w_2) are nontrivial Frobenius solutions at both end points simultaneously. These cases lead to 5-cyclidic harmonics of the first, second and third kind. If the remaining function w_i in case (i) is chosen appropriately, we obtain internal or external 5-cyclidic harmonics.

3. 5-Cyclidic Harmonics of the Second Kind

In [13, Sec. VII] we introduced special solutions $w_i(s_i) = E^{(2)}_{i,\mathbf{n},\mathbf{p}}(s_i)$ to Eq. (2.10) for eigenvalues $\lambda_j = \lambda^{(2)}_{j,\mathbf{n},\mathbf{p}}, j=1,2$, for every $\mathbf{n} \in \mathbb{N}_0^2, \mathbf{p} = (p_0,p_1,p_2,p_3) \in \{0,1\}^4$. If $\mathbf{n} = (n_1,n_3)$ then n_i denotes the number of zeros of $E^{(2)}_{i,\mathbf{n},\mathbf{p}}$ in (a_{i-1},a_i) for $i=1,3$. The subscript p_j describes the behavior of the solutions at the endpoint a_j: We have

$$E^{(2)}_{i,\mathbf{n},\mathbf{p}}(s_i) = (s_i - a_{i-1})^{p_{i-1}/2}(a_i - s_i)^{p_i/2}\mathcal{E}^{(2)}_{i,\mathbf{n},\mathbf{p}}(s_i), \quad s_i \in (a_{i-1}, a_i),$$

where $\mathcal{E}^{(2)}_{1,\mathbf{n},\mathbf{p}}$ is analytic on $[a_0,a_1]$, $\mathcal{E}^{(2)}_{2,\mathbf{n},\mathbf{p}}$ is analytic on $[a_1,a_2)$ (but not at a_2), and $\mathcal{E}^{(2)}_{3,\mathbf{n},\mathbf{p}}$ is analytic on $[a_2,a_3]$.

According to (2.9) the function

$$G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2+1)^{-1/2}E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1)E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)E^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3), \quad \mathbf{r} \in R, \tag{3.1}$$

is harmonic on R. In order to analytically extend $G^{(2)}_{\mathbf{n},\mathbf{p}}$ we use the functions χ_j introduced in Sec. 2. We set

$$G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2+1)^{-1/2}\prod_{j=0}^{3}(\chi_j(\mathbf{r}))^{p_j}\prod_{i=1}^{3}\mathcal{E}^{(2)}_{i,\mathbf{n},\mathbf{p}}(s_i) \quad \text{if } s_2 \neq a_2 \tag{3.2}$$

which is consistent with (3.1). The condition $s_2 \neq a_2$ is equivalent to $\mathbf{r} \in \mathbb{R}^3 \backslash L_2$. We call $G^{(2)}_{\mathbf{n},\mathbf{p}}$ an internal 5-cyclidic harmonic of the second kind.

Theorem 3.1. *Let* $\mathbf{n} \in \mathbb{N}_0^2$ *and* $\mathbf{p} \in \{0,1\}^4$. *Then* $G^{(2)}_{\mathbf{n},\mathbf{p}}$ *is harmonic on* $\mathbb{R}^3 \backslash L_2$. *Moreover*,

$$G^{(2)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j}G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) \quad \textit{for } j=1,2,3, \tag{3.3}$$

and

$$G^{(2)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) = (-1)^{p_0}\|\mathbf{r}\|G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}). \tag{3.4}$$

Proof. By (3.2) and Lemma 2.1, $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is a composition of continuous functions, and thus it is continuous on $\mathbb{R}^3\backslash L_2$. As a composition of analytic functions, $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is analytic and thus harmonic on $\mathbb{R}^3\backslash(A_1 \cup L_2)$. The set A_1 is a removable line singularity of $G^{(2)}_{\mathbf{n},\mathbf{p}}$. This can be seen in two different ways. (1) We may appeal to the general theory of harmonic functions. A_1 is a polar set, and we may apply [2, Corollary 5.2.3]. (2) We can show directly that $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is analytic at each point of A_1 by the method used in the proof of [13, Lemma 6.1]. For example, take the simplest case $\mathbf{p} = (0,0,0,0)$. Then (3.1) holds for all $\mathbf{r} \in \mathbb{R}^3\backslash L_2$, and the product $E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1)E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)$ is analytic at each point of A_1. This is because $E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s)$ and $E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s)$ are analytic extensions of each other, and s_1, s_2 enter symmetrically. Note that $s_1 s_2$ and s_1+s_2 are analytic at each point of A_1 although s_1, s_2 are not analytic there.

The symmetry properties of $G^{(2)}_{\mathbf{n},\mathbf{p}}$ also follow from (3.2) and Lemma 2.1. □

If $U(\mathbf{r})$ is a harmonic function then its Kelvin transformation

$$V(\mathbf{r}) := \|\mathbf{r}\|^{-1}U(\sigma_0(\mathbf{r}))$$

is also harmonic [20, p. 232]. Equation (3.4) states that $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is invariant or changes sign under the Kelvin transformation if $p_0 = 0$ or $p_0 = 1$, respectively. We see that L_2 is a "surface singularity" of $G^{(2)}_{\mathbf{n},\mathbf{p}}$ which is not removable (it is not a polar set). In fact, $G^{(2)}_{\mathbf{n},\mathbf{p}}$ cannot be harmonic on $\mathbb{R}^3$ because it would be identically zero otherwise.

Let $F^{(2)}_{2,\mathbf{n},\mathbf{p}}$ be the Frobenius solution to the Fuchsian equation (2.10) (with $\lambda_j = \lambda^{(2)}_{j,\mathbf{n},\mathbf{p}}$) on (a_1, a_2) belonging to the exponent $\frac{p_2}{2}$ at $s_2 = a_2$, uniquely determined by the Wronskian condition

$$\omega(s)\left(E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)\frac{d}{ds_2}F^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2) - F^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)\frac{d}{ds_2}E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)\right) = 1, \tag{3.5}$$

where

$$\omega(s) := |(s-a_0)(s-a_1)(s-a_2)(s-a_3)|^{1/2}. \tag{3.6}$$

This definition is possible because we know that $E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)$ is not a Frobenius solution belonging to the exponent $\frac{p_2}{2}$ at $s_2 = a_2$. Now we define external 5-cyclidic harmonics of the second kind by

$$H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2+1)^{-1/2}E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1)F^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)E^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3), \quad \mathbf{r} \in R. \tag{3.7}$$

In order to analytically extend $H^{(2)}_{\mathbf{n},\mathbf{p}}$ we write

$$F^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2) = (s_2-a_1)^{p_1/2}(a_2-s_2)^{p_2/2}\mathcal{F}^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2), \quad s_2 \in (a_1, a_2),$$

where $\mathcal{F}^{(2)}_{2,\mathbf{n},\mathbf{p}}$ is analytic on $(a_1, a_2]$ (but not at a_1). Then we define

$$H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2+1)^{-1/2}\prod_{j=0}^{3}(\chi_j(\mathbf{r}))^{p_j}\mathcal{E}^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1)\mathcal{F}^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)\mathcal{E}^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3) \quad \text{if } s_2 \neq a_1. \tag{3.8}$$

The condition $s_2 \neq a_1$ is equivalent to $\mathbf{r} \in \mathbb{R}^3 \backslash L_1$.

Theorem 3.2. *Let* $\mathbf{n} \in \mathbb{N}_0^2$ *and* $\mathbf{p} \in \{0,1\}^4$. *Then* $H^{(2)}_{\mathbf{n},\mathbf{p}}$ *is harmonic on* $\mathbb{R}^3 \backslash L_1$. *The functions* $H^{(2)}_{\mathbf{n},\mathbf{p}}$ *share the symmetries* (3.3), (3.4) *with* $G^{(2)}_{\mathbf{n},\mathbf{p}}$. *Moreover,*

$$H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) = O(\|\mathbf{r}\|^{-1}) \quad \text{as } \|\mathbf{r}\| \to \infty, \tag{3.9}$$

and

$$\|\nabla H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r})\| = O(\|\mathbf{r}\|^{-2}) \quad \text{as } \|\mathbf{r}\| \to \infty. \tag{3.10}$$

Proof. The proof of analyticity and symmetry of $H^{(2)}_{\mathbf{n},\mathbf{p}}$ is similar to that given for $G^{(2)}_{\mathbf{n},\mathbf{p}}$ in Theorem 3.1, and is omitted. Estimates (3.9) and (3.10) follow easily from the observation that the Kelvin transformation of $H^{(2)}_{\mathbf{n},\mathbf{p}}$ is $\pm H^{(2)}_{\mathbf{n},\mathbf{p}}$ which is analytic at $\mathbf{0} \notin L_1$. □

4. Expansion of the Reciprocal Distance in 5-Cyclidic Harmonics of Second Kind

For given $d_2 \in (a_1, a_2)$ we consider the "5-cyclidic ring"

$$D_2 := \{\mathbf{r} \in \mathbb{R}^3 : s_2 < d_2\}, \tag{4.1}$$

or, equivalently,

$$D_2 = \left\{\mathbf{r} : \frac{(\|\mathbf{r}\|^2 - 1)^2}{d_2 - a_0} + \frac{4x^2}{d_2 - a_1} + \frac{4y^2}{d_2 - a_2} + \frac{4z^2}{d_2 - a_3} < 0\right\}. \tag{4.2}$$

Note that each internal 5-cyclidic harmonic $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is harmonic in D_2 (and on its boundary), and each external 5-cyclidic harmonic is harmonic on $\mathbb{R}^3 \backslash D_2$ (and on its boundary).

We represent external harmonics in terms of internal harmonics by a surface integral over the boundary ∂D_2 of the ring D_2 as follows.

Theorem 4.1. *Let* $d_2 \in (a_1, a_2), \mathbf{n} \in \mathbb{N}_0^2, \mathbf{p} \in \{0,1\}^4$. *Then*

$$H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}') = \frac{1}{4\pi\omega(d_2)\{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)\}^2} \int_{\partial D_2} \frac{G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r})}{h_2(\mathbf{r})\|\mathbf{r} - \mathbf{r}'\|}\, dS(\mathbf{r}) \tag{4.3}$$

for all $\mathbf{r}' \in \mathbb{R}^3 \backslash \bar{D}_2$. *The scale factor* h_2 *is given by*

$$16\{h_2(\mathbf{r})\}^2 = \frac{(\|\mathbf{r}\|^2 - 1)^2}{(d_2 - a_0)^2} + \frac{4x^2}{(d_2 - a_1)^2} + \frac{4y^2}{(d_2 - a_2)^2} + \frac{4z^2}{(d_2 - a_3)^2}. \tag{4.4}$$

Proof. Let D be an open bounded subset of $\mathbb{R}^3$ with smooth boundary. For $u, v \in C^2(\bar{D})$, Green's formula states that

$$\int_D (u\Delta v - v\Delta u)\, d\mathbf{r} = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS, \tag{4.5}$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary ∂D of D.

We apply (4.5) to the domain $D = D_2$, and functions $u = G = G^{(2)}_{\mathbf{n},\mathbf{p}}$, $v(\mathbf{r}) = \frac{1}{4\pi\|\mathbf{r}-\mathbf{r}'\|}$. Since u, v are harmonic on an open set containing $\bar{D}_2$ we obtain

$$0 = \int_{\partial D_2} \left(G\frac{\partial v}{\partial \nu} - v\frac{\partial G}{\partial \nu} \right) dS. \tag{4.6}$$

We now use (4.5) a second time. We choose $R > 0$ so large that the ball $B_R(\mathbf{0})$ contains $\mathbf{r}'$ and $\bar{D}_2$. Then we take $D = B_R(\mathbf{0}) - \bar{D}_2 - B_\epsilon(\mathbf{r}')$ with small radius $\epsilon > 0$. Take $u = H = H^{(2)}_{\mathbf{n},\mathbf{p}}$ and v as before. Note that u, v are harmonic on an open set containing $\bar{D}$. By a standard argument [23, Theorem 1, p. 109], taking the limit $\epsilon \to 0$, we obtain

$$H(\mathbf{r}') = \int_{\partial B_R(\mathbf{0})} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu} \right) dS - \int_{\partial D_2} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu} \right) dS, \tag{4.7}$$

where, in the second integral, $\frac{\partial}{\partial \nu}$ denotes the same derivative as in (4.6). The first integral in (4.7) tends to 0 as $R \to \infty$ by (3.9), (3.10). Therefore,

$$H(\mathbf{r}') = -\int_{\partial D_2} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu} \right) dS. \tag{4.8}$$

We now multiply (4.6) by $F_2(d_2), F_2 := F^{(2)}_{2,\mathbf{n},\mathbf{p}}$, then multiply (4.8) by $E_2(d_2), E_i := E^{(2)}_{i,\mathbf{n},\mathbf{p}}$, and add these equations. By (3.1) and (3.7) we have

$$F_2(d_2)G(\mathbf{r}) = E_2(d_2)H(\mathbf{r}), \quad \mathbf{r} \in \partial D_2,$$

first for $\mathbf{r} \in \partial D_2 \cap R$ but then for all $\mathbf{r} \in \partial D_2$ by shared symmetries (3.3), (3.4) of G, H. Therefore, we find

$$E_2(d_2)H(\mathbf{r}') = \int_{\partial D_2} v \left(E_2(d_2)\frac{\partial H}{\partial \nu} - F_2(d_2)\frac{\partial G}{\partial \nu} \right) dS. \tag{4.9}$$

The normal derivative and the derivative with respect to s_2 are related by

$$\frac{\partial}{\partial \nu} = \frac{1}{h_2}\frac{\partial}{\partial s_2},$$

where h_2 is the scale factor of the 5-cyclidic coordinate s_2 given by (4.4); see [13, (22)]. Let $\mathbf{r} \in \partial D_2 \cap R$ with 5-cyclidic coordinates $s_1, s_2 = d_2, s_3$. Then

$$\begin{aligned}
&\left(E_2(d_2)\frac{\partial H}{\partial \nu} - F_2(d_2)\frac{\partial G}{\partial \nu} \right)(\mathbf{r}) \\
&= E_2(d_2)\frac{\partial(\|\mathbf{r}\|^2+1)^{-1/2}}{\partial \nu} E_1(s_1)F_2(d_2)E_3(s_3) \\
&\quad + E_2(d_2)(\|\mathbf{r}\|^2+1)^{-1/2}h_2^{-1}E_1(s_1)F_2'(d_2)E_3(s_3) \\
&\quad - F_2(d_2)\frac{\partial(\|\mathbf{r}\|^2+1)^{-1/2}}{\partial \nu} E_1(s_1)E_2(d_2)E_3(s_3) \\
&\quad - F_2(d_2)(\|\mathbf{r}\|^2+1)^{-1/2}h_2^{-1}E_1(s_1)E_2'(d_2)E_3(s_3) \\
&= h_2^{-1}(\|\mathbf{r}\|^2+1)^{-1/2}E_1(s_1)\left\{E_2(d_2)F_2'(d_2) - E_2'(d_2)F_2(d_2)\right\}E_3(s_3).
\end{aligned}$$

We now use (3.5) and obtain

$$\left(E_2(d_2)\frac{\partial H}{\partial \nu} - F_2(d_2)\frac{\partial G}{\partial \nu}\right)(\mathbf{r}) = \frac{G(\mathbf{r})}{h_2(\mathbf{r})\omega(d_2)E_2(d_2)}, \tag{4.10}$$

which holds for all $\mathbf{r} \in \partial D_2$ because G and H share the symmetries (3.3), (3.4). When we substitute (4.10) in (4.9) we arrive at (4.3). □

We obtain the expansion of the reciprocal distance in 5-cyclidic harmonics.

Theorem 4.2. *Let* $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ *with* 5-*cyclidic coordinates* s_2, s_2', *respectively. If* $s_2 < s_2'$ *then*

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \pi \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^4} G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}'). \tag{4.11}$$

Proof. We pick d_2 such that $s_2 < d_2 < s_2'$, and consider the domain D_2 defined in (4.1). The function $f(\mathbf{q}) := \|\mathbf{q}-\mathbf{r}'\|^{-1}$ is harmonic on an open set containing $\bar{D}_2$. Therefore, by [13, (95), (97)], we have

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^4} d^{(2)}_{\mathbf{n},\mathbf{p}} G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{4.12}$$

where

$$d^{(2)}_{\mathbf{n},\mathbf{p}} := \frac{1}{4\omega(d_2)\{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)\}^2} \int_{\partial D_2} \frac{G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{q})}{h_2(\mathbf{q})\|\mathbf{q}-\mathbf{r}'\|}\, dS(\mathbf{q}).$$

Using Theorem 4.1, we obtain (4.11). □

5. 5-Cyclidic Harmonics of the First Kind

In [13, Sec. V] we introduced special solutions $w_i(s_i) = E^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i)$ to Eq. (2.10) for eigenvalues $\lambda_j = \lambda^{(1)}_{j,\mathbf{n},\mathbf{p}}, j = 1, 2$, for every $\mathbf{n} \in \mathbb{N}_0^2, \mathbf{p} = (p_1, p_2, p_3) \in \{0,1\}^3$. These functions have the form

$$E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) = (a_1 - s_1)^{p_1/2} \mathcal{E}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1), \quad s_1 \in (a_0, a_1),$$

$$E^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i) = (s_i - a_{i-1})^{p_{i-1}/2}(a_i - s_i)^{p_i/2} \mathcal{E}^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i), \quad s_i \in (a_{i-1}, a_i),\ i = 2, 3,$$

where $\mathcal{E}^{(1)}_{1,\mathbf{n},\mathbf{p}}$ is analytic on $(a_0, a_1]$ (but not at a_0) while $\mathcal{E}^{(1)}_{i,\mathbf{n},\mathbf{p}}$ is analytic on $[a_{i-1}, a_i]$ for $i = 2, 3$. As in [13, Sec. VI] we define the internal 5-cyclidic harmonic of the first kind by

$$G^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2} E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3), \quad \mathbf{r} \in R. \tag{5.1}$$

According to (2.9), $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is a harmonic function in the region R. In order to analytically extend $G^{(1)}_{\mathbf{n},\mathbf{p}}$ to a larger domain of definition, some preparations are necessary.

Let $P^{(1)}_{1,\mathbf{n},\mathbf{p}}$ be the solution to (2.10) (with $\lambda_j = \lambda^{(1)}_{j,\mathbf{n},\mathbf{p}}$) on (a_0, a_1) belonging to the exponent 0 at $s = a_0$ and uniquely determined by the condition $P^{(1)}_{1,\mathbf{n},\mathbf{p}}(a_0) = 1$. We write

$$P^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) = (a_1 - s_1)^{p_1/2}\mathcal{P}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1), \quad s_1 \in (a_0, a_1),$$

where $\mathcal{P}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1)$ is analytic on $[a_0, a_1)$. Then using the functions χ_j from Sec. 2 we define

$$I^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2} \prod_{j=1}^{3} (\chi_j(\mathbf{r}))^{p_j} \mathcal{P}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) \prod_{i=2}^{3} \mathcal{E}^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i) \quad \text{if } s_1 \neq a_1. \tag{5.2}$$

The condition $s_1 \neq a_1$ is equivalent to $\mathbf{r} \in \mathbb{R}^3 \backslash (K_1 \cup M_1)$; see Fig. 1.

Similarly, let $Q^{(1)}_{1,\mathbf{n},\mathbf{p}}$ be the solution to (2.10) (with $\lambda_j = \lambda^{(1)}_{j,\mathbf{n},\mathbf{p}}$) on (a_0, a_1) belonging to the exponent $\frac{1}{2}$ at $s = a_0$ and uniquely determined by the condition $\lim_{s_1 \to a_0^+} \omega(s_1) \frac{d}{ds_1} Q^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) = 1$. We write

$$Q^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) = (s_1 - a_0)^{1/2}(a_1 - s_1)^{p_1/2}\mathcal{Q}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1), \quad s_1 \in (a_0, a_1),$$

where $\mathcal{Q}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1)$ is analytic on $[a_0, a_1)$. Then we define

$$J^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2} \chi_0(\mathbf{r}) \prod_{j=1}^{3} (\chi_j(\mathbf{r}))^{p_j} \mathcal{Q}^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) \prod_{i=2}^{3} \mathcal{E}^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i) \quad \text{if } s_1 \neq a_1. \tag{5.3}$$

Lemma 5.1. *The functions $I^{(1)}_{\mathbf{n},\mathbf{p}}$ and $J^{(1)}_{\mathbf{n},\mathbf{p}}$ are harmonic on $\mathbb{R}^3 \backslash (K_1 \cup M_1)$. They have the symmetries*

$$I^{(1)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) = \|\mathbf{r}\| I^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{5.4}$$

$$I^{(1)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j} I^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \quad j = 1, 2, 3, \tag{5.5}$$

$$J^{(1)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) = -\|\mathbf{r}\| J^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{5.6}$$

$$J^{(1)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j} J^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \quad j = 1, 2, 3. \tag{5.7}$$

Proof. By definition (5.2), $I^{(1)}_{\mathbf{n},\mathbf{p}}$ is a composition of continuous functions provided $s_1 \neq a_1$, that is, $I^{(1)}_{\mathbf{n},\mathbf{p}}$ is continuous on $\mathbb{R}^3 \backslash (K_1 \cup M_1)$. $I^{(1)}_{\mathbf{n},\mathbf{p}}$ is also a composition of analytic functions provided $s_1 \neq a_1$ and $s_2 \neq s_3$, that is, $I^{(1)}_{\mathbf{n},\mathbf{p}}$ is analytic on $\mathbb{R}^3 \backslash (K_1 \cup M_1 \cup A_2)$. Thus it is also harmonic on $\mathbb{R}^3 \backslash (K_1 \cup M_1 \cup A_2)$. By the same argument as in the proof of Theorem 3.1, A_2 is a removable singularity of $I^{(1)}_{\mathbf{n},\mathbf{p}}$. Thus $I^{(1)}_{\mathbf{n},\mathbf{p}}$ is harmonic on $\mathbb{R}^3 \backslash (K_1 \cup M_1)$. The proof that $J^{(1)}_{\mathbf{n},\mathbf{p}}$ is harmonic on $\mathbb{R}^3 \backslash (K_1 \cup M_1)$ is analogous. The symmetry properties follow from (5.2), (5.3) and Lemma 2.1.

□

Since $P^{(1)}_{1,\mathbf{n},\mathbf{p}}, Q^{(1)}_{1,\mathbf{n},\mathbf{p}}$ form a fundamental system of solutions to (2.10) (with $\lambda_j = \lambda^{(1)}_{j,\mathbf{n},\mathbf{p}}$) on (a_0, a_1), there are (nonzero) scalars $\alpha^{(1)}_{\mathbf{n},\mathbf{p}}, \beta^{(1)}_{\mathbf{n},\mathbf{p}}$ such that

$$E^{(1)}_{1,\mathbf{n},\mathbf{p}} = \alpha^{(1)}_{\mathbf{n},\mathbf{p}} P^{(1)}_{1,\mathbf{n},\mathbf{p}} + \beta^{(1)}_{\mathbf{n},\mathbf{p}} Q^{(1)}_{1,\mathbf{n},\mathbf{p}}.$$

This leads us to the global definition of internal 5-cyclidic harmonics of the first kind

$$G^{(1)}_{\mathbf{n},\mathbf{p}} := \alpha^{(1)}_{\mathbf{n},\mathbf{p}} I^{(1)}_{\mathbf{n},\mathbf{p}} + \beta^{(1)}_{\mathbf{n},\mathbf{p}} J^{(1)}_{\mathbf{n},\mathbf{p}} \tag{5.8}$$

which is consistent with (5.1). We also note that, if $\|\mathbf{r}\| < 1$ and $\mathbf{r} \notin K_1$, then (5.2), (5.3), (5.8) imply that

$$G^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) = (\|\mathbf{r}\|^2 + 1)^{-1/2} \prod_{j=1}^{3} (\chi_j(\mathbf{r}))^{p_j} \prod_{i=1}^{3} \mathcal{E}^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i). \tag{5.9}$$

Theorem 5.1. *Let* $\mathbf{n} \in \mathbb{N}_0^2$ *and* $\mathbf{p} = (p_1, p_2, p_3) \in \{0,1\}^3$. *Then* $G^{(1)}_{\mathbf{n},\mathbf{p}}$ *extends continuously to a harmonic function on* $\mathbb{R}^3 \backslash M_1$. *Moreover,*

$$G^{(1)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j} G^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) \quad \textit{for } j = 1, 2, 3. \tag{5.10}$$

Proof. By Lemma 5.1, $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is harmonic on $\mathbb{R}^3 \backslash (K_1 \cup M_1)$. If $\|\mathbf{r}\| < 1$ we have $s_1 \neq a_0$. Therefore, the right-hand side of (5.9) is continuous on the ball $B_1(\mathbf{0})$ and harmonic on $B_1(\mathbf{0}) \backslash (A_1 \cup A_2)$. Thus it is harmonic on $B_1(\mathbf{0})$ which proves the first part of the statement of the theorem. The symmetries follow from (5.5), (5.7). □

It will be useful to introduce another solution to (2.10) by

$$F^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) := \gamma^{(1)}_{\mathbf{n},\mathbf{p}} \left(\alpha^{(1)}_{\mathbf{n},\mathbf{p}} P^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) - \beta^{(1)}_{\mathbf{n},\mathbf{p}} Q^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) \right), \quad s_1 \in (a_0, a_1). \tag{5.11}$$

We determine $\gamma^{(1)}_{\mathbf{n},\mathbf{p}}$ from the Wronskian

$$\omega(s_1) \left(E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) \frac{d}{ds_1} F^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) - F^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) \frac{d}{ds_1} E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) \right) = 1, \tag{5.12}$$

which is equivalent to

$$\gamma^{(1)}_{\mathbf{n},\mathbf{p}} = \frac{-1}{2\alpha^{(1)}_{\mathbf{n},\mathbf{p}} \beta^{(1)}_{\mathbf{n},\mathbf{p}}}.$$

We define external 5-cyclidic harmonics of the first kind by

$$H^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := \gamma^{(1)}_{\mathbf{n},\mathbf{p}} \|\mathbf{r}\|^{-1} G^{(1)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) \quad \text{for } \mathbf{r} \in \mathbb{R}^3 \backslash K_1. \tag{5.13}$$

The reason to include the factor $\gamma^{(1)}_{\mathbf{n},\mathbf{p}}$ is that we aim for a simple form of the expansion formula (6.4). In particular, we have

$$H^{(1)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) = (\|\mathbf{r}\|^2 + 1)^{-1/2} F^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3) \quad \text{for } \mathbf{r} \in R. \tag{5.14}$$

We notice an important difference between 5-cyclidic harmonics of the first and second kind (considered in Sec. 3). The external 5-cyclidic harmonics of the first kind are simply the Kelvin transformations of the internal 5-cyclidic harmonics of the first kind up to a constant factor. There is no such simple relationship between internal and external 5-cyclidic harmonics of the second kind.

Theorem 5.2. *Let* $\mathbf{n} \in \mathbb{N}_0^2$ *and* $\mathbf{p} = (p_1, p_2, p_3) \in \{0,1\}^3$. *Then* $H_{\mathbf{n},\mathbf{p}}^{(1)}$ *is harmonic on* $\mathbb{R}^3 \backslash K_1$. *The functions* $H_{\mathbf{n},\mathbf{p}}^{(1)}$ *share the symmetries* (5.10) *with* $G_{\mathbf{n},\mathbf{p}}^{(1)}$. *Moreover,*

$$H_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r}) = O(\|\mathbf{r}\|^{-1}) \quad \textit{as } \|\mathbf{r}\| \to \infty, \tag{5.15}$$

and

$$\|\nabla H_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r})\| = O(\|\mathbf{r}\|^{-2}) \quad \textit{as } \|\mathbf{r}\| \to \infty. \tag{5.16}$$

Proof. The proof of analyticity and symmetry follows directly from (5.13) and Theorem 5.1. Estimates (5.15) and (5.16) follow from the fact that the Kelvin transformation of $H_{\mathbf{n},\mathbf{p}}^{(1)}$ is analytic at the origin. □

6. Expansion of the Reciprocal Distance in 5-Cyclidic Harmonics of First Kind

For fixed $s \in (a_0, a_1)$ the coordinate surface (2.1) consists of two closed surfaces of genus 0. One lies inside the unit ball $B_1(\mathbf{0})$ and the other one is obtained from it by inversion σ_0. We consider the region D_1 interior to the coordinate surface $s = d_1$ which lies in $B_1(\mathbf{0})$:

$$D_1 := \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r}\| < 1, s_1 > d_1\}. \tag{6.1}$$

Theorem 6.1. *Let* $d_1 \in (a_0, a_1), \mathbf{n} \in \mathbb{N}_0^2, \mathbf{p} \in \{0,1\}^3$. *Then*

$$H_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r}') = \frac{1}{4\pi\omega(d_1)\{E_{1,\mathbf{n},\mathbf{p}}^{(1)}(d_1)\}^2} \int_{\partial D_1} \frac{G_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r})}{h_1(\mathbf{r})\|\mathbf{r}-\mathbf{r}'\|}\, dS(\mathbf{r}) \tag{6.2}$$

for all $\mathbf{r}' \in \mathbb{R}^3 \backslash \bar{D}_1$. *The scale factor* h_1 *is given by*

$$16\{h_1(\mathbf{r})\}^2 = \frac{(\|\mathbf{r}\|^2 - 1)^2}{(d_1 - a_0)^2} + \frac{4x^2}{(d_1 - a_1)^2} + \frac{4y^2}{(d_1 - a_2)^2} + \frac{4z^2}{(d_1 - a_3)^2}. \tag{6.3}$$

Proof. The proof is similar to the proof of Theorem 4.1. We use (5.1), (5.14) and the Wronskian (5.12). □

We obtain the expansion of the reciprocal distance in 5-cyclidic harmonics of first kind.

Theorem 6.2. *Let* $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ *with* 5*-cyclidic coordinates* s_1, s_1', *respectively. If either* (a) $\|\mathbf{r}\|, \|\mathbf{r}'\| \le 1, s_1 > s_1'$, *or* (b) $\|\mathbf{r}\| < 1 < \|\mathbf{r}'\|$, *or* (c) $\|\mathbf{r}\|, \|\mathbf{r}'\| \ge 1, s_1 < s_1'$, *then*

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = 2\pi \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^3} G_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r}) H_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r}'). \tag{6.4}$$

Proof. Suppose (a) or (b) holds. Pick d_1 such that $s_1' < d_1 < s_1$ if (a) holds, or such that $a_0 < d_1 < s_1$ if (b) holds. Then consider the domain D_1 defined in (6.1). The function $f(\mathbf{q}) := \|\mathbf{q} - \mathbf{r}'\|^{-1}$ is harmonic on an open set containing $\bar{D}_1$. Therefore, by [13, (71), (73)], we have

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^3} d_{\mathbf{n},\mathbf{p}}^{(1)} G_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{r}), \tag{6.5}$$

where

$$d_{\mathbf{n},\mathbf{p}}^{(1)} := \frac{1}{2\omega(d_1)\{E_{1,\mathbf{n},\mathbf{p}}^{(1)}(d_1)\}^2} \int_{\partial D_1} \frac{G_{\mathbf{n},\mathbf{p}}^{(1)}(\mathbf{q})}{h_1(\mathbf{q})\|\mathbf{q} - \mathbf{r}'\|}\, dS(\mathbf{q}).$$

Using Theorem 6.1, we obtain (6.4).

Now suppose (c) holds. Then the points $\sigma_0(\mathbf{r}'), \sigma_0(\mathbf{r})$ in place of $\mathbf{r}, \mathbf{r}'$ satisfy (a), so, by what we already proved,

$$\frac{1}{\|\sigma_0(\mathbf{r}) - \sigma_0(\mathbf{r}')\|} = 2\pi \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^3} G_{\mathbf{n},\mathbf{p}}^{(1)}(\sigma_0(\mathbf{r}'))H_{\mathbf{n},\mathbf{p}}^{(1)}(\sigma_0(\mathbf{r})).$$

This gives (6.4) by using (5.13) and observing that

$$\|\mathbf{r} - \mathbf{r}'\| = \|\mathbf{r}\|\|\mathbf{r}'\|\|\sigma_0(\mathbf{r}) - \sigma_0(\mathbf{r}')\|. \qquad \square$$

7. 5-Cyclidic Harmonics of the Third Kind

The 5-cyclidic harmonics of the third kind are treated analogously to the harmonics of the first kind. Therefore, we will omit all proofs in the following two sections.

In [13, Sec. IX] we introduced special solutions $w_i(s_i) = E_{i,\mathbf{n},\mathbf{p}}^{(3)}(s_i)$ to Eq. (2.10) for eigenvalues $\lambda_j = \lambda_{j,\mathbf{n},\mathbf{p}}^{(3)}, j = 1, 2$, for every $\mathbf{n} \in \mathbb{N}_0^2, \mathbf{p} = (p_0, p_1, p_2) \in \{0,1\}^3$. These functions have the form

$$E_{i,\mathbf{n},\mathbf{p}}^{(3)}(s_i) = (s_i - a_{i-1})^{p_{i-1}/2}(a_i - s_i)^{p_i/2}\mathcal{E}_{i,\mathbf{n},\mathbf{p}}^{(3)}(s_i), \quad s_i \in (a_{i-1}, a_i), \ \ i = 1, 2,$$

$$E_{3,\mathbf{n},\mathbf{p}}^{(3)}(s_3) = (s_3 - a_2)^{p_2/2}\mathcal{E}_{3,\mathbf{n},\mathbf{p}}^{(3)}(s_3), \quad s_3 \in (a_2, a_3),$$

where $\mathcal{E}_{i,\mathbf{n},\mathbf{p}}^{(3)}$ is analytic on $[a_{i-1}, a_i]$ for $i = 1, 2$ while $\mathcal{E}_{3,\mathbf{n},\mathbf{p}}^{(3)}$ is analytic on $[a_2, a_3)$. As in [13, Sec. X] we define the internal 5-cyclidic harmonic of the third kind by

$$G_{\mathbf{n},\mathbf{p}}^{(3)}(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2} E_{1,\mathbf{n},\mathbf{p}}^{(3)}(s_1)E_{2,\mathbf{n},\mathbf{p}}^{(3)}(s_2)E_{3,\mathbf{n},\mathbf{p}}^{(3)}(s_3), \quad \mathbf{r} \in R. \tag{7.1}$$

Let $P_{3,\mathbf{n},\mathbf{p}}^{(3)}(s_3)$ be the solution to (2.10) (with $\lambda_j = \lambda_{j,\mathbf{n},\mathbf{p}}^{(3)}$) on (a_2, a_3) belonging to the exponent 0 at $s = a_3$ and uniquely determined by the condition $P_{3,\mathbf{n},\mathbf{p}}^{(3)}(a_3) = 1$. We write

$$P_{3,\mathbf{n},\mathbf{p}}^{(3)}(s_3) = (s_3 - a_2)^{p_2/2}\mathcal{P}_{3,\mathbf{n},\mathbf{p}}^{(3)}(s_3), \quad s_3 \in (a_2, a_3),$$

where $\mathcal{P}^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ is analytic on $(a_2, a_3]$. Then we define

$$I^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2} \prod_{j=0}^{2} (\chi_j(\mathbf{r}))^{p_j} \prod_{i=1}^{2} \mathcal{E}^{(3)}_{i,\mathbf{n},\mathbf{p}}(s_i)\mathcal{P}^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \quad \text{if } s_3 \neq a_2. \tag{7.2}$$

The condition $s_3 \neq a_2$ is equivalent to $\mathbf{r} \in \mathbb{R}^3 \backslash (K_2 \cup M_2)$; see Fig. 2.

Similarly, let $Q^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ be the solution to (2.10) (with $\lambda_j = \lambda^{(3)}_{j,\mathbf{n},\mathbf{p}}$) on (a_2, a_3) belonging to the exponent $\frac{1}{2}$ at $s = a_3$ and uniquely determined by the condition $\lim_{s_3 \to a_3^-} \omega(s_3) \frac{d}{ds_3} Q^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) = 1$. We write

$$Q^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) = (a_3 - s_3)^{1/2}(s_3 - a_2)^{p_2/2}\mathcal{Q}^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3), \quad s_3 \in (a_2, a_3),$$

where $\mathcal{Q}^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ is analytic on $(a_2, a_3]$. Then we define

$$J^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := (\|\mathbf{r}\|^2 + 1)^{-1/2}\chi_3(\mathbf{r}) \prod_{j=0}^{2} (\chi_j(\mathbf{r}))^{p_j} \prod_{i=1}^{2} \mathcal{E}^{(3)}_{i,\mathbf{n},\mathbf{p}}(s_i)\mathcal{Q}^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \quad \text{if } s_3 \neq a_2. \tag{7.3}$$

Lemma 7.1. *The functions $I^{(3)}_{\mathbf{n},\mathbf{p}}$ and $J^{(3)}_{\mathbf{n},\mathbf{p}}$ are harmonic on $\mathbb{R}^3 \backslash (K_2 \cup M_2)$. They have the symmetries*

$$I^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) = (-1)^{p_0}\|\mathbf{r}\| I^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{7.4}$$

$$I^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j} I^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \quad j = 1, 2, \tag{7.5}$$

$$I^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_3(\mathbf{r})) = I^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{7.6}$$

$$J^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) = (-1)^{p_0}\|\mathbf{r}\| J^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{7.7}$$

$$J^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j} J^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \quad j = 1, 2, \tag{7.8}$$

$$J^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_3(\mathbf{r})) = -J^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}). \tag{7.9}$$

Since $P^{(3)}_{3,\mathbf{n},\mathbf{p}}, Q^{(3)}_{3,\mathbf{n},\mathbf{p}}$ form a fundamental system of solutions to (2.10) (with $\lambda_j = \lambda^{(3)}_{j,\mathbf{n},\mathbf{p}}$) on (a_2, a_3), there are (nonzero) scalars $\alpha^{(3)}_{\mathbf{n},\mathbf{p}}$, $\beta^{(3)}_{\mathbf{n},\mathbf{p}}$ such that

$$E^{(3)}_{3,\mathbf{n},\mathbf{p}} = \alpha^{(3)}_{\mathbf{n},\mathbf{p}} P^{(3)}_{3,\mathbf{n},\mathbf{p}} + \beta^{(3)}_{\mathbf{n},\mathbf{p}} Q^{(3)}_{3,\mathbf{n},\mathbf{p}}.$$

This leads to the global definition of internal 5-cyclidic harmonics of the third kind

$$G^{(3)}_{\mathbf{n},\mathbf{p}} := \alpha^{(3)}_{\mathbf{n},\mathbf{p}} I^{(3)}_{\mathbf{n},\mathbf{p}} + \beta^{(3)}_{\mathbf{n},\mathbf{p}} J^{(3)}_{\mathbf{n},\mathbf{p}}. \tag{7.10}$$

If $z > 0$, we can write $G^{(3)}_{\mathbf{n},\mathbf{p}}$ as follows

$$G^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) = (\|\mathbf{r}\|^2 + 1)^{-1/2} \prod_{j=0}^{2} (\chi_j(\mathbf{r}))^{p_j} \prod_{i=1}^{3} \mathcal{E}^{(3)}_{i,\mathbf{n},\mathbf{p}}(s_i). \tag{7.11}$$

Theorem 7.1. *Let* $\mathbf{n} \in \mathbb{N}_0^2$ *and* $\mathbf{p} = (p_0, p_1, p_2) \in \{0,1\}^3$. *Then* $G^{(3)}_{\mathbf{n},\mathbf{p}}$ *extends continuously to a harmonic function on* $\mathbb{R}^3 \backslash M_2$. *Moreover*

$$G^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_0(\mathbf{r})) = (-1)^{p_0}\|\mathbf{r}\| G^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \tag{7.12}$$

$$G^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_j(\mathbf{r})) = (-1)^{p_j} G^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}), \quad j = 1, 2. \tag{7.13}$$

We introduce another solution of (2.10) by

$$F^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) = \gamma^{(3)}_{\mathbf{n},\mathbf{p}} \left(\alpha^{(3)}_{\mathbf{n},\mathbf{p}} P^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) - \beta^{(3)}_{\mathbf{n},\mathbf{p}} Q^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \right), \quad s_3 \in (a_2, a_3). \tag{7.14}$$

We determine $\gamma^{(3)}_{\mathbf{n},\mathbf{p}}$ from the Wronskian

$$\omega(s_3) \left(E^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \frac{d}{ds_3} F^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) - F^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \frac{d}{ds_3} E^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \right) = 1, \tag{7.15}$$

which is equivalent to

$$\gamma^{(3)}_{\mathbf{n},\mathbf{p}} = \frac{-1}{2\alpha^{(3)}_{\mathbf{n},\mathbf{p}} \beta^{(3)}_{\mathbf{n},\mathbf{p}}}.$$

We define external 5-cyclidic harmonics of the third kind by

$$H^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) := \gamma^{(3)}_{\mathbf{n},\mathbf{p}} G^{(3)}_{\mathbf{n},\mathbf{p}}(\sigma_3(\mathbf{r})) \quad \text{for } \mathbf{r} \in \mathbb{R}^3 \backslash K_2. \tag{7.16}$$

In particular, we have

$$H^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) = (\|\mathbf{r}\|^2 + 1)^{-1/2} E^{(3)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(3)}_{2,\mathbf{n},\mathbf{p}}(s_2) F^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \quad \text{for } \mathbf{r} \in R. \tag{7.17}$$

Theorem 7.2. *Let* $\mathbf{n} \in \mathbb{N}_0^2$ *and* $\mathbf{p} = (p_0, p_1, p_2) \in \{0,1\}^3$. *Then* $H^{(3)}_{\mathbf{n},\mathbf{p}}$ *is harmonic on* $\mathbb{R}^3 \backslash K_2$. *The functions* $H^{(3)}_{\mathbf{n},\mathbf{p}}$ *share the symmetries* (7.12), (7.13) *with* $G^{(3)}_{\mathbf{n},\mathbf{p}}$. *Moreover,*

$$H^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) = O(\|\mathbf{r}\|^{-1}) \quad \text{as } \|\mathbf{r}\| \to \infty, \tag{7.18}$$

and

$$\|\nabla H^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r})\| = O(\|\mathbf{r}\|^{-2}) \quad \text{as } \|\mathbf{r}\| \to \infty. \tag{7.19}$$

8. Expansion of the Reciprocal Distance in 5-Cyclidic Harmonics of Third Kind

For fixed $s \in (a_2, a_3)$ the coordinate surface (2.1) consists of two closed surfaces of genus 0. One lies in the half-space $z > 0$ and the other one is obtained from it by reflection at the plane $z = 0$. We consider the region interior to the coordinate surface $s = d_3$ which lies in the half-space $\{\mathbf{r} : z > 0\}$:

$$D_3 := \{\mathbf{r} \in \mathbb{R}^3 : z > 0, s_3 < d_3\}. \tag{8.1}$$

Theorem 8.1. *Let* $d_3 \in (a_2, a_3), \mathbf{n} \in \mathbb{N}_0^2, \mathbf{p} \in \{0,1\}^3$. *Then*

$$H^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}') = \frac{1}{4\pi\omega(d_3)\{E^{(3)}_{3,\mathbf{n},\mathbf{p}}(d_3)\}^2} \int_{\partial D_3} \frac{G^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r})}{h_3(\mathbf{r})\|\mathbf{r}-\mathbf{r}'\|}\, dS(\mathbf{r}) \tag{8.2}$$

for all $\mathbf{r}' \in \mathbb{R}^3 \backslash \bar{D}_3$. *The scale factor* h_3 *is given by*

$$16\{h_3(\mathbf{r})\}^2 = \frac{(\|\mathbf{r}\|^2-1)^2}{(d_3-a_0)^2} + \frac{4x^2}{(d_3-a_1)^2} + \frac{4y^2}{(d_3-a_2)^2} + \frac{4z^2}{(d_3-a_3)^2}. \tag{8.3}$$

We obtain the expansion of the reciprocal distance in 5-cyclidic harmonics of the third kind.

Theorem 8.2. *Let* $\mathbf{r} = (x, y, z), \mathbf{r}' = (x', y', z') \in \mathbb{R}^3$ *with 5-cyclidic coordinates* s_3, s_3', *respectively. If either* (a) $z, z' \geq 0, s_3 < s_3'$, *or* (b) $z' < 0 < z$, *or* (c) $z, z' \leq 0, s_3' < s_3$, *then*

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = 2\pi \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^3} G^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) H^{(3)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}'). \tag{8.4}$$

9. Applications

So far there exists no software to compute the particular solutions E and F of Eq. (2.10) introduced in this paper. However, there are methods for the numerical computation of solutions to multiparameter eigenvalue problems, for example, a two-dimensional bisection method is proposed in [1]. Expansion (4.11) may be used to represent the Dirichlet Green function of the ring-cyclide D_2 defined by (4.1) as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\|\mathbf{r}-\mathbf{r}'\|} - \pi \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^4} d_{\mathbf{n},\mathbf{p}} G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}) G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}'), \tag{9.1}$$

for $\mathbf{r}, \mathbf{r}' \in D_2$ with $\mathbf{r} \neq \mathbf{r}'$, where the constants

$$d_{\mathbf{n},\mathbf{p}} = \frac{F^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)}{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)}$$

are determined such that $H^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}') = d_{\mathbf{n},\mathbf{p}} G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r}')$ on ∂D_2.

A given surface density κ on the surface ∂D_2 generates the Newtonian potential

$$\int_{\partial D_2} \frac{\kappa(\mathbf{r})}{\|\mathbf{r}-\mathbf{r}'\|}\, dS(\mathbf{r}). \tag{9.2}$$

If $\mathbf{r}'$ is outside the closure of D_2, we can use (4.11) and interchange sum and integral to expand the potential in a series of external 5-cyclidic harmonics. Such an expansion is useful when the coefficients

$$\pi \int_{\partial D_2} \kappa(\mathbf{r}) G^{(2)}_{\mathbf{n},\mathbf{p}}(\mathbf{r})\, dS(\mathbf{r}) \tag{9.3}$$

can be represented more explicitly. The coefficients (9.3) are essentially the same as those appearing in the expansion of $h_2(\mathbf{r})\kappa(\mathbf{r})$ by [13, Theorem 8.2]. Similarly, a volume distribution k on the ring-cyclide D_2 generates the Newtonian potential

$$\int_{D_2} \frac{k(\mathbf{r})}{\|\mathbf{r}-\mathbf{r}'\|}\, dV(\mathbf{r}). \tag{9.4}$$

If $\mathbf{r}'$ is outside the closure of D_2, one can use (4.11) to expand the potential in a series of external 5-cyclidic harmonics. The coefficients are

$$\pi \int_{D_2} k(\mathbf{r}) G^{(2)}_{\mathbf{n},\mathbf{p}}\, dV(\mathbf{r}). \tag{9.5}$$

It is interesting to note that for toroidal coordinates (a limiting case of 5-cyclidic coordinates when $a_0 \to a_1 = 0, a_3 \to a_2 = 1, s_1 = a_0 \sin^2\beta, s_2 = 1/\cosh^2\alpha, s_3 = \sin^2\phi + a_3\cos^2\phi$), the coefficients (9.5) can be evaluated explicitly when $k = 1$; see [21, Exercise 499]. Dixon [16] represented the potential (9.4) as a double integral for special choices of k. The authors do not know whether such special choices of k also simplify the coefficients (9.5).

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