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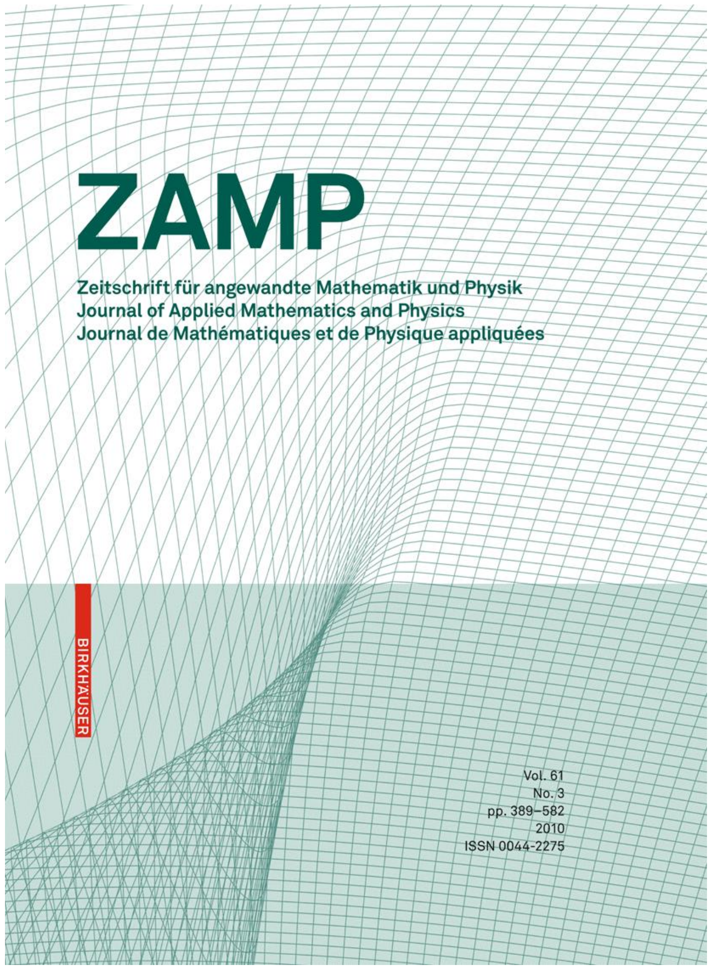

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Exact Fourier expansion in cylindrical coordinates for the three-dimensional Helmholtz Green function

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Abstract. A new method is presented for Fourier decomposition of the Helmholtz Green function in cylindrical coordinates, which is equivalent to obtaining the solution of the Helmholtz equation for a general ring source. The Fourier coefficients of the Green function are split into their half advanced + half retarded and half advanced–half retarded components, and closed form solutions for these components are then obtained in terms of a Horn function and a Kampé de Fériet function respectively. Series solutions for the Fourier coefficients are given in terms of associated Legendre functions, Bessel and Hankel functions and a hypergeometric function. These series are derived either from the closed form 2-dimensional hypergeometric solutions or from an integral representation, or from both. A simple closed form far-field solution for the general Fourier coefficient is derived from the Hankel series. Numerical calculations comparing different methods of calculating the Fourier coefficients are presented. Fourth order ordinary differential equations for the Fourier coefficients are also given and discussed briefly.

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1. Introduction and overview

The inhomogeneous Helmholtz wave equation is

$$\left(\nabla^2 + \beta^2\right) \Phi\left(\beta, \mathbf{r}\right) = \rho\left(\mathbf{r}\right) \tag{1}$$

and this has the well known free-space retarded Green function [1, p. 284]

$$G_H\left(\beta, \mathbf{r} - \mathbf{r}'\right) = -\frac{\exp\left(i\beta \mid \mathbf{r} - \mathbf{r}' \mid\right)}{4\pi \mid \mathbf{r} - \mathbf{r}' \mid} \tag{2}$$

where $\mathbf{r}$ is a field point, $\mathbf{r}'$ is a source point and β is the wave number, here considered to be a general complex number. The free-space Green function (2) is restricted to values of β such that $\mid G_H(\beta, \mathbf{r} - \mathbf{r}') \mid \to 0$ as $\mid \mathbf{r} - \mathbf{r}' \mid \to \infty$. For general dispersive waves with $\beta = \alpha + i\sigma$ where α and σ are real, then $\sigma \geqslant 0$ is a condition for this to hold. In the limit as $\beta \to 0$ these equations reduce to the Poisson equation and its corresponding Green function $G_P(\mathbf{r} - \mathbf{r}')$. The general retarded solution $\Phi(\beta, \mathbf{r})$ of the Helmholtz equation at a field point $\mathbf{r}$ for a general source density $\rho(\mathbf{r})$, subject to the boundary condition that $\Phi(\beta, \mathbf{r}) \to 0$ as $\mid \mathbf{r} \mid \to \infty$, is given in terms of the Green function as

$$\Phi\left(\beta, \mathbf{r}\right) = \iiint G_H\left(\beta, \mathbf{r} - \mathbf{r}'\right) \rho\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}' \tag{3}$$

where the volume integral is to be taken over all regions of space where the source density $\rho(\mathbf{r})$ is non zero.

Many problems of practical interest have some element of axial symmetry and are best treated in cylindrical coordinates (r, ϕ, z), the Cartesian components (x, y, z) of $\mathbf{r}$ being related to the cylindrical components by $(x, y, z) = (r\cos\phi, r\sin\phi, z)$. It follows immediately from this relation that the distance between a source point and a field point is given by

$$\mid \mathbf{r} - \mathbf{r}' \mid = \sqrt{r^2 + r'^2 + \left(z - z'\right)^2 - 2rr'\cos\left(\phi - \phi'\right)}. \tag{4}$$

The solution for $\Phi(\beta,\mathbf{r})$ when $\rho(\mathbf{r})$ is a general circular ring source is of particular interest, with applications such as circular loop antennas [2–4], the acoustics of rotating machinery [5] and acoustic and electromagnetic scattering [6]. For the simpler Poisson equation most of the analytical solutions found in the literature for cylindrical geometry are either ring source solutions or can be easily constructed from them by integration or summation. Examples are gravitating rings and disks, ring vortices and vortex disks, and circular current loops and solenoids.

The source density $\rho_c(\mathbf{r},R,z)$ for a thin circular ring of radius R located in the plane $z=Z$ is of the form

$$\rho_c\left(\mathbf{r},R,Z\right)=f\left(\phi\right)\delta\left(r-R\right)\delta\left(z-Z\right) \tag{5}$$

where $f(\phi)$ is the angular distribution of the source strength around the ring. This can be most conveniently described by a Fourier series of the form

$$f\left(\phi\right)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\left(m\phi\right)+b_m\sin\left(m\phi\right)\right)\ , \tag{6}$$

where the Fourier coefficients a_m and b_m are given by [7, p. 1066]

$$a_m=\frac{1}{2\pi}\int_0^{2\pi}f\left(\phi\right)\cos\left(m\phi\right)\mathrm{d}\phi \tag{7}$$

$$b_m=\frac{1}{2\pi}\int_0^{2\pi}f\left(\phi\right)\sin\left(m\phi\right)\mathrm{d}\phi. \tag{8}$$

From Eq. (4), the Green function (3) is even in the variable $\psi\equiv\phi'-\phi$, where ϕ is the angular coordinate of $\mathbf{r}$ and ϕ' is the angular coordinate of $\mathbf{r}'$. It is convenient to exploit this symmetry when substituting Eqs. (5) and (6) into (3). From the identity $f(\phi+\psi)\equiv f(\phi')$ we obtain

$$\begin{aligned}f\left(\phi'\right)=\frac{a_0}{2}&+\sum_{m=1}^{\infty}\left(a_m\cos\left(m\phi\right)+b_m\sin\left(m\phi\right)\right)\cos\left(m\psi\right)\\&+\sum_{m=1}^{\infty}\left(-a_m\sin\left(m\phi\right)+b_m\cos\left(m\phi\right)\right)\sin\left(m\psi\right)\end{aligned} \tag{9}$$

and on substituting Eqs. (5) and (9) into (3) and performing the volume integration, the odd terms proportional to $\sin(m\psi)$ in Eq. (9) do not contribute to the solution $\Phi(\beta,\mathbf{r})$ as $G_H(\beta,\mathbf{r}-\mathbf{r}')$ is even in ψ. The remaining integrals from the even terms can be calculated over the reduced interval from 0 to π. This gives the solution $\Phi_c(\beta,\mathbf{r},R,Z)$ of the Helmholtz equation for a circular ring source with general $f(\phi)$ in the form

$$\begin{aligned}\Phi_c\left(\beta,\mathbf{r},R,Z\right)=&-\frac{a_0}{2}G_H^0\left(\beta,r,R,z-Z\right)\\&-\sum_{m=1}^{\infty}\left(a_m\cos\left(m\phi\right)+b_m\sin\left(m\phi\right)\right)G_H^m\left(\beta,r,R,z-Z\right)\end{aligned} \tag{10}$$

where

$$G_H^m\left(\beta,r,R,z-Z\right)=\frac{1}{\pi}\int_0^{\pi}\frac{\exp\left(i\beta\sqrt{r^2+R^2+\left(z-Z\right)^2-2rR\cos\psi}\right)}{\sqrt{r^2+R^2+\left(z-Z\right)^2-2rR\cos\psi}}\cos\left(m\psi\right)\mathrm{d}\psi \tag{11}$$

and where the explicit dependence of the solution on the constant ring parameters R and Z has been introduced in these definitions. Introducing the Neumann factor ϵ_m such that $\epsilon_m = 1$ for $m = 0$ and $\epsilon_m = 2$ for $m > 0$, and defining $b_0 = 0$ allows (10) to be expressed more concisely as

$$\Phi_c(\beta, \mathbf{r}, R, Z) = -\frac{1}{2}\sum_{m=0}^{\infty}(a_m \cos(m\phi) + b_m \sin(m\phi))\,\epsilon_m G_H^m(\beta, r, R, z-Z). \quad (12)$$

Apart from a constant factor, the terms $\epsilon_m G_H^m(\beta, r, R, z-Z)$ in (12) are also the coefficients in the Fourier expansion of the Green function (2) itself, when the source point is given by $\mathbf{r}' = (R, \phi', Z)$. From Eqs. (6), (7) and (8) this is given by

$$G_H(\beta, \mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\sum_{m=0}^{\infty}\epsilon_m G_H^m(\beta, r, R, z-Z)\cos(m(\phi-\phi')). \quad (13)$$

Thus the solution $\Phi_c(\beta, \mathbf{r}, R, Z)$ of the Helmholtz equation for a general ring source can be constructed directly from the coefficients $G_H^m(\beta, r, R, Z-z)$ in the Fourier expansion of the Green function (3). This provides in large measure the motivation to analytically construct the Fourier series for the Helmholtz Green function.

For the Poisson equation with $\beta = 0$ the corresponding Fourier expansion of the Green function has already been given in closed form as [8]:

$$G_P(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi^2\sqrt{rR}}\sum_{m=0}^{\infty}\epsilon_m Q_{m-1/2}(\omega)\cos(m(\phi-\phi')) \quad (14)$$

where

$$\omega = \frac{r^2 + R^2 + (z-Z)^2}{2rR} \quad (15)$$

is a toroidal variable such that $\omega \geqslant 1$ and the $Q_{m-1/2}(\omega)$ are the Legendre functions of the second kind and half integral degree, which are also toroidal harmonics. The Fourier expansion given by Eqs. (14) and (15) can be obtained immediately by writing the Green function (2) for $\beta = 0$ in the form

$$G_P(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi\sqrt{rR}\sqrt{2\omega - 2\cos(\phi-\phi')}} \quad (16)$$

where ω is given by (15), and noting that the function $Q_{m-1/2}(\omega)$ has the simple integral representation [7, eqn 8.713]

$$Q_{m-1/2}(\omega) = \int_0^{\pi}\frac{\cos(m\psi)\,\mathrm{d}\psi}{\sqrt{2\omega - 2\cos\psi}}. \quad (17)$$

An alternative derivation of (14) employs the Lipschitz integral [7, eqn 6.611 1]

$$\int_0^{\infty} J_0(sa)\exp(-s\mid b\mid)\,\mathrm{d}s = \frac{1}{\sqrt{a^2+b^2}} \quad (18)$$

and Neumann's addition theorem [9, eqn 11.2 1]

$$J_0\left(s\sqrt{r^2 + R^2 - 2Rr\cos\psi}\right) = \sum_{m=0}^{\infty}\epsilon_m\cos(m\psi)\,J_m(sr)\,J_m(sR) \quad (19)$$

to obtain the well known eigenfunction expansion

$$G_P(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\sum_{m=0}^{\infty}\epsilon_m\cos(m(\phi-\phi'))\int_0^{\infty} J_m(sr)\,J_m(sR)\exp(-s\mid z-Z\mid)\,\mathrm{d}s. \quad (20)$$

This reduces to (14) on employing the integral [7, eqn 6.612 3], [9, eqn 13.22]:

$$\int_0^\infty J_m(sr)\,J_m(sR)\exp(-s\mid Z-z\mid)\,\mathrm{d}s = \frac{1}{\pi\sqrt{rR}}Q_{m-1/2}(\omega)\,. \tag{21}$$

The generalization of (20) for the Helmholtz case is also well known [10, p. 888]

$$G_H(\beta,\mathbf{r}-\mathbf{r}') = -\frac{i}{4\pi}\sum_{m=0}^{\infty}\epsilon_m\cos(m(\phi-\phi'))$$
$$\times\int_0^\infty \exp\left(i\mid Z-z\mid\sqrt{\beta^2-s^2}\right)J_m(sr)\,J_m(sR)\,\frac{s\mathrm{d}s}{\sqrt{\beta^2-s^2}}. \tag{22}$$

This can be similarly obtained from Neumann's theorem by employing the integral [7, eqn 6.616 2]

$$\int_1^\infty \exp(-ax)\,J_0\left(b\sqrt{x^2-1}\right)\mathrm{d}x = \frac{\exp\left(-\sqrt{a^2+b^2}\right)}{\sqrt{a^2+b^2}} \tag{23}$$

instead of the Lipschitz integral. Equation (22) gives the Fourier coefficients of the Helmholtz Green function in the form

$$G_H^m(\beta,r,R,z-Z) = i\int_0^\infty \exp\left(i\mid Z-z\mid\sqrt{\beta^2-s^2}\right)J_m(sr)\,J_m(sR)\,\frac{s\mathrm{d}s}{\sqrt{\beta^2-s^2}}. \tag{24}$$

This reduces to (20) in the limit as $\beta\to 0$ but unfortunately the integral in (24) is not given in standard tables for $\beta\neq 0$. Numerical evaluation of this integral requires care, as the integrand is oscillatory and singular in an infinite range of integration, though the integrand tends exponentially to zero as $s\to\infty$. Equation (11) is a convenient alternative numerical evaluation of the Fourier coefficients, provided m is not too large.

The integrals (11) and (24) contain the additional parameter β which is not contained in (17) and (21). As a consequence of this, the closed form generalization of (14) for the Helmholtz case involves two multidimensional Gaussian hypergeometric series, and the main purpose of this article is to present these solutions and various related results. The core idea leading to the solution is expansion of the exponential in (2) as the absolutely convergent power series [4]

$$G_H(\beta,\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(i\beta)^n\mid\mathbf{r}-\mathbf{r}'\mid^{n-1}}{n!} \tag{25}$$

where

$$\mid\mathbf{r}-\mathbf{r}'\mid^{n-1} = \left(r^2+R^2+(z-Z)^2-2rR\cos\psi\right)^{(n-1)/2}. \tag{26}$$

Hence

$$G_H^m(\beta,r,R,z-Z) = \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}I_{m,n}(r,R,z-Z) \tag{27}$$

where

$$I_{m,n}(r,R,z-Z) = \frac{1}{\pi}\int_0^\pi \left(r^2+R^2+(z-Z)^2-2rR\cos\psi\right)^{(n-1)/2}\cos(m\psi)\,\mathrm{d}\psi. \tag{28}$$

The integral (28) can be evaluated as a series by binomial expansion and this gives a double series for the Fourier coefficient $G_H^m(\beta,r,R,z-Z)$. The expansion of (28) gives an infinite number of terms for n even and a finite number of terms for n odd. These two cases are best treated separately and it is therefore convenient to split the summation over n in (25) into odd and even terms. This is equivalent to splitting the Green function (2) such that

$$G_H(\beta, \mathbf{r}-\mathbf{r}') = \Lambda_+(\beta, \mathbf{r}-\mathbf{r}') + \Lambda_-(\beta, \mathbf{r}-\mathbf{r}') \tag{29}$$

where

$$\Lambda_+(\beta, \mathbf{r}-\mathbf{r}') = -\frac{1}{8\pi}\left(\frac{\exp(i\beta \mid \mathbf{r}-\mathbf{r}' \mid)}{\mid \mathbf{r}-\mathbf{r}' \mid} + \frac{\exp(-i\beta \mid \mathbf{r}-\mathbf{r}' \mid)}{\mid \mathbf{r}-\mathbf{r}' \mid}\right) \tag{30}$$

is the half advanced + half retarded Green function and

$$\Lambda_-(\beta, \mathbf{r}-\mathbf{r}') = -\frac{1}{8\pi}\left(\frac{\exp(i\beta \mid \mathbf{r}-\mathbf{r}' \mid)}{\mid \mathbf{r}-\mathbf{r}' \mid} - \frac{\exp(-i\beta \mid \mathbf{r}-\mathbf{r}' \mid)}{\mid \mathbf{r}-\mathbf{r}' \mid}\right) \tag{31}$$

is the half advanced−half retarded Green function. The corresponding Fourier coefficients are split in the same manner such that

$$\Lambda_+^m(\beta, r, R, z-Z) = \frac{1}{2}\left(G_H^m(\beta, r, R, z-Z) + G_H^m(-\beta, r, R, z-Z)\right) \tag{32}$$

$$\Lambda_-^m(\beta, r, R, z-Z) = \frac{1}{2}\left(G_H^m(\beta, r, R, z-Z) + G_H^m(-\beta, r, R, z-Z)\right). \tag{33}$$

For real β, splitting the Green function in this way is equivalent to dividing it into its real and imaginary parts, but this is not the case for general complex β. It is shown in Sect. 2 that the Fourier coefficients in (32) and (33) are given respectively by

$$\bar{\Lambda}_+^m(rR, \gamma, k) = \frac{\Gamma(m+1/2)}{m!\sqrt{\pi r R}}\left(\frac{k}{2}\right)^{2m+1} \mathrm{H}_3\left(m+1/2, m+1/2, 2m+1, k^2, \frac{\gamma^2}{4}\right) \tag{34}$$

and

$$\bar{\Lambda}_-^m(rR, \gamma, k) = \frac{i}{\sqrt{rR}\,(2m+1)!}\left(\frac{\gamma k}{2}\right)^{2m+1} F^{0:1;0}_{1:1;0}\left[\begin{matrix} -: & m+1/2; & -; & \dfrac{\gamma^2 k^2}{4}, \dfrac{-\gamma^2}{4} \\ m+3/2: & 2m+1; & -; & \end{matrix}\right] \tag{35}$$

where

$$k = \sqrt{\frac{4rR}{(r+R)^2 + (z-Z)^2}} \tag{36}$$

$$\gamma = \beta\sqrt{(r+R)^2 + (z-Z)^2} \tag{37}$$

and

$$\bar{\Lambda}_\pm^m(rR, \gamma, k) \equiv \Lambda_\pm^m(\beta, r, R, z-Z). \tag{38}$$

The variable k is the usual modulus contained in elliptic integral solutions of elementary ring problems and is related to the toroidal variable ω by

$$\omega = \frac{2-k^2}{k^2}. \tag{39}$$

The function H_3 in Eq. (34) is one of the standard Horn functions [11, eqn 5.7.1 31] and is equivalent to the double hypergeometric series

$$\mathrm{H}_3\left(m+1/2, m+1/2, 2m+1, k^2, \frac{\gamma^2}{4}\right) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(m+1/2)_{n-p}\,(m+1/2)_n}{(2m+1)_n\, n!p!}\left(k^2\right)^n\left(\frac{\gamma^2}{4}\right)^p \tag{40}$$

where

$$(a)_p \equiv \frac{\Gamma(a+p)}{\Gamma(a)} \tag{41}$$

is the Pochhammer symbol for the rising factorial. The Kampé de Fériet function [12, p. 27] in (35) is equivalent to the double hypergeometric series

$$F^{0:1;0}_{1:1;0}\left[\begin{matrix} -: & m+1/2; & -; \\ m+3/2: & 2m+1; & -; \end{matrix} \quad \frac{\gamma^2k^2}{4}, \frac{-\gamma^2}{4}\right] = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(m+1/2)_n}{(m+3/2)_{n+p}(2m+1)_n\, n!p!}\left(\frac{\gamma^2k^2}{4}\right)^n\left(\frac{-\gamma^2}{4}\right)^p. \tag{42}$$

The integral (28) can also be evaluated using an integral representation for the associated Legendre function of the first kind, and it is shown in the Appendix that this gives the series expansion:

$$\hat{G}^m_H(rR,\lambda,\omega) = \frac{(-1)^m}{(\omega^2-1)^{1/4}\sqrt{2rR}}\sum_{n=0}^{\infty}\frac{\left(i\lambda\left(\omega^2-1\right)^{1/4}\right)^n}{n!}\frac{\Gamma((n+1)/2)}{\Gamma(m+(n+1)/2)}P^m_{(n-1)/2}\left(\frac{\omega}{\sqrt{\omega^2-1}}\right) \tag{43}$$

where

$$\lambda = \beta\sqrt{2rR} \tag{44}$$

and

$$\hat{G}^m_H(rR,\lambda,\omega) \equiv G^m_H(\beta,r,R,z-Z). \tag{45}$$

The Legendre function in Eq. (43) reduces to an associated Legendre polynomial for odd n. The series in (43) can be split into even and odd terms such that

$$\hat{G}^m_H(rR,\lambda,\omega) = \frac{1}{2}\left(\hat{\Lambda}^m_+(rR,\lambda,k) + \hat{\Lambda}^m_-(rR,\lambda,\omega)\right) \tag{46}$$

where

$$\hat{\Lambda}^m_\pm(rR,\lambda,\omega) \equiv \Lambda^m_\pm(\beta,r,R,z-Z) \tag{47}$$

and it is shown in the Appendix that the even and odd series can be expressed respectively as:

$$\hat{\Lambda}^m_+(rR,\lambda,\omega) = \frac{(-1)^m}{\sqrt{rR}}\sum_{p=0}^{\infty}\left(\frac{\lambda^2\sqrt{\omega^2-1}}{4}\right)^p\frac{Q^p_{m-1/2}(\omega)}{p!\Gamma(p-m+1/2)\Gamma(p+m+1/2)} \tag{48}$$

$$\hat{\Lambda}^m_-(rR,\lambda,\omega) = \frac{(-1)^m}{\sqrt{rR}}\sum_{p=0}^{\infty}\left(\frac{\lambda^2\sqrt{\omega^2-1}}{4}\right)^{p+m+1/2}\frac{Q^{p+m+1/2}_{m-1/2}(\omega)}{p!\Gamma(p+m+3/2)\Gamma(p+2m+1)}. \tag{49}$$

In Eq. (49) the Legendre function is purely imaginary for real λ. In the static limit as $\lambda \to 0$ then $\hat{\Lambda}^m_-(rR,0,\omega) = 0$ and from the Gamma function identity [7, eqn 8.334 2]

$$\Gamma(1/2-m)\Gamma(1/2+m) = (-1)^m\pi \quad [m \in \mathbb{N}_0] \tag{50}$$

where $\mathbb{N}_0 \equiv \{0,1,2,\ldots\}$. Equation then (48) reduces to

$$\hat{\Lambda}^m_+(rR,0,\omega) = \frac{Q_{m-1/2}(\omega)}{\pi\sqrt{rR}} \quad [m \in \mathbb{N}_0] \tag{51}$$

as it must do for consistency with (14).

The solutions in terms of two-dimensional hypergeometric functions defined by Eqs. (34)–(38) and (40)–(42) can be summed over either index to give the solutions as series of special functions. It is shown in Sect. 3 that summation over the index n in Eq. (40) gives Eq. (48), exactly as given by the integral representation. However, summation over the index n in Eq. (42) gives instead the series solution

$$\bar{\Lambda}_{-}^{m}(rR,\gamma,k) = \frac{i\sqrt{\pi}}{m!\sqrt{rR}}\left(\frac{\gamma k}{4}\right)^{2m+1} \times \sum_{p=0}^{\infty} \frac{1}{\Gamma(p+m+3/2)\,p!}\left(\frac{-\gamma^2}{4}\right)^{p} {}_1F_2\left(m+1/2; n+m+3/2, 2m+1; \frac{\gamma^2 k^2}{4}\right). \quad (52)$$

A hypergeometric identity to reduce the hypergeometric function in Eq. (52) to other well known special functions does not seem to be available in standard tabulations. It might nevertheless be conjectured that (52) could somehow be reducible to Eq. (49), but this is not in fact the case. It is easily verified numerically that although Eqs. (49) and (52) both converge rapidly to the same limit, the individual terms do not match. Hence, Eq. (52) is a distinct series from Eq. (49). It is shown in Sect. 3 that summation over the index p in Eqs. (40) and (42) gives the Bessel function series:

$$\bar{\Lambda}_{+}^{m}(rR,\gamma,k) = \frac{-1}{2\sqrt{rR}}\sum_{n=0}^{\infty}\frac{\Gamma(n+m+1/2)}{\Gamma(n+2m+1)\,n!}\left(\frac{\gamma k^2}{2}\right)^{n+m+1/2} Y_{n+m+\frac{1}{2}}(\gamma) \quad (53)$$

$$\bar{\Lambda}_{-}^{m}(rR,\gamma,k) = \frac{i}{2\sqrt{rR}}\sum_{n=0}^{\infty}\frac{\Gamma(n+m+1/2)}{\Gamma(n+2m+1)\,n!}\left(\frac{\gamma k^2}{2}\right)^{n+m+1/2} J_{n+m+\frac{1}{2}}(\gamma) \quad (54)$$

and these two series can be conveniently combined to give a series of Hankel functions of the first kind:

$$\bar{G}_{H}^{m}(rR,\gamma,k) = \frac{i}{2\sqrt{rR}}\sum_{n=0}^{\infty}\frac{\Gamma(n+m+1/2)}{\Gamma(n+2m+1)\,n!}\left(\frac{\gamma k^2}{2}\right)^{n+m+1/2} H_{n+m+\frac{1}{2}}^{(1)}(\gamma). \quad (55)$$

It is also shown in Sect. 3 that Eq. (55) leads to a very simple far field formula for the Fourier coefficient $\bar{G}_{H}^{m}(rR,\gamma,k)$:

$$\bar{G}_{H}^{m}(rR,\gamma,k) \simeq \frac{k}{2\sqrt{rR}}\exp\left(i\left[\gamma\left(1-k^2/4\right)-m\pi/2\right]\right) J_m\left(\gamma k^2/4\right) \qquad \text{for } \gamma >> 1. \quad (56)$$

From the solutions (34) and (35) it can be seen that dimensionless Fourier coefficients defined by

$$g_{\pm}^{m}\left(k^2,\gamma^2/4\right) \equiv \sqrt{rR}\,\bar{\Lambda}_{\pm}^{m}(rR,\gamma,k) \quad (57)$$

depend only on the two dimensionless variables $x \equiv k^2$ and $y \equiv \gamma^2/4$. The functions $g_{\pm}^{m}(x,y)$ are given explicitly by Eqs. (34) and (35) as:

$$g_{+}^{m}(x,y) = \frac{\Gamma(m+1/2)}{2^{2m+1}m!\sqrt{\pi}}x^{\alpha}\mathrm{H}_3(\alpha,\alpha,2\alpha,x,y) \quad (58)$$

and

$$g_{-}^{m}(x,y) = \frac{i}{(2m+1)!}(xy)^{\alpha} F_{1:1;0}^{0:1;0}\left[\begin{matrix} -: & \alpha; & -; & \\ \alpha+1: & 2\alpha; & -; & \end{matrix} xy, -y\right] \quad (59)$$

where

$$\alpha = m + 1/2. \quad (60)$$

Two-dimensional hypergeometric series such as those occurring in (58) and (59) are associated with pairs of partial differential equations [11, section 5.9] and these can be used to construct ordinary differential equations for $g_{\pm}^{m}(x,y)$ with y fixed and x as the independent variable. The equations associated with the confluent Horn function $z(\alpha,\beta,\delta,x,y) \equiv \mathrm{H}_3(\alpha,\beta,\delta,x,y)$ are, after correcting typographical errors in [11]:

$$x(1-x)r + xys + [\delta - (\alpha+\beta+1)x]p + \beta yq - \alpha\beta z = 0 \tag{61}$$
$$yt - xs + (1-\alpha)q + z = 0 \tag{62}$$

where

$$p = \frac{\partial z}{\partial x}; \quad q = \frac{\partial z}{\partial y}; \quad r = \frac{\partial^2 z}{\partial x^2}; \quad s = \frac{\partial^2 z}{\partial x \partial y}; \quad t = \frac{\partial^2 z}{\partial y^2}.$$

The corresponding pair of equations associated with the Kampé de Fé riet function

$$\bar{z}(\alpha,u,v) \equiv F^{0:1;0}_{1:1;0}\left[\begin{matrix} - : & \alpha; & -; \\ \alpha+1 : & 2\alpha; & -; \end{matrix} u, v\right]$$

can be shown by the methods in [11] to be:

$$\bar{z} = v\bar{t} + u\bar{s} + (\alpha+1)\bar{q} \tag{63}$$
$$\bar{z} = 2\alpha\bar{p} + u\bar{r} + \bar{q} + v\bar{t} \tag{64}$$

where

$$\bar{p} = \frac{\partial \bar{z}}{\partial u}; \quad q = \frac{\partial \bar{z}}{\partial v}; \quad \bar{r} = \frac{\partial^2 \bar{z}}{\partial u^2}; \quad \bar{s} = \frac{\partial^2 \bar{z}}{\partial u \partial v}; \quad \bar{t} = \frac{\partial^2 \bar{z}}{\partial v^2}.$$

Using Eqs. (61)–(64) It can be shown [13], though the details are very intricate, that for constant y the coefficients $g^m_\pm(x,y)$ both satisfy the same fourth order ordinary differential equation in x:

$$(1-x)x^4\frac{\mathrm{d}^4 g^m_\pm}{\mathrm{d}x^4} + (6-9x)x^3\frac{\mathrm{d}^3 g^m_\pm}{\mathrm{d}x^3} + (\alpha(1-\alpha) + 6 - 18x - xy + 2y)x^2\frac{\mathrm{d}^2 g^m_\pm}{\mathrm{d}x^2}$$
$$+ (2\alpha(1-\alpha) - 2x(3-y))x\frac{\mathrm{d}g^n_\pm}{\mathrm{d}x} + \left(y^2 + 2\alpha(1-\alpha)y - 3\alpha(\alpha+1)(\alpha+2)\right)g^m_\pm = 0. \tag{65}$$

The integral representation (11) can be expressed as

$$\hat{G}^m_H(rR,\lambda,\omega) = \frac{\hat{y}_m(\lambda,\omega)}{\pi\sqrt{2rR}} \tag{66}$$

where

$$\hat{y}_m(\lambda,\omega) = \int_0^\pi \frac{\exp\left(i\lambda\sqrt{\omega - \cos\psi}\right)}{\sqrt{\omega-\cos\psi}}\cos(m\psi) \tag{67}$$

and (67) can be used to derive an alternative integral representation [13]

$$\hat{y}_m(\lambda,\omega) = \sqrt{\pi}(-i)^{m+1/2}\int_0^\infty \exp\left[i\left(\omega s + \frac{\lambda^2}{4s}\right)\right]J_m(s)s^{-1/2}\mathrm{d}s. \tag{68}$$

Repeated differentiation of (68), employing Bessel's equation of order m and integration by parts [13] gives a fourth order ordinary differential equation for $\hat{y}_m(\lambda,\omega)$ in terms of the toroidal variable ω:

$$(1-\omega^2)\frac{\mathrm{d}^4\hat{y}_m}{\mathrm{d}\omega^4} - 6\omega\frac{\mathrm{d}^3\hat{y}_m}{\mathrm{d}\omega^3} + \left(m^2 - \frac{\lambda^2\omega}{2} - \frac{25}{4}\right)\frac{\mathrm{d}^2\hat{y}_m}{\mathrm{d}\omega^2} - \lambda^2\frac{\mathrm{d}\hat{y}_m}{\mathrm{d}\omega} - \left(\frac{\lambda^2}{4}\right)^2\hat{y}_m = 0 \tag{69}$$

and another fourth order differential equation in terms of the wave number parameter λ can be similarly derived. In the static limit as $\lambda \to 0$, Eq. (69) reduces to:

$$\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left[\left(1-\omega^2\right)\frac{\mathrm{d}^2\hat{y}_m}{\mathrm{d}\omega^2} - 2\omega\frac{\mathrm{d}\hat{y}_m}{\mathrm{d}\omega} + \left(m^2 - \frac{1}{4}\right)\hat{y}_m\right] = 0 \tag{70}$$

Table 1. *Special functions used*

Symbol	Special function
$(a)_n$	Pochhammer symbol
$B\left(x,y\right)$	Beta function
${}_2F_1\left(a,b;c;x\right)$	Gauss hypergeometric function
${}_1F_2(a;b,c;x)$	A hypergeometric function
${}_1F_1\left(a;b;x\right)$	Confluent hypergeometric function
$F^{0:1;0}_{1:1;0}\left[\begin{matrix} - : a; -; \\ b : c; -; \end{matrix} x,y\right]$	A Kampé de Fériet function
$H^{(1)}_\nu\left(x\right)$	Hankel function of the first kind
$\mathrm{H}_3\left(a,b,c,x,y\right)$	The H_3 confluent Horn function
$J_\nu(x)$	Bessel function of the first kind
$P^\mu_\nu(x)$	Associated Legendre function of the first kind
$Q^\mu_\nu(x)$	Associated Legendre function of the second kind
$Y_\nu\left(x\right)$	Bessel function of the second kind
$\Gamma(x)$	Gamma function
$\delta\left(x\right)$	Dirac delta function

where

$$\left(1-\omega^2\right)\frac{\mathrm{d}^2\hat{y}_m}{\mathrm{d}\omega^2}-2\omega\frac{\mathrm{d}\hat{y}_m}{\mathrm{d}\omega}+\left(m^2-\frac{1}{4}\right)\hat{y}_m=0 \tag{71}$$

is Legendre's equation of degree $m-1/2$ [7, eqn 8.820]. Equation (69) must necessarily reduce to Eq. (71) for consistency with Eq. (14). It can be shown that the differential equations (65) and (69) obtained by quite different routes are equivalent, though the details are rather intricate [13].

Recurrence relations for the Fourier coefficients for the Helmholtz equation were investigated by Matviyenko [6], but the closed form solutions and differential equations presented here appear to be new. Werner [3] presented an expansion of the Fourier coefficient as a series of spherical Hankel functions, superficially similar to Eq. (55), but the two expansions are distinct. The two-dimensional hypergeometric series approach applied here to obtain the Fourier expansion for the Helmholtz Green function has recently been applied to obtain the Fourier expansion in terms of the amplitude ϕ for the Legendre incomplete elliptic integral of the third kind [14].

The numerical performance of the various expressions for the Fourier coefficients, including the far field formula (56), was investigated using Mathematica® [15] and this is presented in Sect. 4. The special functions used in the analyses presented here are given in Table 1.

2. Solution in terms of two-dimensional hypergeometric series

The power series expansion (27) for the Fourier coefficient $G^m_H\left(\beta,r,R,z-Z\right)$ can be expressed in the form

$$\bar{G}^m_H\left(rR,\gamma,k\right)=\frac{\left(-1\right)^m k}{\pi\sqrt{rR}}\sum_{n=0}^{\infty}\frac{\left(i\gamma\right)^n}{n!}\int\limits_0^{\pi/2}\cos\left(2m\theta\right)\left(1-k^2\sin^2\theta\right)^{(n-1)/2}\mathrm{d}\theta \tag{72}$$

where

$$G^m_H\left(\beta,r,R,z-Z\right)=\bar{G}^m_H\left(rR,\gamma\left(\beta,r,R,z-Z\right),k\left(r,R,z-Z\right)\right) \tag{73}$$

and k and γ are defined by (36) and (37). The term $\left(1-k^2\sin^2\theta\right)^{(n-1)/2}$ in (72) can be expanded binomially to give $\bar{G}_H^m\left(rR,\gamma,k\right)$ as a double series containing integrals of the form

$$\bar{I}_{m,p}=\int\limits_0^{\pi/2}\sin^{2p}\theta\cos\left(2m\theta\right)\mathrm{d}\theta. \tag{74}$$

This integral is given by Gradshteyn and Ryzhik [7, eqns 3.631 8,12] in a form which can be recast as

$$I_{m,p}=\frac{(-1)^m\pi}{2^{2p+1}\left(2p+1\right)}\frac{1}{B\left(p+m+1,p-m+1\right)}\quad\text{for } p\geqslant m \tag{75}$$

$$I_{m,p}=0\quad\text{for } p<m \tag{76}$$

and expressing the Beta function in (75) in terms of Gamma functions and employing the duplication theorem [7, eqn 8.335 1]

$$\Gamma\left(2x\right)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma\left(x\right)\Gamma\left(x+1/2\right) \tag{77}$$

gives after some reduction the alternative form

$$I_{m,p}=\frac{(-1)^m\sqrt{\pi}}{2}\frac{\Gamma\left(p+1/2\right)p!}{\Gamma\left(p+m+1\right)\Gamma\left(p-m+1\right)}\quad\text{for } p\geqslant m. \tag{78}$$

The binomial expansion of (72) gives an infinite series for n zero or even and a finite sum for n odd, and these two cases must be treated separately. It is therefore convenient to split the series for $\bar{G}_H^m$ such that

$$\bar{G}_H^m\left(rR,\gamma,k\right)=\bar{\Lambda}_+^m\left(rR,\gamma,k\right)+\bar{\Lambda}_-^m\left(rR,\gamma,k\right) \tag{79}$$

and on employing Eq. (77) the divided series are given by

$$\bar{\Lambda}_+^m\left(rR,\gamma,k\right)=\frac{(-1)^m k}{\sqrt{\pi rR}}\sum_{n=0}^{\infty}\frac{\left(-\gamma^2/4\right)^n}{\Gamma\left(n+1/2\right)n!}\int\limits_0^{\pi/2}\cos\left(2m\theta\right)\left(1-k^2\sin^2\theta\right)^{n-1/2}\mathrm{d}\theta \tag{80}$$

$$\bar{\Lambda}_-^m\left(rR,\gamma,k\right)=\frac{i(-1)^m k\gamma}{2\sqrt{\pi rR}}\sum_{n=m}^{\infty}\frac{\left(-\gamma^2/4\right)^n}{\Gamma\left(n+3/2\right)n!}\int\limits_0^{\pi/2}\cos\left(2m\theta\right)\left(1-k^2\sin^2\theta\right)^{n}\mathrm{d}\theta. \tag{81}$$

Binomial expansion of the integrals in Eqs. (80) and (81) gives, respectively

$$\bar{\Lambda}_+^m\left(rR,\gamma,k\right)=\frac{(-1)^m k}{\sqrt{\pi rR}}\sum_{n=0}^{\infty}\sum_{s=m}^{\infty}\frac{\left(-\gamma^2/4\right)^n}{\Gamma\left(n+1/2\right)n!}\frac{\Gamma\left(s-n+1/2\right)}{\Gamma\left(1/2-n\right)}\frac{\left(k^2\right)^s}{s!}\bar{I}_{m,s} \tag{82}$$

$$\bar{\Lambda}_-^m\left(rR,\gamma,k\right)=\frac{i(-1)^m k\gamma}{2\sqrt{\pi rR}}\sum_{n=0}^{\infty}\sum_{s=m}^{n}\frac{\left(-\gamma^2/4\right)^n}{\Gamma\left(n+3/2\right)}\frac{\left(-k^2\right)^s}{s!\left(n-s\right)!}I_{m,s} \tag{83}$$

and employing the explicit formula (78) for $I_{m,s}$ in (82) and (83) gives respectively

$$\bar{\Lambda}_+^m\left(rR,\gamma,k\right)=\frac{k}{2\pi\sqrt{rR}}\sum_{n=0}^{\infty}\sum_{s=m}^{\infty}\frac{\Gamma\left(s-n+1/2\right)\Gamma\left(s+1/2\right)}{\Gamma\left(s+m+1\right)\Gamma\left(s-m+1\right)}\frac{\left(\gamma^2/4\right)^n\left(k^2\right)^s}{n!} \tag{84}$$

$$\bar{\Lambda}_-^m\left(rR,\gamma,k\right)=\frac{ik\gamma}{4\sqrt{rR}}\sum_{n=m}^{\infty}\sum_{s=m}^{n}\frac{\Gamma\left(s+1/2\right)\left(-\gamma^2/4\right)^n\left(-k\right)^s}{\left(n-s\right)!\Gamma\left(n+3/2\right)\Gamma\left(s+m+1\right)\Gamma\left(s-m+1\right)} \tag{85}$$

where the Gamma identity (50) has been used to simplify equation (84).

2.1. The series for $\bar{\Lambda}_{+}^{m}(rR,\gamma,k)$

The substitution $s=p+m$ in Eq. (84) yields after some reduction the double series:

$$\bar{\Lambda}_{+}^{m}(rR,\gamma,k)=\frac{\Gamma(m+1/2)}{m!\sqrt{\pi rR}}\left(\frac{k}{2}\right)^{2m+1}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(m+1/2)_{p-n}(m+1/2)_{p}}{(2m+1)_{p}}\frac{\left(k^2\right)^{p}\left(\gamma^2/4\right)^{n}}{n!p!} \quad (86)$$

The double hypergeometric function in (86) can be identified as one of the confluent Horn functions [11, eqn 5.7.1 31] and hence

$$\bar{\Lambda}_{+}^{m}(rR,\gamma,k)=\frac{\Gamma(m+1/2)}{m!\sqrt{\pi rR}}\left(\frac{k}{2}\right)^{2m+1}\mathrm{H}_3\left(m+1/2,m+1/2,2m+1,k^2,\frac{\gamma^2}{4}\right). \quad (87)$$

The convergence condition given in [11, eqn 5.7.1 31] for the double series in Eq. (86) is $k^2<1$, which always holds. The order of summation in Eq. (86) can be reversed, but the order of the arguments k^2 and $\gamma^2/4$ in Eq. (87) cannot be exchanged.

2.2. The series for $\bar{\Lambda}_{-}^{m}(rR,\gamma,k)$

Equation (85) can be converted to a doubly infinite series by reversing the order of summation, which gives

$$\bar{\Lambda}_{-}^{m}(rR,\gamma,k)=\frac{ik\gamma}{4\sqrt{rR}}\sum_{s=m}^{\infty}\sum_{n=s}^{\infty}\frac{\Gamma(s+1/2)\left(-\gamma^2/4\right)^{n}\left(-k^2\right)^{s}}{(n-s)!\Gamma(n+3/2)\Gamma(s+m+1)\Gamma(s-m+1)}. \quad (88)$$

The substitution $n=s+p$ in (88) gives

$$\bar{\Lambda}_{-}^{m}(rR,\gamma,k)=\frac{ik\gamma}{4\sqrt{rR}}\sum_{s=m}^{\infty}\sum_{p=0}^{\infty}\frac{\Gamma(s+1/2)}{p!\Gamma(p+s+3/2)\Gamma(s+m+1)\Gamma(s-m+1)}\left(\frac{\gamma^2k^2}{4}\right)^{s}\left(-\frac{\gamma^2}{4}\right)^{p} \quad (89)$$

and the further substitution $s=n+m$ in (89) gives

$$\bar{\Lambda}_{-}^{m}(rR,\gamma,k)=\frac{i}{2\sqrt{rR}}\left(\frac{\gamma^2k^2}{4}\right)^{m+1/2}\times\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\Gamma(n+m+1/2)}{\Gamma(p+n+m+3/2)\Gamma(n+2m+1)}\frac{1}{n!p!}\left(\frac{\gamma^2k^2}{4}\right)^{n}\left(-\frac{\gamma^2}{4}\right)^{p} \quad (90)$$

Expressing Eq. (90) in terms of Pochhammer symbols gives after some reduction the double hypergeometric series

$$\bar{\Lambda}_{-}^{m}(rR,\gamma,k)=\frac{i}{\sqrt{rR}(2m+1)!}\left(\frac{\gamma k}{2}\right)^{2m+1}\times\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(m+1/2)_{n}}{(m+3/2)_{n+p}(2m+1)_{n}}\frac{1}{n!p!}\left(\frac{\gamma^2k^2}{4}\right)^{n}\left(\frac{-\gamma^2}{4}\right)^{p} \quad (91)$$

and this can be expressed as a Kampé de Fériet function as defined by Srivastava and Karlsson [12, p. 27]:

$$F_{l:m;n}^{p:q;r}\left[\begin{matrix}\{a_p\}:\{b_q\};\{c_r\};\\ \{\alpha_l\}:\{\beta_m\};\{\gamma_n\};\end{matrix}x,y\right]=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{\prod_{u=1}^{p}(a_u)_{j+i}\prod_{u=1}^{q}(b_u)_{j}\prod_{u=1}^{r}(c_u)_{i}}{\prod_{u=1}^{l}(\alpha_u)_{j+i}\prod_{u=1}^{m}(\beta_u)_{j}\prod_{u=1}^{n}(\gamma_u)_{i}}\frac{x^j}{j!}\frac{y^i}{i!}. \quad (92)$$

In the definition (92), $\{a_p\} \equiv a_1, ..a_p$ and $\{b_q\}$ and so on, are the lists of the arguments of the Pochhammer symbols of the various types which appear in the products on the right-hand side of the equation. If a list has no members, it is represented by a hyphen. Comparing (91) with (92) gives

$$\bar{\Lambda}_-^m(rR,\gamma,k) = \frac{i}{\sqrt{rR}\,(2m+1)!}\left(\frac{\gamma^2k^2}{4}\right)^{m+1/2} F^{0:1;0}_{1:1;0}\left[\begin{matrix} -: & m+1/2; & -; \\ m+3/2: & 2m+1; & -; \end{matrix}\ \frac{\gamma^2k^2}{4}, \frac{-\gamma^2}{4}\right]. \tag{93}$$

3. Consequences of the hypergeometric formulas

In the static limit as $\beta \to 0$ then Eq. (34) reduces to

$$\bar{\Lambda}_+^m(rR,0,k) = \frac{\Gamma(m+1/2)}{m!\sqrt{\pi rR}}\left(\frac{k}{2}\right)^{2m+1} {}_2F_1\left(m+1/2, m+1/2; 2m+1; k^2\right) \tag{94}$$

and from (37) and the quadratic hypergeometric transformation [16, eqn 7.3.1 71]:

$${}_2F_1(a,b;2b;z) = \frac{2^{2b}}{\sqrt{\pi}}\frac{\Gamma(b+1/2)}{\Gamma(2b-a)}z^{-b}(1-z)^{(b-a)/2}\exp(i\pi(a-b))\,Q^{b-a}_{b-1}\left(\frac{2}{z}-1\right) \tag{95}$$

this reduces to

$$\bar{\Lambda}_+^m(rR,0,k) = \frac{1}{\pi\sqrt{rR}}Q_{m-1/2}\left(\frac{2-k^2}{k^2}\right) \tag{96}$$

in agreement with Eqs. (14) and (37).

3.1. Fourier coefficients as series of special functions

The double series given by Eq. (34) can be summed with respect to either the index n or the index p in the definition (40). Summing with respect to n in (40) gives a series of Bessel functions of the second kind:

$$\bar{\Lambda}_+^m(rR,\gamma,k) = \frac{-1}{2\sqrt{rR}}\left(\frac{k^2\gamma}{2}\right)^{m+1/2}\sum_{p=0}^{\infty}\frac{\Gamma(m+1/2+p)}{\Gamma(2m+1+p)\,p!}\left(\frac{k^2\gamma}{2}\right)^p Y_{m+\frac{1}{2}+p}(\gamma) \tag{97}$$

where the Gamma function identity (50) and the Bessel function identity

$$Y_\nu(\gamma) = \frac{1}{\sin\nu\pi}\left[\cos(\nu\pi) - J_{-\nu}(\gamma)\right] \tag{98}$$

have been employed to obtain Eq. (97). Summing instead over the index p in (40) gives the alternative series

$$\bar{\Lambda}_+^m(rR,\gamma,k) = \frac{1}{m!\sqrt{\pi rR}}\left(\frac{k}{2}\right)^{2m+1} \times\sum_{n=0}^{\infty}\Gamma(m-n+1/2)\frac{\left(\gamma^2/4\right)^n}{n!}{}_2F_1\left(m+1/2-n, m+1/2; 2m+1; k^2\right). \tag{99}$$

This can be reduced using (95) and (50) to give:

$$\bar{\Lambda}_+^m(rR,\gamma,k) = \frac{(-1)^m}{\sqrt{rR}}\sum_{n=0}^{\infty}\left(\frac{\gamma^2\sqrt{1-k^2}}{4}\right)^n\frac{Q^n_{m-1/2}\left(\left(2-k^2\right)/k^2\right)}{n!\Gamma(n-m+1/2)\Gamma(n+m+1/2)}. \tag{100}$$

The dimensionless variables γ and λ defined by Eqs. (37) and (44) respectively are related by

$$\gamma = \frac{\sqrt{2}\lambda}{k} \tag{101}$$

and substituting this equation and Eq. (39) in Eq. (100) gives immediately Eq. (48).

Summing with respect to p in Eq. (90) gives the Bessel series

$$\bar{\Lambda}_{-}^{m}(rR, \gamma, k) = \frac{i}{2\sqrt{rR}} \sum_{p=0}^{\infty} \frac{\Gamma(p+m+1/2)}{\Gamma(p+2m+1)\,p!} \left(\frac{\gamma k^2}{2}\right)^{p+m+1/2} J_{m+\frac{1}{2}+p}(\gamma) \tag{102}$$

The corresponding summation over the index n gives

$$\begin{aligned} \bar{\Lambda}_{-}^{m}(rR, \gamma, k) = {}& \frac{i\sqrt{\pi}}{m!\sqrt{rR}} \left(\frac{\gamma k}{4}\right)^{2m+1} \\ & \times \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+m+3/2)\,n!} \left(\frac{-\gamma^2}{4}\right)^{n} {}_1F_2\left(m+1/2; n+m+3/2, 2m+1; \frac{\gamma^2 k^2}{4}\right). \end{aligned} \tag{103}$$

There seems to be no hypergeometric transformation listed in standard tables suitable for directly reducing the hypergeometric function in this equation.

Equations (97) and (102) can be conveniently combined to give a series of Hankel functions of the first kind:

$$\bar{G}_{H}^{m}(rR, \gamma, k) = \frac{i}{2\sqrt{rR}} \sum_{p=0}^{\infty} \frac{\Gamma(p+m+1/2)}{\Gamma(p+2m+1)\,p!} \left(\frac{\gamma k^2}{2}\right)^{p+m+1/2} H_{m+p+\frac{1}{2}}^{(1)}(\gamma), \tag{104}$$

where

$$H_{m+p+\frac{1}{2}}^{(1)}(\gamma) \equiv J_{m+p+\frac{1}{2}}(\gamma) + iY_{m+p+\frac{1}{2}}(\gamma). \tag{105}$$

The Hankel functions in (104) are of half integral order and can therefore be expressed as finite sums of elementary functions [7, 8.466 1]:

$$H_{m+p+\frac{1}{2}}^{(1)}(\gamma) = \sqrt{\frac{2}{\pi\gamma}} \frac{\exp(i\gamma)}{(i)^{m+p+1}} \sum_{j=0}^{m+p} \frac{(m+p+j)!}{(-2i\gamma)^{j}\, j!\,(m+p-j)!}. \tag{106}$$

It is also to be noted that the asymptotic series [7, 8.451 3] for the function $H_{m+p+1/2}^{(1)}(\gamma)$ terminates and reduces to (106).

3.2. Approximate far field formula

In the far field where $\gamma \equiv \beta\sqrt{(r+R)^2 + (z-Z)^2} >> 1$ then only the first term in the sum in (106) need be retained and Eq. (104) then reduces to

$$\bar{G}_{H}^{m}(rR, \gamma, k) = \frac{k\exp(i\gamma)}{2\sqrt{\pi rR}} \left(\frac{\gamma k^2}{2i}\right)^{m} \sum_{p=0}^{\infty} \frac{\Gamma(p+m+1/2)}{\Gamma(p+2m+1)\,p!} \left(\frac{\gamma k^2}{2i}\right)^{p}. \tag{107}$$

The summation with respect to p in Eq. (107) can be expressed in terms of a confluent hypergeometric function:

$$\sum_{p=0}^{\infty} \frac{\Gamma(p+m+1/2)}{\Gamma(p+2m+1)\,p!} \left(\frac{\gamma k^2}{2i}\right)^{p} = \frac{\Gamma(m+1/2)}{(2m)!} {}_1F_1\left(m+1/2; 2m+1; \frac{\gamma k^2}{2i}\right) \tag{108}$$

and employing the identity

$$(2m)! = \frac{2^{2m}\Gamma\left(m+1/2\right)m!}{\sqrt{\pi}} \tag{109}$$

this can be alternatively expressed as

$$\sum_{p=0}^{\infty}\frac{\Gamma\left(p+m+1/2\right)}{\Gamma\left(p+2m+1\right)p!}\left(\frac{\gamma k^2}{2i}\right)^p = \frac{\sqrt{\pi}}{2^{2m}m!}{}_1F_1\left(m+1/2;2m+1;\frac{\gamma k^2}{2i}\right). \tag{110}$$

Substituting (110) into (107) then gives

$$\bar{G}_H^m\left(rR,\gamma,k\right) = \frac{k\exp\left(i\gamma\right)}{2\sqrt{rR}m!}\left(\frac{\gamma k^2}{8i}\right)^m {}_1F_1\left(m+1/2;2m+1;\frac{\gamma k^2}{2i}\right). \tag{111}$$

The confluent hypergeometric function in (111) can be expressed in terms of a Bessel function of the first kind using the standard hypergeometric identity [7, 9.238 1]:

$${}_1F_1\left(\nu+1/2;2\nu+1;2ix\right) = \frac{2^{\nu}\Gamma\left(\nu+1\right)\exp\left(ix\right)}{x^{\nu}}J_{\nu}\left(x\right) \tag{112}$$

and employing (112) in (111) gives after some reduction the simple far field formula:

$$\bar{G}_H^m\left(rR,\gamma,k\right) \simeq \frac{k}{2\sqrt{rR}}\exp\left(i\left[\gamma\left(1-k^2/4\right)-m\pi/2\right]\right)J_m\left(\gamma k^2/4\right). \tag{113}$$

The accuracy of Eq. (113) is investigated numerically in the following section.

4. Numerical results

The series solutions for the Fourier coefficients given by Eq. (55) and Eqs. (48)–(49) were evaluated using Mathematica and the numerical performance was explored for various geometric parameters and wave numbers. For comparison, the two integrals (11) and (22) for the Fourier coefficients were also evaluated numerically for the same parameters. All four methods give identical results at locations which are neither too far away nor too close to the ring source. The numerical integration (11) performs very well at all distances from the ring, whereas the numerical integration (22) fails when either very close to the ring or too far away. No cases were identified where Eq. (22) was superior. The Hankel function series (55) requires fewer and fewer terms for convergence as the distance from the loop increases, and conversely performance decreases as the ring is approached. The associated Legendre function series (48) and (49) have precisely the opposite performance, with great accuracy close to the ring and failure at large distances from the ring. The two series (48) and (49) are well suited to calculations close to the ring as the associated Legendre functions themselves each contain the ring singularity as $\omega \to 1$. By contrast, the Hankel functions (55) are not singular at the ring and hence an increasing number of terms are required to model the singularity as the ring is approached. In all cases, there is are always at least one numerical integration and one series solution which can be used to cross check each other.

Sample numerical results are given in Table 2 for moderate distances from the ring source, and shows the number of terms required by each series to match the numerical integrations exactly. Table 3 shows the performance of the Hankel series with increasing distance from the ring. The number of Hankel terms decreases to very few at large distances from the ring. Table 4 shows the performance of the two associated Legendre series (48) and (49) as the ring is approached. The real part of $G_H^m\left(\beta,r,R,z-Z\right)$ diverges logarithmically as $z \to Z$, whereas the imaginary part tends to a finite limit. It can be seen immediately from Table 4 that only 8 terms in each Legendre series is sufficient to calculate $G_H^m\left(\beta,r,R,z-Z\right)$ for the range $\{0 < z \leqslant 1\}$.

Table 2. *Numerical solution for the Fourier coefficient* $G_H^m(\beta, r, R, z-Z)$*,* $R = 1$*,* $Z = 0$ *and the other parameters are given*

m	β	r	z	N_1	N_2	$G_H^m(\beta, r, R, z-Z)$
0	2	1/2	1/2	94	10	$-0.4332208795 + 0.6063507453\ i$
0	2	1/2	3/2	2 4	12	$-0.4324244083 - 0.2593676946\ i$
0	2	1/2	5	9	24	$-0.1320141290 - 0.1413052175\ i$
0	2	1/2	10	8	37	$0.02892833221 + 0.09482283693\ i$
0	2	1/2	20	6	63	$-0.03552779935 + 0.03502697525\ i$
3	5	3/2	1/2	224	24	$-0.2152817201 - 0.2085849956\ i$
3	5	3/2	1	97	25	$0.1382226177 - 0.2010980843\ i$
3	5	3/2	5	17	47	$-0.009794158906 - 0.000546039281\ i$
3	5	3/2	10	13	80	$-0.0004846328044 + 0.0006340532520\ i$
3	5	3/2	20	10	147	$0.00000356468968 + 0.00005347913049\ i$

N_1 is the number of terms in the Hankel function series (55) needed to give the accuracy given. N_2 is the number of terms in the associated Legendre series (48) and (49) to provide the accuracy given. Of the two associated Legendre series, (48) requires more terms than (49) for the stated accuracy

Table 3. *Series solution for the Fourier coefficient* $G_H^m(\beta, r, R, z-Z)$*,* $m = 1$*,* $\beta = 6$*,* $R = 1$*,* $Z = 0$*,* $r = 3/2$ *and* z *are given*

z	N_1	$G_H^m(\beta, r, R, z-Z)$
1/2	227	$0.0785417676 - 0.2281496125\ i$
1	99	$0.1318397799 + 0.0959755332\ i$
5	20	$0.04717552085 - 0.09819984770\ i$
50	7	$-0.001762722093 - 0.000313668126\ i$
100	6	$-0.0000246835014 + 0.0004487208692\ i$
200	5	$-4.36384289 \times 10^{-6} - 0.00011237773355\ i$
1000	4	$-1.884281670 \times 10^{-6} - 4.086433833 \times 10^{-6}\ i$
5000	3	$-1.446923432 \times 10^{-7} + 1.070704963 \times 10^{-7}\ i$
10000	2	$4.307310056 \times 10^{-8} + 1.302718185 \times 10^{-8}\ i$
10^7	1	$-2.303180610 \times 10^{-14} + 3.865922798 \times 10^{-14}\ i$

N_1 is the number of terms in the Hankel function series (55) needed for the accuracy given

Table 4. *Series solution for the Fourier coefficient* $G_H^m(\beta, r, R, z-Z)$*,* $m = 2$*,* $\beta = 1$*,* $R = 1$*,* $Z = 0$ *and* $r = 1$

z	N_2	$G_H^m(\beta, r, R, z-Z)$
1	8	$0.1874175169 + 0.1222388714\ i$
10^{-1}	8	$0.8955546890 + 0.1360159497\ i$
10^{-2}	8	$1.628566013 + 0.136158894\ i$
10^{-3}	7	$2.361506874 + 0.136160324\ i$
10^{-4}	7	$3.094442571 + 0.136160339\ i$
10^{-5}	7	$3.827378171 + 0.136160339\ i$
10^{-6}	7	$4.560313770 + 0.136160339\ i$
10^{-7}	7	$5.293249369 + 0.136160339\ i$
10^{-8}	7	$6.026184968 + 0.136160339\ i$
10^{-9}	7	$6.759120567 + 0.136160339\ i$

z is given and as $z \to 0$, the ring is approached. N_2 is the number of terms in the associated Legendre function series (48) and (49) needed for the accuracy given

4.1. Far field expression

The variation in accuracy with distance from the ring of the far field expression given by Eq. (113) has been investigated. Results for sample cases calculated using the exact series (104) and the far field solution (113) are given in Tables 5 and 6. Table 5 shows the effect on accuracy of increasing the distance

TABLE 5. *Comparison of the Fourier coefficients* $\bar{G}_H^m(rR, \gamma, k)$ *obtained from the exact series solution* (104) *and the far field solution* (113)

z	$\bar{G}_H^m(rR, \gamma, k)$ from (104)	$G_H^m(rR, \gamma, k)$ from (113)
1	$0.131840 + 0.0959755\ i$	$0.0184152 - 0.0731540\ i$
5	$0.0471755 - 0.0981998\ i$	$0.0503084 - 0.0888675\ i$
10	$-0.0378640 + 0.0111158\ i$	$-0.0368642 + 0.0109045\ i$
50	$-0.00176272 - 0.000313668\ i$	$-0.00176161 - 0.000307639\ i$
10^2	$-0.0000246835 + 0.000448721\ i$	$-0.0000239353 + 0.000448626\ i$
5×10^2	$3.59616 \times 10^{-6} + 0.0000176361\ i$	$3.60199 \times 10^{-6} + 0.0000176347\ i$
10^3	$-1.88428 \times 10^{-6} - 4.08643 \times 10^{-6}\ i$	$-1.88496 \times 10^{-6} - 4.08611 \times 10^{-6}\ i$
5×10^3	$-1.44692 \times 10^{-7} + 1.07070 \times 10^{-7}\ i$	$-1.44689 \times 10^{-7} + 1.07075 \times 10^{-7}\ i$
10^4	$4.30731 \times 10^{-8} + 1.30272 \times 10^{-8}\ i$	$4.30733 \times 10^{-8} + 1.30265 \times 10^{-8}\ i$
5×10^4	$1.92360 \times 10^{-10} + 1.78969 \times 10^{-9}\ i$	$1.92366 \times 10^{-10} + 1.78969 \times 10^{-9}\ i$

The calculations are at fixed radial coordinate for increasing values of the axial coordinate z. The fixed parameters are: $m = 1,\ \beta = 6,\ R = 1,\ Z = 0$ and $r = 3/2$

TABLE 6. *Comparison of the Fourier coefficients* $\bar{G}_H^m(rR, \gamma, k)$ *obtained from the exact series solution* (104) *and the far field solution* (113)

r	$\bar{G}_H^m(rR, \gamma, k)$ from (104)	$\bar{G}_H^m(rR, \gamma, k)$ from (113)
3/2	$-0.216391 - 0.0171359\ i$	$-0.0700107 + 0.163569\ i$
5	$0.0147730 - 0.0473316\ i$	$-0.00709918 + 0.00313568\ i$
10	$-0.0226381 - 0.00913182\ i$	$-0.00617763 - 0.00710630\ i$
50	$-0.0000358188 - 0.00485831\ i$	$0.000408112 - 0.00425824\ i$
10^2	$-0.00242721 + 0.0000894443\ i$	$-0.00228501 - 0.0000347556\ i$
5×10^2	$-0.000474091 + 0.000105773\ i$	$-0.000469615 + 0.0000996081\ i$
10^3	$0.000219612 - 0.000103720\ i$	$0.000218868 - 0.000101965\ i$
5×10^3	$-0.0000289656 - 0.0000389935\ i$	$-0.0000288908 - 0.0000389784\ i$
10^4	$-7.00966 \times 10^{-6} + 0.0000232538\ i$	$-7.01780 \times 10^{-6} + 0.0000232365\ i$
5×10^4	$-4.82955 \times 10^{-6} + 5.19987 \times 10^{-7}\ i$	$-4.82904 \times 10^{-6} + 5.19418 \times 10^{-7}\ i$

The calculations are in the plane of the ring for increasing values of the radial coordinate r. The fixed parameters are: $m = 2,\ \beta = 6,\ R = 1,\ Z = 0$ and $z = 0$

$\mid z - Z \mid$ from the ring plane, whereas Table 6 shows the effect of increasing the radial distance r within the ring plane. At very large distances from the ring the far field solution is a very good approximation in both directions.

5. Comments and conclusions

The Fourier coefficients for the Helmholtz Green function have been split into their half advanced+half retarded and half advanced–half retarded components, and these components have been given in closed form in terms of two-dimensional hypergeometric functions. These solutions generalize the well known solutions of Poisson's equation for ring sources, and reduce to them in the static limit when the wave number $\beta \to 0$.

The two-dimensional hypergeometric functions can be considered as double series, with the order of summation arbitrary. The two summation choices give different series of special functions for each of the Fourier components. One series is given in terms of Hankel functions, and only a few terms are need far from the ring source for accurate results. This Hankel series leads to a simple closed form far field formula in terms of a Bessel function of the first kind. A second series in terms of associated Legendre functions only requires a few terms in the neighborhood of the ring to give accurate results. At intermediate distances from the ring both series have comparable performance.

The systems of partial differential equations associated with each of the two generalized hypergeometric functions allow a fourth order ordinary differential equation in terms of $x \equiv k^2$ to be derived for the Fourier coefficients. A different approach using an integral representation gives an equivalent fourth order ordinary differential equation in terms of the toroidal variable ω and another equation in terms of the wave number parameter λ can also be derived. It is intended to further investigate the applications of these differential equations in future work.

All of the results presented here have been numerically verified. The numerical performance of the series of associated Legendre functions (48) and (49), the Hankel series (55) and the far field formula (56) has been presented in Sect. 4.

Appendix A: Series from an integral representation

The Fourier coefficient $G_H^m(\beta, r, R, z-Z) \equiv \hat{G}_H^m(rR, \lambda, \omega)$ given by Eqs. (27) and (28) can be expressed in the form

$$\hat{G}_H^m(rR, \lambda, \omega) = \frac{1}{\pi\sqrt{2rR}} \sum_{n=0}^{\infty} \int_0^{\pi} \frac{(i\lambda)^n}{n!} (\omega - \cos\psi)^{(n-1)/2} \cos(m\psi)\, \mathrm{d}\psi \tag{114}$$

where ω and λ are defined by Eqs. (15) and (44) respectively. Evaluation of (114) requires the integral

$$\hat{I}_{m,n}(\omega) = \int_0^{\pi} \cos(m\psi)(\omega - \cos\psi)^{(n-1)/2}\, \mathrm{d}\psi \tag{115}$$

which can be evaluated for $m \in \mathbb{N}_0$ using the integral representation [7, eqn 8.711 2]:

$$P_\nu^m(\xi) = \frac{\Gamma(\nu+m+1)}{\pi\Gamma(\nu+1)} \int_0^{\pi} \cos(m\theta) \left(\xi + \sqrt{\xi^2-1}\cos\theta\right)^{\nu} \mathrm{d}\theta \tag{116}$$

which is equivalent to

$$P_\nu^m(\xi) = \frac{(-1)^m \Gamma(\nu+m+1)\left(\xi^2-1\right)^{\nu/2}}{\pi\Gamma(\nu+1)} \int_0^{\pi} \cos(m\psi) \left(\frac{\xi}{\sqrt{\xi^2-1}} - \cos\psi\right)^{\nu} \mathrm{d}\psi. \tag{117}$$

The substitutions

$$\omega = \frac{\xi}{\sqrt{\xi^2-1}} \tag{118}$$

and

$$\nu = \frac{n-1}{2} \tag{119}$$

then give

$$\hat{I}_{m,n}(\omega) = \frac{(-1)^m \pi\Gamma((n+1)/2)\left(\omega^2-1\right)^{(n-1)/4}}{\Gamma(m+(n+1)/2)} P_{(n-1)/2}^m\left(\frac{\omega}{\sqrt{\omega^2-1}}\right) \tag{120}$$

and (114) becomes

$$\hat{G}_H^m(rR, \lambda, \omega) = \frac{(-1)^m}{\left(\omega^2-1\right)^{1/4}\sqrt{2rR}} \times \sum_{n=0}^{\infty} \frac{\left(i\lambda\left(\omega^2-1\right)^{1/4}\right)^n}{n!} \frac{\Gamma((n+1)/2)}{\Gamma(m+(n+1)/2)} P_{(n-1)/2}^m\left(\frac{\omega}{\sqrt{\omega^2-1}}\right). \tag{121}$$

Splitting the series (121) into even and odd terms gives after some reduction

$$\hat{\Lambda}_{+}^{m}\left(rR,\lambda,\omega\right)=\frac{\left(-1\right)^{m}\sqrt{\pi}}{\left(\omega^{2}-1\right)^{1/4}\sqrt{2rR}}\sum_{p=0}^{\infty}\left(\frac{-\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{p}\frac{1}{p!\Gamma\left(p+m+1/2\right)}P_{p-1/2}^{m}\left(\frac{\omega}{\sqrt{\omega^{2}-1}}\right) \tag{122}$$

$$\hat{\Lambda}_{-}^{m}\left(rR,\lambda,\omega\right)=\frac{\left(-1\right)^{m}\sqrt{\pi}i\lambda}{2\sqrt{2rR}}\sum_{s=m}^{\infty}\left(\frac{-\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{s}\frac{1}{\Gamma\left(s+3/2\right)\Gamma\left(m+s+1\right)}P_{s}^{m}\left(\frac{\omega}{\sqrt{\omega^{2}-1}}\right), \tag{123}$$

where the factorial formulas

$$(2p)!=\frac{2^{2p}\Gamma\left(p+1/2\right)p!}{\sqrt{\pi}} \tag{124}$$

$$(2s+1)!=\frac{2^{2s+1}\Gamma\left(s+3/2\right)s!}{\sqrt{\pi}} \tag{125}$$

have been used to obtain (122) and (123). The index in (123) runs from $s=m$ rather than from $s=0$ as the associated Legendre polynomial $P_{s}^{m}\left(\xi\right)$ is zero for $s<m$. The substitution $s=p+m$ in (123) gives the alternative form

$$\hat{\Lambda}_{-}^{m}\left(rR,\lambda,\omega\right)=\frac{\sqrt{\pi}i\lambda}{2\sqrt{2rR}}\left(\frac{\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{m}\times\sum_{p=0}^{\infty}\left(\frac{-\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{p}\frac{1}{\Gamma\left(p+m+3/2\right)\Gamma\left(p+2m+1\right)}P_{p+m}^{m}\left(\frac{\omega}{\sqrt{\omega^{2}-1}}\right). \tag{126}$$

The indices in Eqs. (122) and (126) can be switched to negative values using the relations [17, eqns 8.2.5, 8.2.1]:

$$P_{\nu}^{m}\left(\xi\right)=\frac{\Gamma\left(\nu+m+1\right)}{\Gamma\left(\nu-m+1\right)}P_{\nu}^{-m}\left(\xi\right)\quad\left[m\in\mathbb{N}_{0}\right] \tag{127}$$

$$P_{-\nu-1}^{m}\left(\xi\right)=P_{\nu}^{m}\left(\xi\right) \tag{128}$$

which gives

$$\hat{\Lambda}_{+}^{m}\left(rR,\lambda,\omega\right)=\frac{\left(-1\right)^{m}\sqrt{\pi}}{\left(\omega^{2}-1\right)^{1/4}\sqrt{2rR}}\times\sum_{p=0}^{\infty}\left(\frac{-\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{p}\frac{1}{p!\Gamma\left(p-m+1/2\right)}P_{-p-1/2}^{-m}\left(\frac{\omega}{\sqrt{\omega^{2}-1}}\right) \tag{129}$$

$$\hat{\Lambda}_{-}^{m}\left(rR,\lambda,\omega\right)=\frac{\sqrt{\pi}i\lambda}{2\sqrt{2rR}}\left(\frac{\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{m}\times\sum_{p=0}^{\infty}\left(\frac{-\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{p}\frac{1}{\Gamma\left(p+m+3/2\right)\Gamma\left(p+1\right)}P_{-p-m-1}^{-m}\left(\frac{\omega}{\sqrt{\omega^{2}-1}}\right). \tag{130}$$

The kind of Legendre functions in Eqs. (129) and (130) can be switched using the Whipple transformation [17, eqn 8.2.7], [18]:

$$P_{-\mu-1/2}^{-\alpha-1/2}\left(\frac{\omega}{\sqrt{\omega^{2}-1}}\right)=\frac{\left(\omega^{2}-1\right)^{1/4}\exp\left(-i\mu\pi\right)}{\left(\pi/2\right)^{1/2}\Gamma\left(\alpha+\mu+1\right)}Q_{\alpha}^{\mu}\left(\omega\right) \tag{131}$$

which gives after some reduction

$$\hat{\Lambda}_{+}^{m}\left(rR,\lambda,\omega\right)=\frac{(-1)^{m}}{\sqrt{rR}}\sum_{p=0}^{\infty}\left(\frac{\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{p}\frac{Q_{m-1/2}^{p}\left(\omega\right)}{p!\Gamma\left(p-m+1/2\right)\Gamma\left(m+p+1/2\right)} \tag{132}$$

$$\hat{\Lambda}_{-}^{m}\left(rR,\lambda,\omega\right)=\frac{(-1)^{m}}{\sqrt{rR}}\sum_{p=0}^{\infty}\left(\frac{\lambda^{2}\sqrt{\omega^{2}-1}}{4}\right)^{p+m+1/2}\frac{Q_{m-1/2}^{p+m+1/2}\left(\omega\right)}{p!\Gamma\left(p+m+3/2\right)\Gamma\left(p+2m+1\right)}. \tag{133}$$

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