ARAB JOURNAL OF BASIC AND APPLIED SCIENCES
2023, VOL. 30, NO. 1, 583–592
https://doi.org/10.1080/25765299.2023.2268911

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RESEARCH ARTICLE

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Multi-integral representations for Jacobi functions of the first and second kind

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ABSTRACT
One may consider the generalization of Jacobi polynomials and the Jacobi function of the second kind to a general function where the degree is allowed to be a complex number instead of a non-negative integer. These functions are referred to as Jacobi functions. In a similar fashion as associated Legendre functions, these break into two categories, functions which are analytically continued from the real line segment $(-1,1)$ and those analytically continued from the real ray $(1,\infty)$. Using properties of Gauss hypergeometric functions, we derive multi-derivative and multi-integral representations for the Jacobi functions of the first and second kind.

ARTICLE HISTORY
Received 5 June 2023
Revised 2 October 2023
Accepted 5 October 2023

KEYWORDS
Generalized hypergeometric functions; integral representations; Jacobi functions; Jacobi polynomials; multi-integral representations; Rodrigues-type relations

2020 MATHEMATICS SUBJECT CLASSIFICATION
33C45; 44A20; 42C05; 33C05

1. Preliminaries

Throughout this paper we adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and $\mathbb{C}$ represents the complex numbers. The rising factorial (Pochhammer symbol) for $a \in \mathbb{C}$, $n \in \mathbb{N}_0$ is given by (DLMF, (5.2.4), (5.2.5)) $(a)_n := (a)(a+1)\cdots(a+n-1)$. The gamma function (DLMF, Chapter 5) is related to the rising factorial, namely for $a \in \mathbb{C}\setminus -\mathbb{N}_0$, one has

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{1}$$

which allows one to extend the definition to non-positive integer values of n. Some other properties of rising factorials which we will use are $(n, k \in \mathbb{N}_0,\ n \geq k)$

$$\Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(1-a)_n}, \tag{2}$$

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}. \tag{3}$$

One also has the following expression for the generalized binomial coefficient for $z \in \mathbb{C}$, $n \in \mathbb{N}_0$ (DLMF, (1.2.6))

$$\binom{z}{n} = \frac{(-1)^n (-z)_n}{n!}. \tag{4}$$

We will also use the common notational product conventions, $a_l \in \mathbb{C}$, $l \in \mathbb{N}$, $r \in \mathbb{N}_0$, e.g.,

$$(a_1, \ldots, a_r)_k := (a_1)_k (a_2)_k \cdots (a_r)_k, \tag{5}$$

$$\Gamma(a_1, \ldots, a_r) := \Gamma(a_1) \cdots \Gamma(a_r). \tag{6}$$

The generalized hypergeometric function (DLMF, Chapter 16) is defined by the infinite series (DLMF, (16.2.1))

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!}, \tag{7}$$

where $b_j \notin -\mathbb{N}_0$, for $j \in \{1, \ldots, s\}$; and elsewhere by analytic continuation. Further define Olver's (scaled or regularized) generalized hypergeometric series

$${}_r\mathbf{F}_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \frac{1}{\Gamma(b_1, \ldots, b_s)} {}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right)$$

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$$= \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{\Gamma(b_1+k, \ldots, b_s+k)} \frac{z^k}{k!}, \tag{8}$$

which is entire for all $a_l, b_j \in \mathbb{C}$, $l \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$. Both the generalized and Olver's generalized hypergeometric series, if nonterminating, are entire if $r \le s$, convergent for $|z| < 1$ if $r = s+1$ and divergent if $r \ge s+1$.

The Gauss hypergeometric function ${}_2F_1$ has many useful and interesting properties. For instance it satisfies several useful derivative relations which we will rely upon. These are given by cf. (DLMF, (15.5.2), (15.5.4), (15.5.6), and (15.5.9))

$$\frac{d^n}{dw^n}\, {}_2\mathbf{F}_1\left(\begin{matrix} a,b \\ c \end{matrix}; w\right) = (a)_n (b)_n\, {}_2\mathbf{F}_1\left(\begin{matrix} a+n, b+n \\ c+n \end{matrix}; w\right), \tag{9}$$

$$\frac{d^n}{dw^n} w^{c-1}\, {}_2\mathbf{F}_1\left(\begin{matrix} a,b \\ c \end{matrix}; w\right) = w^{c-n-1}\, {}_2\mathbf{F}_1\left(\begin{matrix} a,b \\ c-n \end{matrix}; w\right), \tag{10}$$

$$\frac{d^n}{dw^n} (1-w)^{a+b-c}\, {}_2\mathbf{F}_1\left(\begin{matrix} a,b \\ c \end{matrix}; w\right) = (c-a)_n (c-b)_n (1-w)^{a+b-c-n}\, {}_2\mathbf{F}_1\left(\begin{matrix} a,b \\ c+n \end{matrix}; w\right), \tag{11}$$

$$\frac{d^n}{dw^n} w^{c-1} (1-w)^{a+b-c}\, {}_2\mathbf{F}_1\left(\begin{matrix} a,b \\ c \end{matrix}; w\right) = w^{c-n-1} (1-w)^{a+b-c-n}\, {}_2\mathbf{F}_1\left(\begin{matrix} a-n, b-n \\ c-n \end{matrix}; w\right). \tag{12}$$

The Jacobi polynomial and many of the other special functions which we will study in this paper are given in terms of a terminating Gauss hypergeometric function. In particular, one of the most important classical orthogonal polynomials, the Jacobi polynomial, is defined as (DLMF, (18.5.7))

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right). \tag{13}$$

2. Definitions and properties of the Jacobi functions

Jacobi functions are complex solutions $w = w(z) = w_\gamma^{(\alpha,\beta)}(z)$ to the Jacobi differential equation (DLMF, Table 18.8.1)

$$(1-z^2)\frac{d^2w}{dz^2} + (\beta - \alpha - z(\alpha+\beta+2))\frac{dw}{dz} + \gamma(\alpha+\beta+\gamma+1)w = 0, \tag{14}$$

which is a second order linear homogeneous differential equation. Jacobi functions of the first and second kind are solutions to the Jacobi differential Equation (14) which are regular as $|z| \to 1$ and $|z| \to \infty$, respectively. There are several important references for Jacobi functions such as (Erdélyi, Magnus, Oberhettinger, & Tricomi, 1981, Section 10.8), and (Durand, 1978; Durand, 1979; Flensted-Jensen & Koornwinder, 1973; Koornwinder, 1984; Kuijlaars, Martinez-Finkelshtein, & Orive, 2005; Szegő, 1975; Wimp, McCabe, & Connor, 1997).

2.1. The Jacobi function of the first kind

The Jacobi function of the first kind is a generalization of the Jacobi polynomial where the degree is no longer restricted to be an integer. In the following section we provide some important properties for the Jacobi function of the first kind.

2.1.1. Single Gauss hypergeometric representations

In the following result we present the four single Gauss hypergeometric function representations of the Jacobi function of the first kind.

Theorem 2.1. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha + \gamma \notin -\mathbb{N}$. Then the Jacobi function of the first kind $P_\gamma^{(\alpha,\beta)} : \mathbb{C}\setminus(-\infty, -1] \to \mathbb{C}$ has the following single Gauss hypergeometric function representations*

$$P_\gamma^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\, {}_2\mathbf{F}_1\left(\begin{matrix} -\gamma, \alpha+\beta+\gamma+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2}\right) \tag{15}$$

$$= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)} \left(\frac{2}{z+1}\right)^{\beta}\, {}_2\mathbf{F}_1\left(\begin{matrix} -\beta-\gamma, \alpha+\gamma+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2}\right) \tag{16}$$

$$= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)} \left(\frac{z+1}{2}\right)^{\gamma}\, {}_2\mathbf{F}_1\left(\begin{matrix} -\gamma, -\beta-\gamma \\ \alpha+1 \end{matrix}; \frac{z-1}{z+1}\right) \tag{17}$$

$$= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)} \left(\frac{2}{z+1}\right)^{\alpha+\beta+\gamma+1} \times {}_2\mathbf{F}_1\left(\begin{matrix} \alpha+\gamma+1, \alpha+\beta+\gamma+1 \\ \alpha+1 \end{matrix}; \frac{z-1}{z+1}\right). \tag{18}$$

Proof. Start with (13) and replace the rising factorial by a ratio of gamma functions using (1) and the factorial $n! = \Gamma(n+1)$ and substitute $n \mapsto \gamma \in \mathbb{C}$, $x \mapsto z$. Application of Pfaff's and Euler's transformations (DLMF, (15.8.1)) provides the other three representations. This completes the proof. □

One of the consequences of the definition of the Jacobi function of the first kind is the following special value:

$$P_\gamma^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)}, \tag{19}$$

where $\alpha + \gamma \notin -\mathbb{N}$.

2.2. The Jacobi function of the second kind

Studies of the Jacobi function of the second kind $Q_\gamma^{(\alpha,\beta)}$ traditionally used a degree γ which was integer valued (see for instance Szegő, 1975, §4.61). However, in this paper we treat the Jacobi function of the second kind where γ is not necessarily restricted to be an integer. In the following material we provide some important properties for the Jacobi function of the second kind.

2.2.1. Single Gauss hypergeometric representations

Below we give the four single Gauss hypergeometric function representations of the Jacobi function of the second kind.

Theorem 2.2. *Let $\gamma,\alpha,\beta,z\in\mathbb{C}$ such that $\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$. Then, the Jacobi function of the second kind $Q_\gamma^{(\alpha,\beta)}:\mathbb{C}\backslash[-1,1]\to\mathbb{C}$ has the following single Gauss hypergeometric function representations*

$$Q_\gamma^{(\alpha,\beta)}(z):=\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha+\gamma+1}(z+1)^{\beta}}\times{}_2F_1\left(\begin{matrix}\gamma+1,\alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix};\frac{2}{1-z}\right) \tag{20}$$

$$=\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha+\beta+\gamma+1}}\times{}_2F_1\left(\begin{matrix}\beta+\gamma+1,\alpha+\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix};\frac{2}{1-z}\right) \tag{21}$$

$$=\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha}(z+1)^{\beta+\gamma+1}}\times{}_2F_1\left(\begin{matrix}\gamma+1,\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix};\frac{2}{1+z}\right) \tag{22}$$

$$=\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z+1)^{\alpha+\beta+\gamma+1}}\times{}_2F_1\left(\begin{matrix}\alpha+\gamma+1,\alpha+\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix};\frac{2}{1+z}\right). \tag{23}$$

Proof. Start with (Erdélyi et al., 1981, (10.8.18)) and let $n\mapsto\gamma\in\mathbb{C}$ and $x\mapsto z$. Application of Pfaff's and Euler's transformations (DLMF, (15.8.1)) provides the other three representations. This completes the proof. □

The Jacobi function of the second kind $Q_\gamma^{(\alpha,\beta)}:\mathbb{C}\backslash[-1,1]\to\mathbb{C}$ has the following integral representation (Szegő, 1975, (4.61.1))

$$Q_\gamma^{(\alpha,\beta)}(z)=\frac{1}{2^{\gamma+1}(z-1)^{\alpha}(z+1)^{\beta}}\times\int_{-1}^{1}\frac{(1-t)^{\alpha+\gamma}(1+t)^{\beta+\gamma}}{(z-t)^{\gamma+1}}\mathrm{d}t, \tag{24}$$

provided $\Re(\alpha+\gamma),\Re(\beta+\gamma)>-1$ (Wimp et al., 1997, (2.5)). We will use this integral representation to derive an alternative integral representation for the Jacobi function of the second kind.

Theorem 2.3. *Let $k\in\mathbb{N}_0$, $\alpha,\beta,\gamma,z\in\mathbb{C}$ such that $z\notin[-1,1]$. Decompose $\gamma=\delta+k$, where $\delta=\gamma-k$. Then, the Jacobi function of the second kind $Q_\gamma^{(\alpha,\beta)}:\mathbb{C}\backslash[-1,1]\to\mathbb{C}$ has the following integral representation*

$$Q_\gamma^{(\alpha,\beta)}(z)=\frac{(-1)^k k!}{2^{\gamma+1-k}(-\gamma)_k(z-1)^{\alpha}(z+1)^{\beta}}\times\int_{-1}^{1}\frac{(1-t)^{\alpha+\gamma-k}(1+t)^{\beta+\gamma-k}}{(z-t)^{\gamma-k+1}}P_k^{(\alpha+\gamma-k,\beta+\gamma-k)}(t)\ \mathrm{d}t. \tag{25}$$

Proof. Start with (24) and decompose γ such that it is written as some complex number δ added to a non-negative integer k, $\gamma=\delta+k$ with $k\in\mathbb{N}_0$, this produces

$$Q_\gamma^{(\alpha,\beta)}(z)=\frac{1}{2^{\delta+k+1}(z-1)^{\alpha}(z+1)^{\beta}}\times\int_{-1}^{1}\frac{(1-t)^{\alpha+\delta+k}(1+t)^{\beta+\delta+k}}{(z-t)^{\delta+k+1}}\mathrm{d}t. \tag{26}$$

Along the lines of the derivation of (Szegő, 1975, (4.61.4))

$$Q_n^{(\alpha,\beta)}(z)=\frac{1}{2(z-1)^{\alpha}(z+1)^{\beta}}\times\int_{-1}^{1}\frac{(1-t)^{\alpha}(1+t)^{\beta}}{z-t}P_n^{(\alpha,\beta)}(t)\ \mathrm{d}t, \tag{27}$$

proceed by integrating (26) by parts k-times with the boundary terms vanishing and utilizing the Rodrigues-type formula for Jacobi polynomials (Koekoek, Lesky, & Swarttouw, 2010, (9.8.10))

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(1-t)^{a+k}(1+t)^{b+k}=(-1)^k2^kk!(1-t)^a(1+t)^bP_k^{(a,b)}(t), \tag{28}$$

with $a=\alpha+\delta$, $b=\beta+\delta$. This completes the proof. □

The Jacobi function of the second kind has the following raising and lowering operators (Ismail & Mansour, 2014, Theorem 3.1).

Theorem 2.4. *Let $\gamma,\alpha,\beta\in\mathbb{C}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}z}Q_\gamma^{(\alpha,\beta)}(z)=\frac{(\beta-\alpha-z(\alpha+\beta))}{z^2-1}Q_\gamma^{(\alpha,\beta)}(z)-\frac{2(\gamma+1)}{z^2-1}Q_{\gamma+1}^{(\alpha-1,\beta-1)}(z) \tag{29}$$

$$= -\frac{\alpha+\beta+\gamma+1}{2} Q_{\gamma-1}^{(\alpha+1,\beta+1)}(z). \qquad (30)$$

Proof. We extend the proof given in (Ismail & Mansour, 2014, Theorem 3.1) by replacing integer n by complex γ for the Jacobi functions of the second kind. □

Theorem 2.5. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma, z \in \mathbb{C}$. *Then*

$$\frac{d^n}{dz^n}(z-1)^{\alpha}(1+z)^{\beta} Q_{\gamma}^{(\alpha,\beta)}(z) = (-2)^n(\gamma+1)_n (z-1)^{\alpha-n}(1+z)^{\beta-n} Q_{\gamma+n}^{(\alpha-n,\beta-n)}(z). \qquad (31)$$

Proof. Start with the $n=1$ case. Taking into account (30) and (29) one obtains

$$(1-z)^{-\alpha+1}(1+z)^{-\beta+1}\frac{d}{dz}(1-z)^{\alpha}(1+z)^{\beta}Q_{\gamma}^{(\alpha,\beta)}(z)$$
$$= (1-z^2)\frac{d}{dz}Q_{\gamma}^{(\alpha,\beta)}(z) + (\beta-\alpha-(\alpha+\beta)z)Q_{\gamma}^{(\alpha,\beta)}(z)$$
$$= 2(\gamma+1)Q_{\gamma+1}^{(\alpha-1,\beta-1)}(z).$$

From the above expression, a straightforward calculation allows us to obtain the general n integral case. □

Starting from Theorem 2.5 we can obtain a multi-derivative representation of the Jacobi function of the second kind.

Corollary 2.6. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$ *such that* $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$. *Then*

$$Q_{\gamma}^{(\alpha,\beta)}(z) = \frac{1}{2^n(-\gamma)_n (z-1)^{\alpha}(z+1)^{\beta}} \times \frac{d^n}{dz^n}(z-1)^{\alpha+n}(z+1)^{\beta+n} Q_{\gamma-n}^{(\alpha+n,\beta+n)}(z). \qquad (32)$$

Proof. Starting with Theorem 2.5 and mapping $(\gamma, \alpha, \beta) \mapsto (\gamma-n, \alpha+n, \beta+n)$ completes the proof. □

3. Multi-integral representations for the Jacobi functions

In this section we derive multi-integral representations for the Jacobi functions of the first and second kind. First we will derive a useful lemma.

Lemma 3.1. *Let* $n, r \in \mathbb{N}_0, a, x \in \mathbb{C}, \vec{\mu} \in \mathbb{C}^r$, *and let* $f^{\vec{\mu}}$ *be a function such that*

$$\frac{d}{dz} f^{\vec{\mu}}(z) = \lambda_{\vec{\mu}} f^{\vec{\mu}\pm\vec{1}}(z), \qquad (33)$$

where $\lambda_{\vec{\mu}} \in \mathbb{C}^*$. *Then the following identity holds:*

$$\int_a^x \cdots \int_a^x f^{\mu}(w)(dw)^n = \frac{1}{\lambda_{\vec{\mu}\mp\vec{1}} \cdots \lambda_{\vec{\mu}\mp n\vec{1}}} \times \sum_{k=n}^{\infty} \frac{\lambda_{\vec{\mu}\mp n\vec{1}} \cdots \lambda_{\vec{\mu}\mp n\vec{1}\pm(k-1)\vec{1}} f^{\vec{\mu}\mp n\vec{1}\pm k\vec{1}}(a)(x-a)^k}{k!},$$

where $\vec{1} = (1, 1, \ldots, 1) \in \mathbb{C}^r$.

The proof of this result is analogous to the one given in (Cohl & Costas-Santos, 2020, Lemma 2.3).

3.1. The Jacobi functions of the first kind

Theorem 3.2. *Let* $n \in \mathbb{N}_0, \alpha, \beta, \gamma \in \mathbb{C}$, *such that* $\alpha+\gamma \notin -\mathbb{N}$, $\Re\alpha, \Re\beta > -1$, $(-\gamma)_n \neq 0$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\int_z^1 \cdots \int_z^1 (1-w)^{\alpha}(1+w)^{\beta} P_{\gamma}^{(\alpha,\beta)}(w)\ (dw)^n = \frac{(-1)^n}{2^n(-\gamma)_n}(1-z)^{\alpha+n}(1+z)^{\beta+n} P_{\gamma-n}^{(\alpha+n,\beta+n)}(z). \qquad (34)$$

Proof. Considering the $n=1$ case of (35) and then integrating produces the following definite integral

$$\int_z^1 (1-w)^{\alpha}(1+w)^{\beta} P_{\gamma}^{(\alpha,\beta)}(w)\ dw = \frac{1}{2\gamma}(1-z)^{\alpha+1}(1+z)^{\beta+1} P_{\gamma-1}^{(\alpha+1,\beta+1)}(z).$$

Iterating the above expression n-times completes the proof. □

The Jacobi function of the first kind obeys the following multi-derivative identity.

Theorem 3.3. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\frac{d^n}{dz^n}(1-z)^{\alpha}(1+z)^{\beta}P_{\gamma}^{(\alpha,\beta)}(z) = (-2)^n(\gamma+1)_n(1-z)^{\alpha-n}(1+z)^{\beta-n}P_{\gamma+n}^{(\alpha-n,\beta-n)}(z). \qquad (35)$$

Proof. Starting with (12), (15), one has

$$w = \frac{1-z}{2}, \quad \frac{d}{dw} = -2\frac{d}{dz}, \qquad (36)$$

and from this, we derive

$$\frac{d^n}{dz^n}(1-z)^{c-1}(1+z)^{a+b-c} {}_2\boldsymbol{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}; \frac{1-z}{2}\right) = (-2)^n(1-z)^{c-n-1}(1+z)^{a+b-c-n} \times {}_2\boldsymbol{F}_1\left(\begin{matrix} a-n, b-n \\ c-n \end{matrix}; \frac{1-z}{2}\right). \qquad (37)$$

Substituting the Gauss hypergeometric function (15) into the above expression completes the proof. □

Theorem 3.4. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, *such that* $\alpha + \gamma \notin -\mathbb{N}$, $\Re(\alpha + \beta + \gamma) < -n$, $\Re\gamma > n-1$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\int_z^\infty \cdots \int_z^\infty (1-w)^\alpha (1+w)^\beta P_\gamma^{(\alpha,\beta)}(w)\ (dw)^n = \frac{1}{2^n(-\gamma)_n}(1-z)^{\alpha+n}(1+z)^{\beta+n} P_{\gamma-n}^{(\alpha+n,\beta+n)}(z). \tag{38}$$

Proof. Considering the $n=1$ case of (35), taking into account the constraints considered in the statement and then integrating produces the following improper integral:

$$\int_z^\infty (1-w)^\alpha (1+w)^\beta P_\gamma^{(\alpha,\beta)}(w)\ dw = \frac{1}{2\gamma}(1-z)^{\alpha+1}(1+z)^{\beta+1} P_{\gamma-1}^{(\alpha+1,\beta+1)}(z).$$

Iterating the above expression n-times completes the proof. □

Theorem 3.5. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\frac{d^n}{dz^n}(1-z)^\alpha P_\gamma^{(\alpha,\beta)}(z) = (-\alpha-\gamma)_n (1-z)^{\alpha-n} P_\gamma^{(\alpha-n,\beta+n)}(z). \tag{39}$$

Proof. Adopting (37) and utilizing the Gauss hypergeometric representation (16) completes the proof. □

Theorem 3.6. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, *with* $\Re\alpha > -1$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\int_z^1 \cdots \int_z^1 (1-w)^\alpha P_\gamma^{(\alpha,\beta)}(w)\ (dw)^n = \frac{(1-z)^{\alpha+n}}{(\alpha+\gamma+1)_n} P_\gamma^{(\alpha+n,\beta-n)}(z). \tag{40}$$

Proof. Considering the $n=1$ case of (39) and then integrating produces the following definite integral

$$\int_z^1 (1-w)^\alpha P_\gamma^{(\alpha,\beta)}(w)\ dw = \frac{(1-z)^{\alpha+1}}{\alpha+\gamma+1} P_\gamma^{(\alpha+1,\beta-1)}(z).$$

Iterating the above expression n-times completes the proof. □

Theorem 3.7. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, *with* $\Re(\alpha+\gamma) < -n$, $\Re(\beta+\gamma) > n-1$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\int_z^\infty \cdots \int_z^\infty (1-w)^\alpha P_\gamma^{(\alpha,\beta)}(w)\ (dw)^n = \frac{(1-z)^{\alpha+n}}{(\alpha+\gamma+1)_n} P_\gamma^{(\alpha+n,\beta-n)}(z). \tag{41}$$

Proof. Considering the $n=1$ case of (39), taking into account the constraints considered in the statement and then integrating produces the following improper integral:

$$\int_z^\infty (1-w)^\alpha P_\gamma^{(\alpha,\beta)}(w)\ dw = \frac{(1-z)^{\alpha+1}}{\alpha+\gamma+1} P_\gamma^{(\alpha+1,\beta-1)}(z).$$

Iterating the above expression n-times completes the proof. □

Theorem 3.8. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\frac{d^n}{dz^n}(1+z)^\beta P_\gamma^{(\alpha,\beta)}(z) = (-1)^n(-\beta-\gamma)_n (1+z)^{\beta-n} P_\gamma^{(\alpha+n,\beta-n)}(z). \tag{42}$$

Proof. Adopting (37) and utilizing the Gauss hypergeometric representation (16) completes the proof. □

Theorem 3.9. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, *with* $\Re(\beta+\gamma) < -n$, $\Re(\alpha+\gamma) > n-1$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\int_z^\infty \cdots \int_z^\infty (1+w)^\beta P_\gamma^{(\alpha,\beta)}(w)\ (dw)^n = \frac{(-1)^n(1+z)^{\beta+n}}{(\beta+\gamma+1)_n} P_\gamma^{(\alpha-n,\beta+n)}(z). \tag{43}$$

Proof. Considering the $n=1$ case of (42), taking into account the constraints considered in the statement and then integrating produces the following improper integral:

$$\int_z^\infty (1+w)^\beta P_\gamma^{(\alpha,\beta)}(w)\ dw = \frac{-(1+z)^{\beta+1}}{\beta+\gamma+1} P_\gamma^{(\alpha-1,\beta+1)}(z).$$

Iterating the above expression n-times completes the proof. □

Theorem 3.10. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\frac{d^n}{dz^n} P_\gamma^{(\alpha,\beta)}(z) = 2^{-n}(\alpha+\beta+\gamma+1)_n P_{\gamma-n}^{(\alpha+n,\beta+n)}(z). \tag{44}$$

Proof. Starting with (9) and using (15), (36) completes the proof. □

Theorem 3.11. *Let* $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, *with* $\alpha+\gamma \notin -\mathbb{N}$, $z \in \mathbb{C}\backslash(-\infty, -1]$. *Then*

$$\int_z^1 \cdots \int_z^1 P_\gamma^{(\alpha,\beta)}(w)\ (dw)^n = \frac{2^n}{(-\alpha-\beta-\gamma)_n} P_{\gamma+n}^{(\alpha-n,\beta-n)}(z) + \frac{2\Gamma(\alpha+\gamma+1)(1-z)^{n-1}}{(n-1)!(\alpha+\beta+\gamma)\Gamma(\alpha)\Gamma(\gamma+2)}$$

$$\times {}_3F_2\left(\begin{matrix}-n, 1-\alpha, 1\\ \gamma+2, 1-\alpha-\beta-\gamma\end{matrix}; \frac{2}{1-z}\right) \tag{45}$$

$$= \frac{\Gamma(\alpha+\gamma+1)(1-z)^n}{\Gamma(\alpha+1)\Gamma(\gamma+1)n!} {}_3F_2\left(\begin{matrix}-\gamma, \alpha+\beta+\gamma+1, 1\\ \alpha+1, n+1\end{matrix}; \frac{1-z}{2}\right). \tag{46}$$

Proof. Considering the $n=1$ case of (44) and then integrating using (19) produces the following definite integral

$$\int_z^1 P_\gamma^{(\alpha,\beta)}(w)\, dw = \frac{2}{\alpha+\beta+\gamma}\left(\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha)\Gamma(\gamma+2)} - P_{\gamma+1}^{(\alpha-1,\beta-1)}(z)\right).$$

Iterating the above expression n-times, reversing the order of the sum and utilizing standard properties of rising factorials such as (1)–(3) results in the term involving the terminating generalized hypergeometric ${}_3F_2$ series. The term involving the Jacobi function of the first kind is clear. The second identity is a direct consequence of Lemma 3.1. This completes the proof. □

Remark 3.12. Taking into account Lemma 3.1 as well as the previous result and then setting $(\alpha, \beta, \gamma) \mapsto (\alpha+n, \beta+n, \gamma-n)$, one obtains the following identity for the Taylor series of the Jacobi function of the first kind of order n – *1* at $z=1$:

$$\sum_{k=0}^{n-1} \frac{d^k}{dz^k} P_\gamma^{(\alpha,\beta)}(z)|_{z=1} \frac{(z-1)^k}{k!}$$

$$= \frac{\Gamma(\alpha+\gamma+1)\Gamma(-\alpha-\beta-\gamma)}{\Gamma(\alpha+n)(n-1)!}\left(\frac{z-1}{2}\right)^{n-1}$$

$$\times {}_3\boldsymbol{F}_2\left(\begin{matrix}1-n, 1-\alpha-n, 1\\ \gamma-n+2, 1-\alpha-\beta-\gamma-n\end{matrix}; \frac{2}{1-z}\right). \tag{47}$$

Theorem 3.13. *Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, with $\Re\gamma < -n$, $\Re(\alpha+\beta+\gamma) > n-1$, $z \in \mathbb{C}\backslash(-\infty, -1]$. Then*

$$\int_z^\infty \cdots \int_z^\infty (1+w)^\beta P_\gamma^{(\alpha,\beta)}(w)\, (dw)^n = \frac{2^n}{(-\alpha-\beta-\gamma)_n} P_{\gamma+n}^{(\alpha-n,\beta-n)}(z). \tag{48}$$

Proof. Considering the $n=1$ case of (44), taking into account the constraints considered in the statement and then integrating produces the following improper integral:

$$\int_z^\infty P_\gamma^{(\alpha,\beta)}(w)\, dw = \frac{-2}{\alpha+\beta+\gamma} P_{\gamma+1}^{(\alpha-1,\beta-1)}(z).$$

Iterating the above expression n-times completes the proof. □

If we consider the different hypergeometric representations for the Jacobi function of the first kind (15)–(18) and applying the derivative relations (9)–(12) one obtains the following alternative result.

Theorem 3.14. *Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, with $\alpha+\gamma \notin -\mathbb{N}$, $z \in \mathbb{C}\backslash(-\infty, -1]$. Then*

$$\left[(z-1)^2 \frac{d}{dz}\right]^n (z-1)^{\alpha+\beta+\gamma+1} P_\gamma^{(\alpha,\beta)}(z) = (\alpha+\beta+\gamma+1)_n (z-1)^{\alpha+\beta+\gamma+1+n} P_\gamma^{(\alpha,\beta+n)}(z), \tag{49}$$

$$\left[(z-1)^2 \frac{d}{dz}\right]^n \frac{1}{(z-1)^\gamma} P_\gamma^{(\alpha,\beta)}(z) = \frac{(-\alpha-\gamma)_n}{(z-1)^{\gamma-n}} P_{\gamma-n}^{(\alpha,\beta+n)}(z), \tag{50}$$

$$\left[(z-1)^2 \frac{d}{dz}\right]^n (z+1)^\beta (z-1)^{\alpha+\gamma+1} P_\gamma^{(\alpha,\beta)}(z) = 2^n(\gamma+1)_n (z+1)^{\beta-n} (z-1)^{\alpha+\gamma+1+n} P_{\gamma+n}^{(\alpha,\beta-n)}(z), \tag{51}$$

$$\left[(z-1)^2 \frac{d}{dz}\right]^n \frac{(z+1)^\beta}{(z-1)^{\beta+\gamma}} P_\gamma^{(\alpha,\beta)}(z) = 2^n(-\beta-\gamma)_n \frac{(z+1)^{\beta-n}}{(z-1)^{\beta-n+\gamma}} P_\gamma^{(\alpha,\beta-n)}(z). \tag{52}$$

Proof. First consider the $n=1$ case. Start with the representation (15) multiply it by $(z-1)^{\alpha+\beta+\gamma+1}$ and apply the derivative relation (9), to obtain

$$\frac{d}{dz}(z-1)^{\alpha+\beta+\gamma+1} P_\gamma^{(\alpha,\beta)}(z) = (\alpha+\beta+\gamma+1)(z-1)^{\alpha+\beta+\gamma} P_\gamma^{(\alpha,\beta+1)}(z).$$

Multiplying the expression by $(z-1)^2$ and iterating the identity produces (49). By repeating an analogous process for the expressions (16)–(18), the result follows. □

As a consequence of this result, we have the following derivative relations.

Corollary 3.15. *Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma, z \in \mathbb{C}$. The following identities hold:*

$$\left[(z+1)^2 \frac{d}{dz}\right]^n (z+1)^{\alpha+\beta+\gamma+1} P_\gamma^{(\alpha,\beta)}(z) = (\alpha+\beta+\gamma+1)_n (z+1)^{\alpha+\beta+\gamma+1+n} P_\gamma^{(\alpha+n,\beta)}(z), \tag{53}$$

$$\left[(z+1)^2 \frac{d}{dz}\right]^n \frac{1}{(z+1)^\gamma} P_\gamma^{(\alpha,\beta)}(z) = \frac{(1+\beta+\gamma-n)_n}{(z+1)^{\gamma-n}} P_{\gamma-n}^{(\alpha+n,\beta)}(z), \tag{54}$$

$$\left[(z+1)^2\frac{\mathrm{d}}{\mathrm{d}z}\right]^n (z-1)^{\alpha}(z+1)^{\beta+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(z) = 2^n(\gamma+1)_n(z-1)^{\alpha-n}(z+1)^{\beta+\gamma+n}P_{\gamma+n}^{(\alpha-n,\beta)}(z), \quad (55)$$

$$\left[(z+1)^2\frac{\mathrm{d}}{\mathrm{d}z}\right]^n \frac{(z-1)^{\alpha}}{(z+1)^{\alpha+\gamma}}P_{\gamma}^{(\alpha,\beta)}(z) = (-2)^n(-\alpha-\gamma)_n\frac{(z-1)^{\alpha-n}}{(z+1)^{\alpha-n+\gamma}}P_{\gamma}^{(\alpha-n,\beta)}(z). \quad (56)$$

Proof. By starting with (12) using

$$w=\frac{z-1}{z+1}, \quad \frac{\mathrm{d}}{\mathrm{d}w}=\frac{(z+1)^2}{2}\frac{\mathrm{d}}{\mathrm{d}z}, \quad (57)$$

and substituting in (15)–(17) and (18) in an as we did in the Theorem 3.14, one obtains the above expressions which completes the proof. □

We are also able to obtain some interesting Rodrigues-type relations for Jacobi polynomials.

Corollary 3.16 (Rodrigues-type formula). *Let $n\in\mathbb{N}_0$, $\alpha,\beta,z\in\mathbb{C}$. The Jacobi polynomial admits the following Rodrigues-type relation:*

$$P_n^{(\alpha,\beta)}(z)=\frac{1}{2^n n!}\frac{1}{(z-1)^{\alpha}(z+1)^{\beta+n+1}} \times\left[(z+1)^2\frac{\mathrm{d}}{\mathrm{d}z}\right]^n (z-1)^{\alpha+n}(z+1)^{\beta+1} \quad (58)$$

$$=\frac{1}{2^n n!}\frac{1}{(z-1)^{\alpha+n+1}(z+1)^{\beta}} \times\left[(z-1)^2\frac{\mathrm{d}}{\mathrm{d}z}\right]^n (z-1)^{\alpha}(z+1)^{\beta+n+1}. \quad (59)$$

Proof. Setting $\gamma\mapsto 0$ and $\beta\mapsto\beta+n$ in (51) the first identity follows. Setting $\gamma\mapsto 0$ and $\alpha\mapsto\beta+n$ in (55), the second identity follows. □

Theorem 3.17. *Let $n\in\mathbb{N}_0$, $\alpha,\beta,\gamma\in\mathbb{C}$, with $\alpha+\gamma\notin-\mathbb{N}$, $z\in\mathbb{C}\backslash(-\infty,-1]$. Then one has the following multi-integral representations for the Jacobi function of the first kind:*

$$\int_1^z\cdots\int_1^z (w-1)^{\alpha+\beta+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(w)\left[(w-1)^{-2}\,\mathrm{d}w\right]^n = \frac{(z-1)^{\alpha+\beta+\gamma+1-n}}{(\alpha+\beta+\gamma-n+1)_n}P_{\gamma}^{(\alpha,\beta-n)}(z), \quad (60)$$

$$\int_1^z\cdots\int_1^z (w+1)^{\beta}(w-1)^{\alpha+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(w)\left[(w-1)^{-2}\,\mathrm{d}w\right]^n = \frac{(z+1)^{\beta+n}(z-1)^{\alpha+\gamma-n+1}}{2^n(\gamma-n+1)_n}P_{\gamma-n}^{(\alpha,\beta+n)}(z), \quad (61)$$

$$\int_1^z\cdots\int_1^z (w-1)^{\alpha}(w+1)^{\beta+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(w)\left[(w+1)^{-2}\,\mathrm{d}w\right]^n = \frac{(z-1)^{\alpha+n}(z+1)^{\beta+\gamma-n+1}}{2^n(\gamma-n+1)_n}P_{\gamma-n}^{(\alpha+n,\beta)}(z), \quad (62)$$

$$\int_1^z\cdots\int_1^z \frac{(w-1)^{\alpha}}{(w+1)^{\alpha+\gamma}}P_{\gamma}^{(\alpha,\beta)}(w)\left[(w+1)^{-2}\,\mathrm{d}w\right]^n = \frac{(z-1)^{\alpha+n}}{2^n(1+\alpha+\gamma)_n(z+1)^{\alpha+n+\gamma}}P_{\gamma}^{(\alpha+n,\beta)}(z), \quad (63)$$

where $\Re(\alpha+\beta+\gamma+1)>n$, $\Re(\alpha+\gamma+1)>n$, $\Re(\alpha+n)>0, \Re(\alpha+n)>0$, *respectively.*

Proof. Consider the $n=1$ case of (49) and then integrate both sides using the fundamental theorem of calculus. This produces the following definite integral

$$\int_1^z (w-1)^{\alpha+\beta+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(w)\ (w-1)^{-2}\,\mathrm{d}w = \frac{(z-1)^{\alpha+\beta+\gamma+1}}{\alpha+\beta+\gamma}P_{\gamma}^{(\alpha,\beta-1)}(z).$$

Iterating the above expression n-times completes the proof. The process for the remaining cases, i.e., for the cases starting with (51), (55), and (56), is similar and we will omit their proofs. This completes the proof. □

Theorem 3.18. *Let $n\in\mathbb{N}_0$, $\alpha,\beta,\gamma\in\mathbb{C}$, such that $\alpha+\gamma\notin-\mathbb{N}$, $z\in\mathbb{C}\backslash(-\infty,-1]$. Then one has the following multi-integral representations for the Jacobi function of the first kind:*

$$\int_1^z\cdots\int_1^z \frac{P_{\gamma}^{(\alpha,\beta)}(w)}{(w-1)^{\gamma}}\left[(w-1)^{-2}\,\mathrm{d}w\right]^n = \frac{(-1)^n}{(\alpha+\gamma+1)_n}\frac{P_{\gamma+n}^{(\alpha,\beta-n)}(z)}{(z-1)^{\gamma+n}}, \quad (64)$$

$$\int_1^z\cdots\int_1^z \frac{(w+1)^{\beta}}{(w-1)^{\gamma+\beta}}P_{\gamma}^{(\alpha,\beta)}(w)\left[(w-1)^{-2}\,\mathrm{d}w\right]^n = \frac{(-1)^n}{2^n(\beta+\gamma+1)_n}\frac{(z+1)^{\beta+n}}{(z-1)^{\gamma+\beta}}P_{\gamma}^{(\alpha,\beta)}(z), \quad (65)$$

where $\Re\gamma<-n$, $\Re(\gamma+\beta)<0$, respectively.

The proof is analogous to those carried out previously, and we leave this to the reader.

Theorem 3.19. *Let $n\in\mathbb{N}_0$, $\alpha,\beta,\gamma\in\mathbb{C}$, such that $\alpha+\gamma\notin-\mathbb{N}$, $z\in\mathbb{C}\backslash(-\infty,-1]$. Then one has the following multi-integral representations for the Jacobi function of the first kind:*

$$\int_1^z\cdots\int_1^z (w+1)^{\alpha+\beta+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(w)\left[(w+1)^{-2}\,\mathrm{d}w\right]^n = \frac{(z+1)^{\alpha+\beta+\gamma+1-n}P_{\gamma}^{(\alpha-n,\beta)}(z)}{(\alpha+\beta+\gamma+1-n)_n} - \frac{2^{\alpha+\beta+\gamma+1-n}\Gamma(\alpha+\gamma)}{(\alpha+\beta+\gamma)\Gamma(\alpha,\gamma+1,n)}\left(\frac{z-1}{z+1}\right)^{n-1} \times{}_3F_2\left(\begin{matrix}-n+1,1-\alpha,1\\1-\alpha-\gamma,1-\alpha-\beta-\gamma\end{matrix};\frac{z+1}{z-1}\right), \quad (66)$$

$$\int_1^z \cdots \int_1^z \frac{1}{(w+1)^\gamma} P_\gamma^{(\alpha,\beta)}(w) \left[(w+1)^{-2}\ \mathrm{d}w\right]^n = \frac{P_{\gamma+n}^{(\alpha-n,\beta)}(z)}{(\beta+\gamma+1)_n (z+1)^{\gamma+n}} - \frac{\Gamma(\alpha+\gamma+1)}{2^{\gamma+n}(\beta+\gamma+1)\Gamma(\alpha,\gamma+2,n)} \left(\frac{z-1}{z+1}\right)^{n-1} \times {}_3F_2\left(\begin{matrix} -n+1, 1-\alpha, 1 \\ 2+\gamma, 2+\beta+\gamma \end{matrix}; \frac{z+1}{z-1}\right). \tag{67}$$

Proof. Considering the $n=1$ case of (54) and then integrating produces the definite integral

$$\int_1^z (w+1)^{\alpha+\beta+\gamma+1} P_\gamma^{(\alpha,\beta)}(w) \frac{\mathrm{d}w}{(w+1)^2} = \frac{(w+1)^{\alpha+\beta+\gamma} P_\gamma^{(\alpha-1,\beta)}(z) - 2^{\alpha+\beta+\gamma} P_\gamma^{(\alpha-1,\beta)}(1)}{\beta+\gamma+1},$$

which is due to (19). Iterating the above expression n-times completes the proof of the first identity. Using (1), (2), (19), taking into account

$$\int_1^z \left(\frac{w-1}{w+1}\right)^k \frac{\mathrm{d}w}{(w+1)^2} = \frac{1}{2(k+1)} \left(\frac{z-1}{z+1}\right)^{k+1},$$

where $k = 0, 1, \ldots$, and reversing the finite series, i.e., for any non-negative integer m,

$$\sum_{k=0}^m \frac{(a_1,\ldots,a_{r+1})_k}{(b_1,\ldots,b_r)_k} \frac{z^k}{k!} = \frac{(a_1,\ldots,a_{r+1})_m}{(b_1,\ldots,b_r)_m} \frac{z^m}{m!} \times {}_{r+2}F_{r+1}\left(\begin{matrix} -m, 1-m-b_1, \ldots, 1-m-b_r, 1 \\ 1-m-a_1, \ldots, 1-m-a_{r+1} \end{matrix}; \frac{1}{z}\right),$$

the result follows. □

3.2. The Jacobi functions of the second kind

In this section we derive multi-integral representations for the Jacobi function of the second kind. Since both the Jacobi function of the first kind and the Jacobi function of the second kind are strongly connected (27), we expect to obtain similar multi-integrals to those obtained in the previous section.

Theorem 3.20. *Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash[-1,1]$, with $\Re\alpha, \Re\beta > -1, \Re\gamma > n$. Then*

$$\int_z^\infty \cdots \int_z^\infty (w-1)^\alpha (1+w)^\beta Q_\gamma^{(\alpha,\beta)}(w)\ (\mathrm{d}w)^n = \frac{(z-1)^{\alpha+n}(1+z)^{\beta+n}}{2^n(\gamma-n+1)_n} Q_{\gamma-n}^{(\alpha+n,\beta+n)}(z). \tag{68}$$

Proof. Considering the $n=1$ case of (31) and then integrating produces the following definite integral

$$\int_z^\infty (w-1)^\alpha (1+w)^\beta Q_\gamma^{(\alpha,\beta)}(w) \mathrm{d}w = \frac{1}{2\gamma} \lim_{w\to\infty} \Big((z-1)^{\alpha+1}(1+z)^{\beta+1} Q_{\gamma+1}^{(\alpha+1,\beta+1)}(z) - (w-1)^{\alpha+1}(1+w)^{\beta+1} Q_{\gamma+1}^{(\alpha+1,\beta+1)}(w)\Big) = \frac{(z-1)^{\alpha+1}(1+z)^{\beta+1}}{2\gamma} Q_{\gamma+1}^{(\alpha+1,\beta+1)}(z).$$

Iterating the above expression n-times completes the proof. □

Theorem 3.21. *Let $n \in \mathbb{N}_0$, α, β, γ, $z \in \mathbb{C}$. Then*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} (z-1)^\alpha Q_\gamma^{(\alpha,\beta)}(z) = (-\alpha-\gamma)_n (z-1)^{\alpha-n} Q_\gamma^{(\alpha-n,\beta+n)}(z). \tag{69}$$

Proof. First we prove the $n=1$ case. Consider (22), multiply this expression by $(z-1)^\alpha$ and differentiate with respect to z. This obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(z-1)^\alpha Q_\gamma^{(\alpha,\beta)}(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z+1)^{\beta+\gamma+1}} \times {}_2\boldsymbol{F}_1\left(\begin{matrix} \gamma+1, \beta+\gamma+1 \\ \alpha+\beta+2\gamma+2 \end{matrix}; \frac{2}{1+z}\right) \\
&= -\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+2)}{\Gamma(\alpha+\beta+2\gamma+2)(z+1)^{\beta+\gamma+2}} \times \sum_{k=0}^\infty \frac{(\gamma+1,\beta+\gamma+2)_k}{(\alpha+\beta+2\gamma+2)_k k!} \left(\frac{2}{1+z}\right)^k \\
&= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+2)}{(z+1)^{\beta+\gamma+1}} \times {}_2\boldsymbol{F}_1\left(\begin{matrix} \gamma+1, \beta+\gamma+2 \\ \alpha+\beta+2\gamma+2 \end{matrix}; \frac{2}{1+z}\right) \\
&= -(\alpha+\beta)(z-1)^{\alpha-1} Q_\gamma^{(\alpha-1,\beta+1)}(z).
\end{aligned}$$

The nth derivative case is obtained by iterating the above procedure. □

Theorem 3.22. *Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash[-1,1]$, with $\Re\alpha > -1$, $\Re\beta > n-1$, $\Re(\beta+\gamma+1) > n$. Then*

$$\int_z^\infty \cdots \int_z^\infty (w-1)^\alpha Q_\gamma^{(\alpha,\beta)}(w)\ (\mathrm{d}w)^n = \frac{(z-1)^{\alpha+n}}{(\alpha+\gamma+1)_n} Q_\gamma^{(\alpha+n,\beta-n)}(z). \tag{70}$$

Proof. Considering the $n=1$ case of (69) and then integrating produces the following definite integral

$$\int_z^\infty (w-1)^\alpha Q_\gamma^{(\alpha,\beta)}(w)\ \mathrm{d}w$$

$$= \frac{1}{\alpha+\gamma+1}(z-1)^{\alpha+1}Q_\gamma^{(\alpha+1,\beta-1)}(z).$$

Iterating the above expression n-times completes the proof. □

Theorem 3.23. *Let $n \in \mathbb{N}_0$, $\alpha,\beta,\gamma,z \in \mathbb{C}$. Then*

$$\frac{d^n}{dz^n}Q_\gamma^{(\alpha,\beta)}(z) = (-2)^{-n}(\alpha+\beta+\gamma+1)_n Q_{\gamma-n}^{(\alpha+n,\beta+n)}(z). \tag{71}$$

Proof. The $n=1$ case follows from (30) and for $n>1$, the result is obtained by iterating the above procedure. □

Theorem 3.24. *Let $n \in \mathbb{N}_0$, $\alpha,\beta,\gamma \in \mathbb{C}$, $z \in \mathbb{C}\backslash[-1,1]$, with $\Re\alpha > n-1$, $\Re\beta > n-1$, $\Re(\alpha+\beta+\gamma+1) > n$. Then*

$$\int_z^\infty \cdots \int_z^\infty Q_\gamma^{(\alpha,\beta)}(w)\ (dw)^n = \frac{2^n}{(\alpha+\beta+\gamma-n+1)_n}Q_{\gamma+n}^{(\alpha-n,\beta-n)}(z). \tag{72}$$

Proof. Considering the $n=1$ case of (71) and then integrating produces the following definite integral

$$\int_z^\infty Q_\gamma^{(\alpha,\beta)}(w)\ dw = \frac{2}{\alpha+\beta+\gamma}Q_{\gamma+1}^{(\alpha-1,\beta-1)}(z).$$

Iterating the above expression n-times completes the proof. □

If we consider, as we did in the case of the Jacobi functions of the first kind, the different hypergeometric representations for the Jacobi function of the second kind (20)–(23) and applying the derivative relations (9)–(12) one obtains the following result.

Theorem 3.25. *Let $n \in \mathbb{N}_0$, $\alpha,\beta,\gamma \in \mathbb{C}$, with $\alpha+\gamma \notin -\mathbb{N}$, $z \in \mathbb{C}\backslash[-1,1]$. Then*

$$\left[(z-1)^2\frac{d}{dz}\right]^n (z-1)^{\alpha+\beta+\gamma+1}Q_\gamma^{(\alpha,\beta)}(z) = (\alpha+\beta+\gamma+1)_n(z-1)^{\alpha+\beta+\gamma+1+n}Q_\gamma^{(\alpha,\beta+n)}(z), \tag{73}$$

$$\left[(z-1)^2\frac{d}{dz}\right]^n \frac{1}{(z-1)^\gamma}Q_\gamma^{(\alpha,\beta)}(z) = \frac{(-\alpha-\gamma)_n}{(z-1)^{\gamma-n}}Q_{\gamma-n}^{(\alpha,\beta+n)}(z), \tag{74}$$

$$\left[(z-1)^2\frac{d}{dz}\right]^n (z+1)^\beta(z-1)^{\alpha+\gamma+1}Q_\gamma^{(\alpha,\beta)}(z) = 2^n(\gamma+1)_n(z+1)^{\beta-n}(z-1)^{\alpha+\gamma+1+n}Q_{\gamma+n}^{(\alpha,\beta-n)}(z), \tag{75}$$

$$\left[(z-1)^2\frac{d}{dz}\right]^n \frac{(z+1)^\beta}{(z-1)^{\beta+\gamma}}Q_\gamma^{(\alpha,\beta)}(z) = 2^n(-\beta-\gamma)_n\frac{(z+1)^{\beta-n}}{(z-1)^{\beta-n+\gamma}}Q_\gamma^{(\alpha,\beta-n)}(z). \tag{76}$$

Proof. The proof of these results are analogous to the proof of Theorem 3.14 and we will leave this to the reader. □

Corollary 3.26. *Let $n \in \mathbb{N}_0$, $\alpha,\beta,\gamma,z \in \mathbb{C}$. Then*

$$\left[(z+1)^2\frac{d}{dz}\right]^n (z+1)^{\alpha+\beta+\gamma+1}Q_\gamma^{(\alpha,\beta)}(z) = (\alpha+\beta+\gamma+1)_n(z+1)^{\alpha+\beta+\gamma+1+n}Q_\gamma^{(\alpha+n,\beta)}(z), \tag{77}$$

$$\left[(z+1)^2\frac{d}{dz}\right]^n \frac{1}{(z+1)^\gamma}Q_\gamma^{(\alpha,\beta)}(z) = \frac{(1+\beta+\gamma-n)_n}{(z+1)^{\gamma-n}}Q_{\gamma-n}^{(\alpha+n,\beta)}(z), \tag{78}$$

$$\left[(z+1)^2\frac{d}{dz}\right]^n (z-1)^\alpha(z+1)^{\beta+\gamma+1}Q_\gamma^{(\alpha,\beta)}(z) = 2^n(\gamma+1)_n(z-1)^{\alpha-n}(z+1)^{\beta+\gamma+n}Q_{\gamma+n}^{(\alpha-n,\beta)}(z), \tag{79}$$

$$\left[(z+1)^2\frac{d}{dz}\right]^n \frac{(z-1)^\alpha}{(z+1)^{\alpha+\gamma}}Q_\gamma^{(\alpha,\beta)}(z) = (-2)^n(-\alpha-\gamma)_n\frac{(z-1)^{\alpha-n}}{(z+1)^{\alpha-n+\gamma}}Q_\gamma^{(\alpha-n,\beta)}(z). \tag{80}$$

Proof. The proof is analogous to the proof of Corollary 3.15 and we leave this to the reader. □

Remark 3.27. One should note the interesting work by Loyal Durand which was recently presented in (Durand, 2022). In that paper, the author derives many of the multi-integral representations appearing in (Cohl & Costas-Santos, 2020) for associated Legendre and Ferrers functions from more general relations involving non-integer changes in the order obtained using fractional Lie group operator methods developed earlier for SO(1,2), E(1,2), and its conformal extension SO(3). It is therefore probable that similar Lie group theoretic methods could be used to derive the multi-integral representations contained within the present paper for Jacobi functions of the first and second kind. This could potentially shed some interesting light on the Lie groups which are associated with general Jacobi functions of the first and second kind, as well as these functions.

Disclosure statement

No potential conflict of interest was reported by the authors.

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