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On a generalization of the Rogers generating function

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ABSTRACT

We derive a generalization of the Rogers generating function for the continuous q-ultraspherical/Rogers polynomials whose coefficient is a ${}_2\phi_1$. From that expansion, we derive corresponding specialization and limit transition expansions for the continuous q-Hermite, continuous q-Legendre, Laguerre, and Chebyshev polynomials of the first kind. Using a generalized expansion of the Rogers generating function in terms of the Askey–Wilson polynomials by Ismail & Simeonov whose coefficient is a ${}_8\phi_7$, we derive corresponding generalized expansions for the Wilson, continuous q-Jacobi, and Jacobi polynomials. By comparing the coefficients of the Askey–Wilson expansion to our continuous q-ultraspherical/Rogers expansion, we derive a new quadratic transformation for basic hypergeometric functions which relates an ${}_8\phi_7$ to a ${}_2\phi_1$. We also obtain several definite integral representations which correspond to the above mentioned expansions through the use of orthogonality.

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1. Introduction

In the context of generalized hypergeometric orthogonal polynomials, the first author and collaborators developed in [5, (2.1)] a series rearrangement technique which we utilize in the present context to produce a generalization of the generating function for the continuous q-ultraspherical/Rogers polynomials. This technique is valid for a larger class of hypergeometric orthogonal polynomials. For instance, in [4], we applied this same technique for the Jacobi polynomials and in [7], we extended this technique to many generating functions for the Jacobi, Gegenbauer, Laguerre, and Wilson polynomials.

The series rearrangement technique combines a connection relation with a generating function, resulting in a series with multiple sums. The order of summations are then rearranged and the result often simplifies to produce a generalized generating function whose coefficients are given in terms of generalized or basic

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hypergeometric functions. This technique is especially productive when using connection relations with one free parameter, since the connection coefficient is most often a product of Pochhammer or q-Pochhammer symbols.

Basic hypergeometric orthogonal polynomials with more than one free parameter, such as the Askey–Wilson polynomials, have multi-parameter connection relations. These connection relations are given by single or multiple summation expressions. For the Askey–Wilson polynomials, the connection relation with four free parameters is given as a basic double hypergeometric series. The fact that the four free parameter connection coefficient for the Askey–Wilson polynomials is given by a double sum was known to Askey and Wilson as far back as 1985 (see [17, Section 16.4]). When our series rearrangement technique is applied to cases with more than one free parameter, the resulting coefficients of the generalized generating function are rarely given in terms of a basic hypergeometric series. The more general problem of generalized generating functions with more than one free parameter requires the theory of multiple basic hypergeometric series and is not treated in this paper.

Through analysis of an Askey-Wilson polynomial expansion due to Ismail & Simeonov [19], we construct various expansions as follows. In Section 3, we construct an expansion for the Wilson polynomials. In Section 4, we construct an expansion for the continuous q-Jacobi polynomials. In Section 5, we construct expansions for the continuous q-ultraspherical/Rogers polynomials, derive some specialization and limit transition formulas from the derived expansion, and prove a new quadratic transformation for basic hypergeometric functions. In Section 6, we compute some new definite integrals corresponding to our derived generalized generating function expansions using orthogonality for the orthogonal polynomials we have studied.

In addition to being of independent interest, this investigation was motivated by an application of generalized generating functions in the non-q regime [4,5]. This would be the generation of q-polyspherical addition theorems in terms of a product of q-zonal harmonics. In order to compute these q-analogues, one would need to derive a q-analogue of the addition theorem for the hyperspherical harmonics (see [30]; see also [9, Section 10.2.1])

$$C_n^{\frac{d}{2}-1}(\cos\gamma) = \frac{2(d-2)\pi^{\frac{d}{2}}}{(2n+d-2)\Gamma(\frac{d}{2})}\sum_K Y_n^K(\widehat{\mathbf{x}})\overline{Y_n^K(\widehat{\mathbf{x}}')},$$

where, for a given value of $n \in \mathbb{N}_0 := \{0\}\cup\mathbb{N} := \{0\}\cup\{1,2,\ldots\}$, C_n^μ is the Gegenbauer polynomial, K stands for a set of $(d-2)$-quantum numbers identifying normalized hyperspherical harmonics $Y_n^K : \mathbf{S}^{d-1} \to \mathbb{C}$, and γ is the separation angle between two arbitrary vectors $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$. The Gegenbauer polynomials can be defined using the Gauss hypergeometric function [24, (18.5.9)], and in terms of a symmetric Jacobi polynomial $P_n^{(\alpha,\beta)}$, [20, (9.8.19)],

$$C_n^\mu(x) := \frac{(2\mu)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, 2\mu+n\\ \mu+\frac12\end{matrix}; \frac{1-x}{2}\right) = \frac{(2\mu)_n}{(\mu+\frac12)_n}P_n^{(\mu-\frac12,\mu-\frac12)}(x). \tag{1.1}$$

One would also need q-analogues of a fundamental solution of the polyharmonic equation, and Laplace's expansion

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|^{d-2}} = \sum_{l=0}^\infty \frac{r_<^l}{r_>^{l+d-2}}C_l^{\frac{d}{2}-1}(\cos\gamma),$$

which is the $q \uparrow 1^-$ limit of the generating function for the continuous q-ultraspherical/Rogers polynomials, hereafter referred to as the *Rogers generating function* (see (3.3) below). These analogues do not exist in the literature, however they may be found by using material from [12], [16], [23, Section 3], which we will attempt in future publications. Addition theorems for the continuous q-ultraspherical/Rogers polynomials should also be useful here [22].

2. Preliminaries

Throughout the paper, we adopt the following notation to indicate sequential positive and negative elements, in a list of elements, namely

$$\pm a := \{a, -a\}.$$

If the symbol $\pm$ appears in an expression, but not in a list, it is to be treated as normal.

In order to obtain our derived identities, we rely on properties of the Pochhammer and q-Pochhammer symbols, also called shifted and q-shifted factorials respectively. The Pochhammer symbol for $a, b \in \mathbb{C}$, with $\Re b > 0$, is defined naturally by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)},$$

where $a + b \notin -\mathbb{N}_0$. Note that if $\Re b < 0$ then $(a)_b := 1/(a+b)_{-b}$. For the q-Pochhammer symbol, $a \in \mathbb{C}$, $|q| < 1$, we define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n), \tag{2.1}$$

then for $b \in \mathbb{C}$, [20, (1.8.9)]

$$(a;q)_b := \frac{(a;q)_\infty}{(aq^b;q)_\infty}, \tag{2.2}$$

where the principal value of q^b will always be taken and $(aq^b;q)_\infty \neq 0$. Therefore for $n \in \mathbb{N}_0$, one has [20, (1.8.8)]

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{2.3}$$

where $(aq^n;q)_\infty \neq 0$. We will also use the common notational product conventions

$$(a_1, \ldots, a_k)_b := (a_1)_b \cdots (a_k)_b,$$
$$(a_1, \ldots, a_k; q)_b := (a_1;q)_b \cdots (a_k;q)_b.$$

We define the q-factorial as [10, (1.2.44)]

$$[0]_q! := 1, \ [n]_q! := [1]_q [2]_q \cdots [n]_q, \qquad n \in \mathbb{N},$$

where the q-number is defined as [20, (1.8.1)]

$$[z]_q := \frac{1 - q^z}{1 - q}, \qquad z \in \mathbb{C}.$$

Note that $[n]_q! = (q;q)_n/(1-q)^n$.

The following properties for the q-Pochhammer symbol can be found in Koekoek et al. (2010) [20, (1.8.7), (1.8.10-11), (1.8.14), (1.8.19), (1.8.21-22)], namely for appropriate values of a, and $n, k \in \mathbb{N}_0$,

$$(a;q)_{n+k} = (a;q)_k (aq^k;q)_n = (a;q)_n (aq^n;q)_k, \tag{2.4}$$

$$(a^2;q^2)_n = (\pm a;q)_n. \tag{2.5}$$

Observe that by using (2.3) and (2.5), one has

$$(aq^n;q)_n = \frac{(\pm\sqrt{a}, \pm\sqrt{aq};q)_n}{(a;q)_n}. \tag{2.6}$$

Lemma 2.1. *Let $n \in \mathbb{N}_0$, $q, a, b \in \mathbb{C}$, $0 < |q| < 1$. Then*

$$(a;q)_{n+b} = (a;q)_n (aq^n;q)_b. \tag{2.7}$$

Proof. Follows from the identity (2.2) and (2.3). □

Lemma 2.2. *Let $q, a, b \in \mathbb{C}$, $0 < |q| < 1$. Then*

$$\lim_{q\uparrow 1^-} \frac{(q^a;q)_b}{(1-q)^b} = (a)_b. \tag{2.8}$$

Proof. Define the q-gamma function Γ_q by [20, (1.9.1)]

$$\Gamma_q(x) := \frac{(1-q)^{1-x}(q;q)_\infty}{(q^x;q)_\infty},$$

and the arbitrary q-Pochhammer symbol by (2.2). Observe that, by using (2.7), if $\Re b < 0$ then

$$(a;q)_b := \frac{1}{(aq^b;q)_{-b}}. \tag{2.9}$$

- If $a + b \in -\mathbb{N}_0$ then the result is straightforward by definition since $(-n)_n = 0$ and $(q^{-n};q)_n = 0$ for any $n \in \mathbb{N}_0$.
- If $\Re b > 0$ then

$$\lim_{q\uparrow 1^-} \frac{(q^a;q)_b}{(1-q)^b} = \lim_{q\uparrow 1^-} \frac{(q^a;q)_\infty}{(1-q)^b (q^{a+b};q)_\infty} = \lim_{q\uparrow 1^-} \frac{\Gamma_q(a+b)}{\Gamma_q(a)} = (a)_b,$$

  since [20, Section 1.9] $\lim_{q\uparrow 1^-} \Gamma_q(x) = \Gamma(x)$.
- If $\Re b < 0$ then

$$\lim_{q\uparrow 1^-} \frac{(q^a;q)_b}{(1-q)^b} \overset{(2.9)}{=} \lim_{q\uparrow 1^-} \frac{(1-q)^{-b}}{(q^{a+b};q)_{-b}} = \lim_{q\uparrow 1^-} \frac{(1-q)^{-b}(q^a;q)_\infty}{(q^{a+b};q)_\infty} = \lim_{q\uparrow 1^-} \frac{\Gamma_q(a+b)}{\Gamma_q(a)} = (a)_b.$$

This completes the proof. □

We also take advantage of the q-binomial theorem [20, (1.11.1)]

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, \tag{2.10}$$

where we have used (2.1). The basic hypergeometric series, which we often use, is defined as [20, (1.10.1)]

$$ {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right) := \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, \tag{2.11} $$

where $\binom{k}{2} = \frac{1}{2}k(k-1)$ is a binomial coefficient. Let us now prove some inequalities that we will use in the sequel.

Lemma 2.3. *Let $j \in \mathbb{N}$, $k, n \in \mathbb{N}_0$, $z \in \mathbb{C}$, $\Re u > 0$, $v \geq 0$, and $0 < |q| < 1$. Then*

$$ \left|\frac{(q^u;q)_j}{(1-q)^j}\right| \geq \left|[\Re u]_q [j-1]_q!\right|, \tag{2.12} $$

$$ \left|\frac{(q^u;q)_n}{(q;q)_n}\right| \leq \left|[1+n]_q^u\right|, \tag{2.13} $$

$$ \left|\frac{(q^{v+k};q)_n}{(q^{u+k};q)_n}\right| \leq \left|\frac{[n+1]_q^{v+1}}{[\Re(u)]_q}\right|. \tag{2.14} $$

Proof. If $0 < |q| < 1$ then

$$ \left|\frac{(q^u;q)_j}{(1-q)^j}\right| = \prod_{k=1}^{j-1}\left|\frac{1-q^{u+k-1}}{1-q}\right| \geq \left|\frac{1-q^u}{1-q}\right| \prod_{k=1}^{j-1}\left|\frac{1-q^k}{1-q}\right| \geq \left|[\Re(u)]_q [j-1]_q!\right|. $$

This completes the proof of (2.12). Choose $m \in \mathbb{N}_0$ such that $m \leq u \leq m+1$. Then $q^{m+1} \leq q^u$, so

$$ \left|\frac{(q^u;q)_n}{(q;q)_n}\right| = \prod_{k=0}^{n-1}\left|\frac{1-q^{u+k}}{1-q^{1+k}}\right| \leq \prod_{k=1}^{n}\left|\frac{1-q^{m+k}}{1-q^k}\right| = \prod_{k=1}^{m}\left|\frac{1-q^{n+k}}{1-q^k}\right| \leq \left|[n+1]_q^m\right| \leq \left|[n+1]_q^u\right|. $$

This completes the proof of (2.13). Without loss of generality we assume $u > 0$. If $v \leq u$ then the inequality is clear, so let us assume that $0 < u < v$. Since $0 < |q| < 1$ and for $t \geq 0$,

$$ \frac{t+v}{t+u} \leq \frac{v}{u}, $$

and we have

$$ \left|\frac{(q^{v+k};q)_n}{(q^{u+k};q)_n}\right| \leq \left|\frac{(q^v)_n}{(q^u)_n}\right| \leq \left|\frac{1}{[u]_q}\frac{(q^v)_n}{[n-1]_q!(1-q)^n}\right|. $$

Choose $m \in \mathbb{N}$ so that $m-1 < v \leq m$. Then

$$ \left|\frac{(q^{v+k};q)_n}{(q^{u+k};q)_n}\right| \leq \left|\frac{1}{[u]_q}\frac{[n]_q(q^m;q)_n}{(q;q)_n}\right| = \left|\frac{1}{[u]_q}\frac{[n]_q(q^n;q)_{m-1}}{(q;q)_{m-1}}\right| \leq \left|\frac{1}{[u]_q}[n]_q[n+1]_q^{m-1}\right| \leq \left|\frac{1}{[u]_q}[n+1]_q^{v+1}\right|. $$

This completes the proof of (2.14). □

As we have mentioned previously, we need to assure that one can rearrange certain series expressions. The following result is necessary in order to guarantee the validity of such actions. If an infinite series is absolutely convergent then all of its rearrangements converge to the same sum.

Lemma 2.4. *Let $n, k \in \mathbb{N}_0$, $\mathbf{a}$, $\mathbf{b}$, be sets of parameters associated with the polynomial sequences (p_n) and $(\tilde{p}_n)$. Furthermore, assume that the polynomial sequences satisfy the following identities*

$$\tilde{p}_n(x;\mathbf{a}) = \sum_{k=0}^{n} c_{k,n}(\mathbf{a},\mathbf{b})\, p_k(x;\mathbf{b}), \qquad \sum_{n=0}^{\infty} a_n(\mathbf{a})p_n(x;\mathbf{a}) = F(x,\mathbf{a}),$$

for some coefficients $a_n, c_{k,n} \in \mathbb{C}$*. Then one can justify the rearrangement of the two series as*

$$\sum_{n=0}^{\infty} a_n(\mathbf{a}) \sum_{k=0}^{n} c_{k,n}(\mathbf{a},\mathbf{b})p_k(x;\mathbf{b}) = \sum_{k=0}^{\infty} p_k(x;\mathbf{b}) \sum_{n=k}^{\infty} a_n(\mathbf{a})c_{k,n}(\mathbf{a},\mathbf{b}),$$

if one can verify

$$\sum_{n=0}^{\infty} |a_n(\mathbf{a})| \sum_{k=0}^{n} \left|c_{k,n}(\mathbf{a},\mathbf{b})p_k(x;\mathbf{b})\right| < \infty.$$

3. Expansions for the Askey–Wilson and Wilson polynomials

The Askey–Wilson polynomials can be defined as [20, (14.1.1)]

$$p_n(x;\mathbf{a}|q) := a_1^{-n}(a_1a_2, a_1a_3, a_1a_4;q)_n \; {}_4\phi_3\left(\begin{matrix} q^{-n}, a_1a_2a_3a_4q^{n-1}, a_1e^{i\theta}, a_1e^{-i\theta} \\ a_1a_2, a_1a_3, a_1a_4 \end{matrix} ;q,q\right),$$

where $x = \cos\theta$, $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$. In [19, Theorem 4.2] the following Askey–Wilson polynomial expansion of the Rogers generating function [20, (14.10.27)] is proven.

Theorem 3.1 *(Ismail & Simeonov (2015)). Let* $t, \beta, q \in \mathbb{C}$, $\max\{|a_1|, |a_2|, |a_3|, |a_4|, |t|, |q|\} < 1$, $x = \cos\theta \in (-1, 1)$*. Then*

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta};q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty} c_n(\beta, t, \mathbf{a};q)p_n(x;\mathbf{a}|q), \tag{3.1}$$

where

$$c_n(\beta,t,\mathbf{a};q) := \frac{t^n(\beta;q)_n(q^na_1\beta t, q^na_2\beta t, q^na_3\beta t, q^na_1a_2a_3t;q)_\infty}{(q, q^{n-1}a_1a_2a_3a_4;q)_n(a_1t, a_2t, a_3t, q^{2n}a_1a_2a_3\beta t;q)_\infty} \mathsf{F}_n^{\beta,t,\mathbf{a};q},$$

$$\begin{aligned}\mathsf{F}_n^{\beta,t,\mathbf{a};q} &:= {}_8\phi_7\left(\begin{matrix} q^{2n-1}a_1a_2a_3\beta t, \pm q^{n+\frac12}(a_1a_2a_3\beta t)^{\frac12}, q^na_1a_2, q^na_1a_3, q^na_2a_3, \beta ta_4^{-1}, q^n\beta \\ \pm q^{n-\frac12}(a_1a_2a_3\beta t)^{\frac12}, q^na_1\beta t, q^na_2\beta t, q^na_3\beta t, q^{2n}a_1a_2a_3a_4, q^na_1a_2a_3t \end{matrix} ;q, a_4t\right)\\ &= {}_8W_7(q^{2n-1}a_1a_2a_3\beta t; q^na_1a_2, q^na_1a_3, q^na_2a_3, \beta ta_4^{-1}, q^n\beta; q, a_4t),\end{aligned}$$

and [10, (2.1.11)]

$${}_8W_7(a_1;a_4,\ldots,a_8;q,z) := {}_8\phi_7\left(\begin{matrix} a_1, \pm qa_1^{\frac12}, a_4, \ldots, a_8 \\ \pm a_1^{\frac12}, qa_1/a_4, \ldots, qa_1/a_8 \end{matrix} ;q,z\right), \tag{3.2}$$

defines the very-well poised hypergeometric series ${}_8W_7$*.*

Remark 3.2. Note (3.1) is a generalization of the Rogers generating function (the generating function where the coefficient multiplying t^n is unity) [20, (14.10.27)]

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta};q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty} C_n(x;\beta|q)t^n, \quad x = \cos\theta, \tag{3.3}$$

where $C_n(x;\beta|q)$ is the continuous q-ultraspherical/Rogers polynomial (see Section 5 below).

Remark 3.3. Note that to compute such basic hypergeometric functions, it is convenient to use (2.6).

3.1. The Wilson limit for the Ismail–Simeonov expansion

In this section we obtain a new infinite series over the Wilson polynomials W_n [20, Section 9.1] whose left hand side is given by a ratio of gamma functions. We will see that this identity follows formally from the Ismail–Simeonov expansion over the Askey–Wilson polynomials (3.1) by taking the $q \uparrow 1$ limit.

Let $b, a_k \in \mathbb{C}$, $k = 1, 2, 3, 4$. Define $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$, $\mathbf{a}+b := \{a_1+b, a_2+b, a_3+b, a_3+b\}$, $a_{12} := a_1+a_2$, $a_{13} := a_1 + a_3$, $a_{23} := a_2 + a_3$, $a_{123} := a_1 + a_2 + a_3$, $a_{1234} := a_1 + a_2 + a_3 + a_4$, etc. Note that we use the compact product notation for $a, b \in \mathbb{C}$, $\Gamma(a \pm b) := \Gamma(a + b)\Gamma(a - b)$.

Lemma 3.4. *Let $n \in \mathbb{N}_0$, $t, u, a_k \in \mathbb{C}$, $k = 1, 2, 3, 4$, $\Re(a_{1234} + t - u) > \frac{3}{2}$. Then*

$$\int_0^\infty \frac{\Gamma(t \pm ix)\Gamma(a_1 \pm ix)\cdots\Gamma(a_4 \pm ix)}{\Gamma(u \pm ix)\Gamma(\pm 2ix)} W_n(x^2; \mathbf{a})dx$$
$$= (u-t)_n \int_0^\infty \frac{\Gamma(t-\frac{1}{2}n \pm ix)\Gamma(a_1+\frac{1}{2}n \pm ix)\cdots\Gamma(a_4+\frac{1}{2}n \pm ix)}{\Gamma(u+\frac{1}{2}n \pm ix)\Gamma(\pm 2ix)}dx$$
$$= (u-t)_n \int_0^\infty \frac{\Gamma(t-\frac{1}{2}n \pm ix)\mathsf{W}(x; \mathbf{a}+\frac{1}{2}n)}{\Gamma(u+\frac{1}{2}n \pm ix)}dx. \tag{3.4}$$

Proof. The weight function for the Wilson polynomials is [20, (9.1.2)]

$$\mathsf{W}(x; \mathbf{a}) := \frac{\Gamma(a_1 \pm ix)\Gamma(a_2 \pm ix)\Gamma(a_3 \pm ix)\Gamma(a_4 \pm ix)}{\Gamma(\pm 2ix)}. \tag{3.5}$$

Define $\tilde{\mathsf{W}}(x; \mathbf{a}) := (2ix)^{-1}\mathsf{W}(x; \mathbf{a})$. The Rodrigues-type formula for the Wilson polynomials is [20, (9.1.11)]

$$\tilde{\mathsf{W}}(x; \mathbf{a})\, W_n(x^2; \mathbf{a}) = \mathcal{W}^n\, \tilde{\mathsf{W}}\left(x; \mathbf{a}+\tfrac{1}{2}n\right), \tag{3.6}$$

where $\mathcal{W}$ is the Wilson (divided difference) operator (see e.g., [18], [20, Section 1.16])

$$\mathcal{W}f(x) := \frac{\delta f(x)}{\delta x^2} := \frac{1}{2ix}\left(f\left(x + \tfrac{i}{2}\right) - f\left(x - \tfrac{i}{2}\right)\right). \tag{3.7}$$

Substitute (3.6) in the left-hand side of (3.4) and integrate by parts using (3.7) and [18, Theorem 9.1], along with the identity

$$-\mathcal{W}\frac{\Gamma(t \pm ix)}{\Gamma(u \pm ix)} = \frac{\Gamma(t - \frac{1}{2} \pm ix)}{\Gamma(u + \frac{1}{2} \pm ix)} \implies (-1)^n\, \mathcal{W}^n \frac{\Gamma(t \pm ix)}{\Gamma(u \pm ix)} = (u-t)_n \frac{\Gamma(t - \frac{n}{2} \pm ix)}{\Gamma(u + \frac{n}{2} \pm ix)},$$

which demonstrates (3.4). □

A powerful integral representation of a very-well poised ${}_7F_6(1)$ which we rely on to derive the Wilson polynomial expansion formula below, is the $q \uparrow 1$ limit of the Nassrallah–Rahman integral (6.3), which can be found in [10, (6.3.11)], [25, (1.17)].

Lemma 3.5 *(Rahman (1986)). Let $n \in \mathbb{N}_0$, $t, u, a_k \in \mathbb{C}$, $k = 1, 2, 3, 4$, $\Re(a_4 + t) > 0$, $\Re(a_{1234} + t - u) > \frac{3}{2}$. Then*

$$\int_0^\infty \frac{\Gamma(t \pm ix)\Gamma(a_1 \pm ix) \cdots \Gamma(a_4 \pm ix)}{\Gamma(u \pm ix)\Gamma(\pm 2ix)} dx = \frac{2\pi\Gamma(u + a_{123})\Gamma(a_{12}) \cdots \Gamma(a_{34})\Gamma(t + a_1)\Gamma(t + a_2)\Gamma(t + a_3)}{\Gamma(u + a_1)\Gamma(u + a_2)\Gamma(u + a_3)\Gamma(a_{1234})\Gamma(t + a_{123})} \mathsf{J}(t, u, \mathbf{a}), \tag{3.8}$$

where

$$\mathsf{J}(t, u, \mathbf{a}) := {}_7F_6\left(\begin{matrix} a_{123} + u - 1, \frac{1}{2}(a_{123} + u + 1), a_{12}, a_{13}, a_{23}, u - a_4, u - t \\ \frac{1}{2}(a_{123} + u - 1), u + a_1, u + a_2, u + a_3, a_{1234}, t + a_{123} \end{matrix}; 1\right).$$

Proof. See [10, (6.3.11)], [25, (1.17)]. The condition $\Re(a_4 + t) > 0$ follows from the requirement of uniform convergence of the ${}_7F_6(1)$ [24, (16.2.2)]. The condition $\Re(a_{1234}+t-u) > \frac{3}{2}$ follows since the integrand clearly vanishes at the origin by applying the Stirling formula [24, (5.11.7)] on the integrand as $x \to +\infty$. □

Remark 3.6. Observe that the generalized hypergeometric function ${}_7F_6$ in (3.8) is very-well poised and of argument unity. Using Bailey's W notation for a very-well poised ${}_7F_6$ of argument unity (see for instance, [13, p. 2])

$$W(a; b, c, d, e, f) := {}_7F_6\left(\begin{matrix} a, \frac{a}{2} + 1, b, c, d, e, f \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - f \end{matrix}; 1\right).$$

In Lemma 3.5, the ${}_7F_6(1)$ can be written as

$$W(a_{123} + u - 1; a_{12}, a_{13}, a_{23}, u - a_4, u - t).$$

Theorem 3.7. *Let $x \in (0, \infty)$, $t, u, a_1, a_2, a_3, a_4 \in \mathbb{C}$, $\Re(a_{1234} + t - u) > \frac{3}{2}$, $\Re(a_t + t) > 0$. Then*

$$\frac{\Gamma(t + ix)\Gamma(t - ix)}{\Gamma(u + ix)\Gamma(u - ix)} = \frac{(a_{123})_u(a_1, a_2, a_3)_t}{(a_{123})_t(a_1, a_2, a_3)_u} \times \sum_{n=0}^{\infty} \frac{(u - t, a_{1234} - 1)_n(a_{123} + u)_{2n}\, \mathsf{K}_n(t, u, \mathbf{a})\, W_n(x^2; \mathbf{a})}{n!(a_1 + u, a_2 + u, a_3 + u, a_{123} + t)_n(a_{1234} - 1)_{2n}}, \tag{3.9}$$

where

$$\mathsf{K}_n(t, u, \mathbf{a}) := {}_7F_6\left(\begin{matrix} a_{123}+u+2n-1, \frac{a_{123}+u+2n+1}{2}, a_{12}+n, a_{13}+n, a_{23}+n, u-a_4, u-t+n \\ \frac{a_{123}+u+2n-1}{2}, a_1+u+n, a_2+u+n, a_3+u+n, a_{123}+t+n, a_{1234}+2n \end{matrix}; 1\right)$$
$$= W(a_{123}+u+2n-1, a_{12}+n, a_{13}+n, a_{23}+n, u-a_4, u-t+n).$$

Proof. Consider the Wilson polynomial expansion

$$\frac{\Gamma(t + ix)\Gamma(t - ix)}{\Gamma(u + ix)\Gamma(u - ix)} = \sum_{n=0}^{\infty} \mathsf{c}_n(t, u, \mathbf{a}) W_n(x^2; \mathbf{a}). \tag{3.10}$$

Using orthogonality for the Wilson polynomials [20, (9.1.2)], one can obtain the coefficient of the expansion (3.10), namely

$$\mathsf{c}_n(t,u,\mathbf{a}) = \frac{1}{H_n(\mathbf{a})}\int_0^\infty \frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(u+ix)\Gamma(u-ix)} W_n(x^2;\mathbf{a})\mathsf{W}(x;\mathbf{a})dx, \tag{3.11}$$

where the Wilson square norm is given by [20, (9.1.2)]

$$\begin{aligned} H_n(\mathbf{a}) &:= \int_0^\infty W_n(x^2;\mathbf{a})W_n(x^2;\mathbf{a})\mathsf{W}(x;\mathbf{a})dx \\ &= \frac{2\pi n!\,\Gamma(a_{12}+n)\Gamma(a_{13}+n)\Gamma(a_{14}+n)\Gamma(a_{23}+n)\Gamma(a_{24}+n)\Gamma(a_{34}+n)}{(a_{1234}-1+2n)\Gamma(a_{1234}-1+n)}. \end{aligned} \tag{3.12}$$

The integral in (3.11) can be re-expressed as an integral over a shifted weight function for the Wilson polynomials using Lemma 3.4. Evaluating the resulting definite integral using Lemma 3.5 yields $\mathsf{c}_n(t,u,\mathbf{a})$ in (3.10). Since the Wilson polynomials when normalized represent an orthonormal basis for $L_2(\mathsf{W}(x;\mathbf{a}),(0,\infty))$, and due to Lemma 3.5, and also due to its analyticity,

$$\frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(u+ix)\Gamma(u-ix)} \in L_2(\mathsf{W}(x;\mathbf{a}),(0,\infty)),$$

the definite integral and the series converges in the L_2 sense. The conditions for convergence of Lemma 3.5 are applied to this expansion theorem when the series does not terminate. The series terminates when $u-t\in -\mathbb{N}_0$, and in this case all possible values for the parameters are allowed as long as they are bounded and the functions involved are defined. $\square$

Remark 3.8. Note that Theorem 3.7 can also be derived formally by starting with the Ismail–Simeonov expansion [19, (4.9)]

$$\frac{(ue^{\pm i\theta};q)_\infty}{(te^{\pm i\theta};q)_\infty} := \frac{(ue^{i\theta},ue^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^\infty c_n(t,u,\mathbf{a};q)p_n(x;\mathbf{a}|q),$$

where

$$\begin{aligned} c_n(t,u,\mathbf{a};q) &= \frac{t^n(ut^{-1};q)_n(q^na_1u,q^na_2u,q^na_3u,q^na_1a_2a_3t;q)_\infty}{(q,q^{n-1}a_1a_2a_3a_4;q)_n(a_1t,a_2t,a_3t,qb_1;q)_\infty}\mathsf{G}_n^{t,u,\mathbf{a};q}, \\ \mathsf{G}_n^{t,u,\mathbf{a};q} &:= {}_8\phi_7\left(\begin{matrix} b_1,\pm qb_1^{\frac12},q^na_1a_2,q^na_1a_3,q^na_2a_3,ua_4^{-1},q^nut^{-1} \\ \pm b_1^{\frac12},q^na_3u,q^na_2u,q^na_1u,q^{2n}a_1a_2a_3a_4,q^na_1a_2a_3t \end{matrix};q,a_4t\right) \\ &= {}_8W_7(b_1;q^na_1a_2,q^na_1a_3,q^na_2a_3,ua_4^{-1},q^nut^{-1};q,a_4t), \end{aligned}$$

with $b_1 := q^{2n-1}a_1a_2a_3u$. Observe that $\mathsf{G}_n^{\beta t,t,\mathbf{a};q} = \mathsf{F}_n^{\beta,t,\mathbf{a};q}$, cf. Theorem 3.1. We apply the substitutions $a_k\mapsto q^{a_k}$, for all $k\in\{1,2,3,4\}$, $e^{i\theta}\mapsto q^{ix}$, $t\mapsto q^t$, $u\mapsto q^u$, multiply both sides by $(1-q)^{2(u-t)}$ and take the $q\uparrow 1^-$ limit. We use (2.7), (2.2), (2.8), and apply the relation [20, (14.1.21)]

$$\lim_{q\uparrow 1^-}\frac{p_n((q^{ix}+q^{-ix})/2;q^{a_1},q^{a_2},q^{a_3},q^{a_4}|q)}{(1-q)^{3n}} = W_n(x^2;\mathbf{a}),$$

with [20, (9.1.1)]. Since

$$\frac{1}{(t+ix,t-ix)_{u-t}} = \frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(u+ix)\Gamma(u-ix)},$$

the result follows.

4. Expansion for the continuous q-Jacobi polynomials

We would like to examine specialization and limit transition properties for the Ismail & Simeonov result in terms of the continuous q-Jacobi polynomials. For the continuous q-Jacobi polynomials, we adopt the standard normalization adopted by Rahman et al. in [20, (14.10.1)]. However, in order to simplify our formulae we have further replaced $q^{\alpha+\frac{1}{2}}, q^{\gamma+\frac{1}{2}} \mapsto \alpha, \gamma$. Using this notation one has

$$\begin{aligned} P_n^{(\alpha,\gamma)}(x|q) &:= \frac{\alpha^{\frac{n}{2}}}{(q, -(\alpha\gamma)^{\frac{1}{2}}, -(q\alpha\gamma)^{\frac{1}{2}}; q)_n} p_n\big(x; \alpha^{\frac{1}{2}}, -\gamma^{\frac{1}{2}}, -(q\gamma)^{\frac{1}{2}}, (q\alpha)^{\frac{1}{2}}|q\big) \\ &= \frac{(q^{\frac{1}{2}}\alpha; q)_n}{(q;q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^n\alpha\gamma, \alpha^{\frac{1}{2}}e^{i\theta}, \alpha^{\frac{1}{2}}e^{-i\theta} \\ q^{\frac{1}{2}}\alpha, -(\alpha\gamma)^{\frac{1}{2}}, -(q\alpha\gamma)^{\frac{1}{2}} \end{matrix}; q, q\right). \end{aligned} \tag{4.1}$$

Note that some consequences of this notation are

$$C_n(x;\beta|q) = \beta^{-\frac{n}{2}} \frac{(\beta^2;q)_n}{(q^{\frac{1}{2}}\beta;q)_n} P_n^{(\beta,\beta)}(x|q),$$

and

$$\lim_{q\uparrow 1^-} P_n^{\left(q^{\alpha+\frac{1}{2}}, q^{\gamma+\frac{1}{2}}\right)}(x|q) = P_n^{(\alpha,\gamma)}(x), \tag{4.2}$$

where $P_n^{(\alpha,\gamma)}$ is the Jacobi polynomial [24, (18.5.7)].

Corollary 4.1. *Let $|q|, |t|, |\alpha|, |\beta|, |\gamma| < 1$, $x = \cos\theta \in (-1, 1)$. Then*

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} P_n^{(\alpha,\gamma)}(x|q)\, \mathsf{D}_n^{\beta,t,\alpha,\gamma;q}, \tag{4.3}$$

where

$$\begin{aligned} \mathsf{D}_n^{\beta,t,\alpha,\gamma;q} &:= \frac{(t\alpha^{-\frac{1}{2}})^n(\beta, -(\alpha\gamma)^{\frac{1}{2}}, -(q\alpha\gamma)^{\frac{1}{2}}; q)_n (q^n\alpha^{\frac{1}{2}}\beta t, -q^n\gamma^{\frac{1}{2}}\beta t, -q^{n+\frac{1}{2}}\gamma^{\frac{1}{2}}\beta t, q^{n+\frac{1}{2}}\alpha^{\frac{1}{2}}\gamma t; q)_\infty}{(q^n\alpha\gamma;q)_n(\alpha^{\frac{1}{2}}t, -\gamma^{\frac{1}{2}}t, -(q\gamma)^{\frac{1}{2}}t, q^{2n+\frac{1}{2}}\alpha^{\frac{1}{2}}\gamma\beta t; q)_\infty} \mathsf{H}_n^{\beta,t,\alpha,\gamma;q}, \\ \mathsf{H}_n^{\beta,t,\alpha,\gamma;q} &:= {}_8W_7(q^{2n-\frac{1}{2}}\alpha^{\frac{1}{2}}\gamma\beta t; -q^n(\alpha\gamma)^{\frac{1}{2}}, -q^{n+\frac{1}{2}}(\alpha\gamma)^{\frac{1}{2}}, q^{n+\frac{1}{2}}\gamma, (q\alpha)^{-\frac{1}{2}}\beta t, q^n\beta; q, (q\alpha)^{\frac{1}{2}}t). \end{aligned}$$

Proof. Let $a_1 = \alpha^{\frac{1}{2}}$, $a_2 = -\gamma^{\frac{1}{2}}$, $a_3 = -(q\gamma)^{\frac{1}{2}}$, $a_4 = (q\alpha)^{\frac{1}{2}}$, using (3.1), (4.1), the result follows. □

Note $\mathsf{H}_n^{\beta,t,\alpha,\gamma;q} := \mathsf{F}_n^{\beta,t,\alpha^{\frac{1}{2}},-\gamma^{\frac{1}{2}},-(q\gamma)^{\frac{1}{2}},(q\alpha)^{\frac{1}{2}};q}$, cf. Theorem 3.1. Using (4.2) in (4.3), we obtain a Jacobi generalization of the Gegenbauer generating function

$$\frac{1}{(1+t^2-2tx)^\beta} = \sum_{n=0}^{\infty} \frac{t^n(\beta)_n(\alpha+\gamma+1)_n P_n^{(\alpha,\gamma)}(x)}{\left(\frac{\alpha+\gamma+1}{2}\right)_n \left(\frac{\alpha+\gamma+2}{2}\right)_n (1+t)^{2(n+\beta)}} {}_2F_1\left(\begin{matrix} \gamma+n+1, n+\beta \\ 2n+\alpha+\gamma+2 \end{matrix}; \frac{4t}{(1+t)^2}\right),$$

which is equivalent to [4, (3.1)]

$$\begin{aligned} \frac{1}{(z-x)^\nu} &= \frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z) P_n^{(\alpha,\beta)}(x), \end{aligned} \tag{4.4}$$

where $z = (t + t^{-1})/2$ (see Remark 5.1 below), and $Q_\nu^{(\alpha,\gamma)}$ is the Jacobi function of the second kind. The q-analogue of the specialization of (4.4) with $\nu = 1$ [29, (9.2.1)]

$$\frac{1}{z-x} = \frac{(z-1)^\alpha (z+1)^\beta}{2^{\alpha+\beta}} \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} Q_n^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(x),$$

is (4.3) with $\beta = q$.

5. Expansion for the continuous q-ultraspherical/Rogers polynomials

The continuous q-ultraspherical/Rogers polynomials are defined as [20, (14.10.17)]

$$C_n(x;\beta|q) := \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} {}_2\phi_1\left(\begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n} \end{matrix}; q, q\beta^{-1}e^{-2i\theta}\right), \quad x = \cos\theta.$$

We now derive a generalization of the Rogers generating function (3.3) using the connection relation for the continuous q-ultraspherical/Rogers polynomials [17, (13.3.1)]

$$C_n(x;\beta\,|\,q) = \frac{1}{1-\gamma} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1-\gamma q^{n-2k})\gamma^k (\beta\gamma^{-1};q)_k (\beta;q)_{n-k}}{(q;q)_k (q\gamma;q)_{n-k}} C_{n-2k}(x;\gamma|q). \tag{5.1}$$

Remark 5.1. Note that the functions $x \mapsto (2t)^{-1}(1+t^2-2tx)$ and $x \mapsto z - x$ are identical through the Szegő transformation

$$z = \frac{t+t^{-1}}{2},$$

which maps circles in the complex plane to ellipses with foci at ± 1, with the unit circle being mapped to the line segment $[-1, 1]$. Both of these functions appear in the analysis below. The Rogers generating function (3.3) is a q-analogue of the generating function for the Gegenbauer polynomials [24, (18.12.4)], [11]

$$\frac{1}{(1+t^2-2tx)^\mu} = \sum_{n=0}^{\infty} t^n C_n^\mu(x), \tag{5.2}$$

which has already been generalized in [5, Theorem 2.1]

$$\frac{1}{(z-x)^\nu} = \frac{2^{\mu+\frac12}\Gamma(\mu)e^{i\pi(\mu-\nu+\frac12)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{\frac{\nu-\mu}{2}-\frac14}} \sum_{n=0}^{\infty} (n+\mu) Q_{n+\mu-\frac12}^{\nu-\mu-\frac12}(z) C_n^\mu(x), \tag{5.3}$$

where $Q_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}$ is the associated Legendre function of the second kind defined in terms of the Gauss hypergeometric function, $\nu + \mu + 1 \notin -\mathbb{N}_0$, [24, (14.3.7)]

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi}\,e^{i\pi\mu}\Gamma(\nu+\mu+1)(z^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}\Gamma(\nu+\frac32)z^{\nu+\mu+1}} {}_2F_1\left(\begin{matrix} \frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2} \\ \nu+\frac32 \end{matrix}; \frac{1}{z^2}\right).$$

Theorem 5.2. *Let $x = \cos\theta \in (-1, 1)$, $|t| < 1$, $\beta, \gamma \in (-1, 1) \setminus \{0\}$, $0 < |q| < 1$. Then*

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} \frac{(\beta;q)_n}{(\gamma;q)_n} {}_2\phi_1\left(\begin{matrix} \beta\gamma^{-1}, \beta q^n \\ \gamma q^{n+1} \end{matrix}; q, \gamma t^2\right) C_n(x;\gamma|q)\, t^n. \tag{5.4}$$

Proof. The proof follows as above by starting with (3.3), inserting (5.1), shifting the n index by $2k$, and reversing the order of summation. We use (2.4) through (2.11), since $|a_n| = |t|^n$, $|c_{n,k}| \le K_6[n+1]_q^{\sigma_3}$, $|C_n(x;\beta|q)| \le C_n(1;\beta|q) \le [n+1]_q^{\sigma_4}$, where $\sigma_3 := 2b - c + 2$, $\sigma_4 := 2b + 2$, with $\beta = q^b$, $\gamma = q^c$. Note that $|C_n(x;\beta|q)| \le C_n(1;\beta|q)$, $q, \beta \in (-1, 1)$ is given in [2, (3.19)]. Therefore for n sufficiently large,

$$|C_n(x;\beta|q)| \le [n+1]_q^{\sigma_4} \le (n+1)^{\sigma_4}, \tag{5.5}$$

where $K_6 = 1/[\Re(c+1)]_q$, and σ_3 and σ_4 are independent of n. Then, since

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{\lfloor n/2 \rfloor} |c_{k,n}||C_k(x;\beta|q)| \le K_6 \sum_{n=0}^{\infty} |t|^n (n+1)^{\sigma_3+\sigma_4+1} < \infty,$$

by Lemma 2.4, the result is proven. □

Remark 5.3. Coefficients of derived generalized generating functions such as (5.4) are amenable to situations where summation theorems for the basic hypergeometric functions (see for instance [24, Sections 17.5–17.7]) may be utilized. When applicable, one may use these summation theorems to compute alternative expansions. Some of these expansions may not be interesting, as they no longer represent generating functions. Take for example Theorem 5.2. If you use the q-Gauss sum [24, (17.6.1)]

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, c/(ab)\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/(ab); q)_\infty},$$

on the coefficient of the expansion, and make the appropriate substitutions and simplifications, it becomes

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \frac{(\beta t^2, \beta^3 t^4 q^{-1}; q)_\infty}{(\beta^2 t^4 q^{-1}, \beta^2 t^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(\beta, \beta^2 t^2; q)_n}{(\beta^2 t^2 q^{-1}, \beta^3 t^4 q^{-1}; q)_n} C_n(x; \beta^2 t^2 q^{-1}|q)\, t^n,$$

which is an alternative expansion of the Rogers generating function. However, it is not a generating function since t appears in the parameter of the polynomial as well as in the q-Pochhammer coefficients.

By using Theorem 5.2 as a starting point, there are a number of interesting results which follow.

5.1. The continuous q-Hermite polynomials

One may derive an expansion of a specialized Rogers generating function in terms of the continuous q-Hermite polynomials defined as

$$H_n(x|q) := e^{in\theta}\, {}_2\phi_0\left(\begin{matrix} q^n, 0 \\ - \end{matrix}; q, q^n e^{-2i\theta}\right),$$

where $x = \cos\theta$. Using [20, (14.10.34)]

$$\lim_{\beta \to 0} C_n(x;\beta|q) = \frac{H_n(x|q)}{(q;q)_n},$$

one obtains

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} \frac{(\beta;q)_n}{(q;q)_n} t^n\, {}_1\phi_1\left(\begin{matrix} \beta q^n \\ 0 \end{matrix}; \beta t^2\right) H_n(x|q). \tag{5.6}$$

One can see that by setting $\beta = 0$ in (5.6), that this is a generalization of a generating function for the continuous q-Hermite polynomials, namely [20, (14.26.11)]

$$\frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(x|q). \tag{5.7}$$

5.2. The Chebyshev polynomials of the first kind

We also derive an expansion of the Rogers generating function in terms of the Chebyshev polynomials of the first kind $T_n(\cos\theta) := \cos(n\theta)$. The following corollary is a q-analogue of [6, (3.10)]

$$\frac{1}{(z-x)^\nu} = \sqrt{\frac{2}{\pi}} \frac{e^{i\pi(\frac{1}{2}-\nu)}}{\Gamma(\nu)(z^2-1)^{\frac{\nu}{2}-\frac{1}{4}}} \sum_{n=0}^{\infty} \epsilon_n Q_{n-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) T_n(x), \tag{5.8}$$

which is a generalization of the Heine reciprocal square root identity [15, p. 286]

$$\frac{1}{\sqrt{z-x}} = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \epsilon_n Q_{n-\frac{1}{2}}(z) T_n(x). \tag{5.9}$$

The q-analogue of (5.9) is (5.10) with $\beta = q^{\frac{1}{2}}$. We have used the common convention for Legendre functions of the second kind $Q_\nu := Q_\nu^0$, and $\epsilon_n := 2 - \delta_{n,0}$, is called the Neumann factor.

Corollary 5.4. *Let $x = \cos\theta \in (-1,1)$, $|t| < 1$, $\beta, \gamma \in (-1,1) \setminus \{0\}$, $0 < |q| < 1$. Then*

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} \epsilon_n \frac{(\beta;q)_n}{(q;q)_n} t^n \, {}_2\phi_1\left(\begin{matrix} \beta, \beta q^n \\ q^{n+1} \end{matrix}; t^2\right) T_n(x). \tag{5.10}$$

Proof. Using [20, p. 474]

$$\lim_{\beta \to 0} \frac{(q^{\beta+1};q)_n}{(q^\beta;q)_n} C_n(x; q^\beta|q) = \epsilon_n \, T_n(x),$$

the proof follows. □

5.3. The continuous q-Legendre polynomials

Furthermore, (5.4) produces the following result in terms of the continuous q-Legendre polynomials which can be defined in terms of the continuous q-ultraspherical/Rogers polynomials by [20, p. 478]

$$P_n(x|q) := q^{\frac{n}{4}} C_n(x; q^{\frac{1}{2}}|q).$$

Corollary 5.5. *Let $x = \cos\theta \in (-1,1)$, $|t| < 1$, $\beta, \gamma \in (-1,1) \setminus \{0\}$, $0 < |q| < 1$. Then*

$$\frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} \frac{(\beta;q)_n}{(q^{\frac{1}{2}};q)_n} (tq^{-\frac{1}{4}})^n \, {}_2\phi_1\left(\begin{matrix} \beta q^{-\frac{1}{2}}, \beta q^n \\ q^{n+\frac{3}{2}} \end{matrix}; q^{\frac{1}{2}} t^2\right) P_n(x|q). \tag{5.11}$$

Using [20, (14.10.49)]

$$\lim_{q \uparrow 1^-} P_n(x|q) = P_n(x),$$

where P_n is the Legendre polynomial defined by [20, (9.8.62)]

$$P_n(x) := {}_2F_1\left(\begin{matrix}-n, n+1\\ 1\end{matrix}; \frac{1-x}{2}\right),$$

one can see that (5.11) is a q-analogue of [5, (14)]

$$\frac{1}{(z-x)^\nu} = \frac{e^{i\pi(1-\nu)}(z^2-1)^{(1-\nu)/2}}{\Gamma(\nu)} \sum_{n=0}^\infty (2n+1) Q_n^{\nu-1}(z) P_n(x), \tag{5.12}$$

which in itself is a generalization of the Heine formula [14]

$$\frac{1}{z-x} = \sum_{n=0}^\infty (2n+1) Q_n(z) P_n(x). \tag{5.13}$$

The q-analogue of Heine's formula is (5.11) with $\beta = q$.

The above analysis is summarized as a hierarchical scheme in Figures 1 and 2.

5.4. A quadratic transformation for basic hypergeometric functions

In (3.1), let $a_1 \mapsto \gamma^{\frac{1}{2}}$, $a_2 \mapsto -\gamma^{\frac{1}{2}}$, $a_3 \mapsto -(q\gamma)^{\frac{1}{2}}$, $a_4 \mapsto (q\gamma)^{\frac{1}{2}}$, and specializing the Askey–Wilson polynomials to the continuous q-ultraspherical/Rogers polynomials using [20, p. 472]

$$C_n(x;\gamma|q) = \frac{(\gamma^2;q)_n}{\left(q, -\gamma, \pm q^{\frac{1}{2}}\gamma; q\right)_n} p_n\left(x; \gamma^{\frac{1}{2}}, -\gamma^{\frac{1}{2}}, -(q\gamma)^{\frac{1}{2}}, (q\gamma)^{\frac{1}{2}} | q\right),$$

produces an expansion of the Rogers generating function whose coefficients are an ${}_8\phi_7$. By comparing the coefficients of this expansion with the generalized Rogers generating function (5.4), and further replacing $(\beta,\gamma) \mapsto (q^{-n}\beta, q^{-n}\gamma)$, $t \mapsto (q/\gamma)^{\frac{1}{2}} t$, we derive

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix}\beta/\gamma, \beta\\ q\gamma\end{matrix}; q, qt^2\right) &= \frac{(q(\beta t)^2, q\gamma t, qt; q)_\infty}{(q\beta\gamma t, q\beta t, qt^2; q)_\infty} {}_8\phi_7\left(\begin{matrix}\beta\gamma t, \pm q(\beta\gamma t)^{\frac{1}{2}}, \pm q^{\frac{1}{2}}\gamma, -\gamma, \beta\gamma^{-1}t, \beta\\ \pm(\beta\gamma t)^{\frac{1}{2}}, \pm q^{\frac{1}{2}}\beta t, -q\beta t, q\gamma^2, q\gamma t\end{matrix}; q, qt\right)\\ &= \frac{(q(\beta t)^2, q\gamma t, qt; q)_\infty}{(q\beta\gamma t, q\beta t, qt^2; q)_\infty} {}_8W_7(\beta\gamma t; \pm q^{\frac{1}{2}}\gamma, -\gamma, \beta\gamma^{-1}t, \beta; q, qt). \end{aligned} \tag{5.14}$$

This is a generalization of [19, Corollary 4.4] with $\beta = \gamma$. By re-expressing (5.14), we see that our procedure has produced a new *quadratic transformation* for basic hypergeometric functions (see [26]).

Theorem 5.6. *Let $0 < |q| < 1$, $|qt| < 1$, $|qt^2| < 1$. Then*

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix}a, b\\ qab^{-1}\end{matrix}; q, qt^2\right) &= \frac{\left(q(at)^2, qab^{-1}t, qt; q\right)_\infty}{\left(qa^2b^{-1}t, qat, qt^2; q\right)_\infty} {}_8\phi_7\left(\begin{matrix}a^2b^{-1}t, \pm qab^{-\frac{1}{2}}t^{\frac{1}{2}}, \pm q^{\frac{1}{2}}ab^{-1}, -ab^{-1}, bt, a\\ \pm ab^{-\frac{1}{2}}t^{\frac{1}{2}}, \pm q^{\frac{1}{2}}at, -qat, qa^2b^{-2}, qab^{-1}t\end{matrix}; q, qt\right)\\ &= \frac{\left(q(at)^2, qab^{-1}t, qt; q\right)_\infty}{\left(qa^2b^{-1}t, qat, qt^2; q\right)_\infty} {}_8W_7\left(a^2b^{-1}t; \pm q^{\frac{1}{2}}ab^{-1}, -ab^{-1}, bt, a; q, qt\right), \end{aligned}$$

which is valid under the transformation $t \mapsto -t$.

Proof. Start with (5.14) and replace $(\beta, ab^{-1}) \mapsto (a, b)$. Given (3.2) and the expression for the very-well poised hypergeometric series ${}_8W_7$, this completes the proof. □

$$\frac{1}{(z-x)^{\nu}} = \frac{2^{\mu+\frac{1}{2}}\Gamma(\mu)e^{i\pi(\mu-\nu+\frac{1}{2})}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{\frac{\nu-\mu}{2}-\frac{1}{4}}}\sum_{n=0}^{\infty}(n+\mu)Q^{\nu-\mu-\frac{1}{2}}_{n+\mu-\frac{1}{2}}(z)C_n^{\mu}(x)$$

(5.3) : Theorem 2.1 in Cohl (2013) [5]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\frac{(\beta;q)_n}{(\gamma;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}\beta\gamma^{-1},\beta q^n\\ \gamma q^{n+1}\end{matrix};q,\gamma t^2\right)C_n(x;\gamma|q)$$

(5.4) : q-analogue (continuous q-ultraspherical/Rogers polynomials)

$$\frac{1}{z-x} = \frac{2^{\mu+\frac{1}{2}}\Gamma(\mu)e^{i\pi(\mu-\frac{1}{2})}}{\sqrt{\pi}\,(z^2-1)^{-\frac{\mu}{2}+\frac{1}{4}}}\sum_{n=0}^{\infty}(n+\mu)Q^{-\mu+\frac{1}{2}}_{n+\mu-\frac{1}{2}}(z)C_n^{\mu}(x)$$

(7.2) in Durand et al. (1976) [8]

$$\frac{(tqe^{i\theta},tqe^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\frac{(q;q)_n}{(\gamma;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}q\gamma^{-1},q^{n+1}\\ \gamma q^{n+1}\end{matrix};\gamma t^2\right)C_n(x;\gamma|q)$$

(5.4) with $\beta = q$: q-analogue (continuous q-ultraspherical/Rogers polynomials)

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\frac{(\beta;q)_n}{(q;q)_n}t^n\,{}_1\phi_1\left(\begin{matrix}\beta q^n\\ 0\end{matrix};\beta t^2\right)H_n(x|q)$$

(5.6) : q-expansion (continuous q-Hermite polynomials)

$$\frac{1}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}H_n(x|q)$$

(5.7) : generating function for continuous q-Hermite polynomials

$$\frac{1}{(z-x)^{\nu}} = \sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{-\frac{\nu}{2}+\frac{1}{4}}}{e^{i\pi(\nu-\frac{1}{2})}\Gamma(\nu)}\sum_{n=0}^{\infty}\epsilon_n Q^{\nu-\frac{1}{2}}_{n-\frac{1}{2}}(z)T_n(x)$$

(5.8) : (3.10) in Cohl & Dominici (2011) [6]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\epsilon_n\frac{(\beta;q)_n}{(q;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}\beta,\beta q^n\\ q^{n+1}\end{matrix};t^2\right)T_n(x)$$

(5.10) : q-analogue (Chebyshev polynomial of the first kind)

$$\frac{1}{(z-x)^{\nu}} = \frac{(z^2-1)^{\frac{\nu-1}{2}}}{\Gamma(\nu)e^{i\pi(\nu-1)}}\sum_{n=0}^{\infty}(2n+1)Q_n^{\nu-1}(z)P_n(x)$$

(5.12) : (13) in Cohl (2013) [5]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\frac{(\beta;q)_n}{(q^{\frac{1}{2}};q)_n}(tq^{-\frac{1}{4}})^n\,{}_2\phi_1\left(\begin{matrix}\beta q^{-\frac{1}{2}},\beta q^n\\ q^{n+\frac{3}{2}}\end{matrix};q^{\frac{1}{2}}t^2\right)P_n(x|q)$$

(5.11) : q-analogue (continuous q-Legendre polynomials)

$$\frac{1}{\sqrt{z-x}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\epsilon_n Q_{n-\frac{1}{2}}(z)T_n(x)$$

(5.9) : Heine (1881) [15] reciprocal square root identity (1881)

$$\frac{(tq^{\frac{1}{2}}e^{i\theta},tq^{\frac{1}{2}}e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\epsilon_n\frac{(q^{\frac{1}{2}};q)_n}{(q;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}q^{\frac{1}{2}},q^{n+\frac{1}{2}}\\ q^{n+1}\end{matrix};t^2\right)T_n(x)$$

(5.10) with $\beta = q^{\frac{1}{2}}$: q-analogue (Chebyshev polynomials of the first kind)

$$\frac{1}{(1+t^2-2tx)^{\nu}} = \sum_{n=0}^{\infty}t^nC_n^{\nu}(x)$$

(5.2) : Gegenbauer (1874) [11] generating function

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}t^nC_n(x;\beta|q)$$

(3.3) : Rogers (1893) [28] generating function

$$\frac{1}{z-x} = \sum_{n=0}^{\infty}(2n+1)Q_n(z)P_n(x)$$

(5.13) : Heine (1878) [14] Heine's formula

$$\frac{(tqe^{i\theta},tqe^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty}\frac{(q;q)_n}{(q^{\frac{1}{2}};q)_n}(tq^{-\frac{1}{4}})^n\,{}_2\phi_1\left(\begin{matrix}q^{\frac{1}{2}},q^{n+1}\\ q^{n+\frac{3}{2}}\end{matrix};q^{\frac{1}{2}}t^2\right)P_n(x|q)$$

(5.11) with $\beta = q$: q-analogue (continuous q-Legendre polynomial)

Fig. 1. A hierarchy of generalized generating functions which connect expansions of classical and q-hypergeometric orthogonal polynomials for the continuous q-Legendre, Legendre, continuous q-ultraspherical/Rogers, Gegenbauer, Chebyshev of the first kind, and continuous q-Hermite polynomials [5,6,8,11,14,15,28].

$$\frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(t\beta+ix)\Gamma(t\beta-ix)}=\sum_{n=0}^{\infty}\frac{(t(\beta-1))_n(a_{123}+n+t)_{n+t(\beta-1)}W_n(x^2;\mathbf{a})}{n!(n-1+a_{1234})_n(a_1+t,a_2+t,a_3+t)_{n+t(\beta-1)}}$$

$$\times{}_7F_6\left(\begin{matrix}2n-1+a_{123}+t\beta,\frac{2n+1+a_{123}+t\beta}{2},n+a_{12},n+a_{13},n+a_{23},t\beta-a_4,n+t(\beta-1)\\ \frac{2n-1+a_{123}+t\beta}{2},n+a_1+t\beta,n+a_2+t\beta,n+a_3+t\beta,2n+a_{1234},n+a_{123}+t\end{matrix};1\right)$$

(3.9) : Wilson limit of Ismail-Simeonov generalized generating function

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^{\infty}c_n(\beta,t,\mathbf{a},q)p_n(x;\mathbf{a}|q)$$

$$c_n(\beta,t,\mathbf{a},q):=\frac{t^n(\beta;q)_n(q^na_1u,q^na_2u,q^na_3u,tq^na_1a_2a_3;q)_\infty}{(q,q^{n-1}a_1a_2a_3a_4;q)_n(ta_1,ta_2,ta_3,q^{2n}a_1a_2a_3\beta t;q)_\infty}$$

$$\times{}_8\phi_7\left(\begin{matrix}q^{2n-1}a_1a_2a_3\beta t,\pm q^{n+\frac12}(a_1a_2a_3\beta t)^{\frac12},q^na_1a_2,q^na_1a_3,q^na_2a_3,\beta t/a_4,q^n\beta\\ \pm q^{n-\frac12}(a_1a_2a_3\beta t)^{\frac12},q^na_1\beta t,q^na_2\beta t,q^na_3\beta t,q^{2n}a_1a_2a_3a_4,tq^na_1a_2a_3\end{matrix};q,ta_4\right)$$

(3.1) : expansion of Rogers generating function in Askey-Wilson polynomials

$$\frac{1}{(z-x)^\nu}=\frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}}\sum_{n=0}^{\infty}\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z)P_n^{(\alpha,\beta)}(x)$$

Theorem 1 in Cohl (2013) [4]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^{\infty}d_n(\beta,t,\alpha,\gamma,q)P_n^{(\alpha,\gamma)}(x|q)$$

$$d_n(\beta,t,\alpha,\gamma,q):=\frac{(t\alpha^{-\frac12})^n(\beta,-(\alpha\gamma)^{\frac12},-(q\alpha\gamma)^{\frac12};q)_n(q^n\alpha^{\frac12}\beta t,-q^n\gamma^{\frac12}\beta t,-q^{n+\frac12}\gamma^{\frac12}\beta t,q^{n+\frac12}\alpha^{\frac12}\gamma t;q)_\infty}{(q^n\alpha\gamma;q)_n(\alpha^{\frac12}t,-\gamma^{\frac12}t,-(q\gamma)^{\frac12}t,q^{2n+\frac12}\alpha^{\frac12}\gamma\beta t;q)_\infty}$$

$$\times{}_8\phi_7\left(\begin{matrix}q^{2n-\frac12}\alpha^{\frac12}\gamma\beta t,\pm q^{n+\frac34}\alpha^{\frac14}(\gamma\beta t)^{\frac12},-q^n(\alpha\gamma)^{\frac12},-q^{n+\frac12}(\alpha\gamma)^{\frac12},q^{n+\frac12}\gamma,(q\alpha)^{-\frac12}\beta t,q^n\beta\\ \pm q^{n-\frac14}\alpha^{\frac14}(\gamma\beta t)^{\frac12},q^n\alpha^{\frac12}\beta t,-q^n\gamma^{\frac12}\beta t,-q^{n+\frac12}\gamma^{\frac12}\beta t,q^{2n+1}\alpha\gamma,q^{n+\frac12}\alpha^{\frac12}\gamma t\end{matrix};q,(q\alpha)^{\frac12}t\right)$$

(4.3) : q-analogue (continuous q-Jacobi polynomials)

$$\frac{1}{z-x}=\frac{(z-1)^\alpha(z+1)^\beta}{2^{\alpha+\beta}}\sum_{n=0}^{\infty}\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}Q_n^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(x)$$

(9.2.1) in Szegő (1959) [29]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^{\infty}d_n(q,t,\alpha,\gamma,q)P_n^{(\alpha,\gamma)}(x|q)$$

(4.3) with $\beta=q$: q-analogue (continuous q-Jacobi polynomials)

$$\frac{1}{(z-x)^\nu}=\frac{2^{\mu+\frac12}\Gamma(\mu)e^{i\pi(\mu-\nu+\frac12)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{\frac{\nu-\mu}{2}-\frac14}}\sum_{n=0}^{\infty}(n+\mu)Q_{n+\mu-\frac12}^{\nu-\mu-\frac12}(z)C_n^\mu(x)$$

(5.3) : Theorem 2.1 in Cohl (2013) [5]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^{\infty}\frac{(\beta;q)_n}{(\gamma;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}\beta\gamma^{-1},\beta q^n\\ \gamma q^{n+1}\end{matrix};q,\gamma t^2\right)C_n(x;\gamma|q)$$

(5.4) : q-analogue (continuous q-ultraspherical/Rogers polynomials)

Fig. 2. A hierarchy of generalized Rogers generating functions which connects expansions of classical and q-hypergeometric orthogonal polynomials for the continuous q-ultraspherical/Rogers, Gegenbauer, continuous q-Jacobi, Jacobi, Wilson, and Askey–Wilson polynomials [4,5,29].

This quadratic transformation has some interesting consequences. For $a = 0$, one obtains the q-binomial theorem (2.10). For $t \in \mathbb{C}$, $t = iqb^{-\frac{1}{2}}$, the ${}_2\phi_1$ can be evaluated through the q-Kummer (Bailey–Daum) summation [10, (II.9)] (this leads to a very unusual summation of the ${}_8\phi_7$). It corresponds in the $q \uparrow 1^-$ limit to the quadratic transformation for the Gauss hypergeometric function [24, (15.8.21), (15.8.1)]

$$ {}_2F_1\left(\begin{matrix} a, b \\ a-b+1 \end{matrix}; t^2\right) = \frac{1}{(1 \pm t)^{2a}}\, {}_2F_1\left(\begin{matrix} a, a-b+\frac{1}{2} \\ 2a-2b+1 \end{matrix}; \frac{\pm 4t}{(1 \pm t)^2}\right). \tag{5.15} $$

Note that [26, (4.1)] is a quadratic transformation of basic hypergeometric series which in the limit $q \uparrow 1^-$ yields (5.15), but our new quadratic transformation is altogether different.

Remark 5.7. Our quadratic transformation given in Theorem 5.6 has recently been extended by Rains & Warnaar using Kaneko–Macdonald-type basic hypergeometric series (see [27, Theorem 5.22]).

5.5. Jacobi expansion of $(1-x)^{-\nu}$ and associated expansions

From the Jacobi expansion of $(z-x)^{-\nu}$ (4.4), we can derive an expansion of $(1-x)^{-\nu}$ by using the limit as $z \to 1^+$. Also, this is the corresponding limit of the Wilson polynomial expansion (3.9) to the Jacobi polynomials. In this subsection we derive this and other limiting expansions, which generalize [24, (18.18.15)] for $\nu = -n$, $n \in \mathbb{N}_0$.

Corollary 5.8. *Let $x \in (-1, 1)$, $\nu \in \mathbb{C}$, $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha - \nu + 1) > 0$. Then*

$$ \frac{1}{(1-x)^{\nu}} = \frac{\Gamma(\alpha-\nu+1)}{2^{\nu}} \sum_{n=0}^{\infty} \frac{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+1+n)(\nu)_n}{\Gamma(\alpha+1+n)\Gamma(\alpha+\beta+2-\nu+n)} P_n^{(\alpha,\beta)}(x). \tag{5.16} $$

Proof. Consider the expansion over Jacobi polynomials

$$ \frac{1}{(1-x)^{\nu}} = \sum_{n=0}^{\infty} c_n(\nu, \alpha, \beta) P_n^{(\alpha,\beta)}(x). $$

Using orthogonality for Jacobi polynomials, one can see that the coefficient of the expansion is given as

$$ c_n(\nu, \alpha, \beta) = \frac{1}{h_n(\alpha, \beta)} \int_{-1}^{1} (1-x)^{\alpha-\nu}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)dx, $$

where $h_n(\alpha, \beta)$ can be found in [20, (9.8.2)]. This integral can be computed with the assistance of [24, (18.17.36)] with $z = \alpha-\nu+1$, which implies that for the integral to converge one must have $\Re(\alpha-\nu+1) > 0$. Since the function $x \mapsto (1-x)^{-\nu}$ is analytic (clear from the binomial theorem) on the segment $(-1, 1)$ which is interior to an ellipse with foci at ± 1, then the integrated form implies the expansion by [24, Section 18.18(i)]. $\square$

It is interesting to see that this expansion can also be obtained from more general expansions using a limiting procedure. In order to perform these limits termwise, one must justify the interchange of the limit and the sum. Having already proved the expansion formula, we leave these justification proofs to the reader.

Remark 5.9 *(Formal limit 1).* Start with (4.4) and examine the singular behavior of the Jacobi function of the second kind $Q_\gamma^{(\alpha,\beta)}(z)$ as $z \to 1^+$. Starting with the definition of the Jacobi function of the second kind in terms of the Gauss hypergeometric function, and applying [24, (15.8.2)], results in the identity

$$Q_\gamma^{(\alpha,\beta)}(z) = -\frac{\pi}{2}\csc(\pi\alpha)P_\gamma^{(\alpha,\beta)}(z) + \frac{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)(z-1)^\alpha(z+1)^\beta}\,{}_2F_1\left(\begin{matrix}\gamma+1,-\alpha-\beta-\gamma\\ 1-\alpha\end{matrix};\frac{1-z}{2}\right), \tag{5.17}$$

where $P_\gamma^{(\alpha,\beta)}(z)$ is the Jacobi function of the first kind, $\alpha,\beta,\gamma\in\mathbb{C}$, such that $\alpha+\gamma+1,\alpha+1\notin-\mathbb{N}_0$, is defined by

$$P_\gamma^{(\alpha,\beta)}(z) := \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)}\,{}_2F_1\left(\begin{matrix}-\gamma,\alpha+\beta+\gamma+1\\ \alpha+1\end{matrix};\frac{1-z}{2}\right).$$

Note that $P_\gamma^{(\alpha,\beta)}(z)$ generalizes the Jacobi polynomials for $\gamma=n\in\mathbb{N}_0$. Using (5.17) easily demonstrates that as $z\to 1^+$,

$$(z-1)^{\alpha+1-\nu}Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z)\sim\frac{2^{\alpha-\nu}\Gamma(\alpha+1-\nu)\Gamma(\beta+1-\nu)}{\Gamma(\alpha+\beta-\nu+2+n)},$$

for $\Re(\alpha+1-\nu)>0$, and (5.16) follows.

Lemma 5.10. *Let $a,b\in\mathbb{C}$. Then we have as $0<\tau\to\infty$,*

$$\frac{\Gamma(a\pm i\tau)}{\Gamma(b\pm i\tau)} = e^{\pm\frac{i\pi}{2}(a-b)}\tau^{a-b}\left\{1+\mathcal{O}(\tau^{-1})\right\}, \tag{5.18}$$

where τ^{a-b} takes its principal value.

Proof. Let $\delta\in(0,\pi)$. From [24, (5.11.13)], as $z\to\infty$ with a and b real or complex constants, provided $\arg z\le\pi-\delta(<\pi)$. If one takes $z=\pm i\tau$ with $\tau>0$ then the argument restriction implies $\arg(\pm i\tau)=\pm\pi/2$, and the result follows. □

Remark 5.11 *(Formal limit 2).* Jacobi polynomials are obtained from Wilson polynomials using [20, (9.1.18)]

$$P_n^{(\alpha,\beta)}(x) = \lim_{\tau\to\infty}\frac{1}{\tau^{2n}n!}W_n\left(\frac{(1-x)\tau^2}{2};\frac{\alpha+1}{2},\frac{\alpha+1}{2},\frac{\beta+1}{2}+i\tau,\frac{\beta+1}{2}-i\tau\right). \tag{5.19}$$

Define $\nu=u-t$. Apply (5.19) to (3.9) using (5.18) repeatedly, one obtains

$$\frac{1}{(1-x)^\nu} = \frac{1}{2^\nu}\frac{\left(\frac{\alpha+1}{2},\frac{\alpha+1}{2}\right)_t}{\left(\frac{\alpha+1}{2},\frac{\alpha+1}{2}\right)_{t+\nu}}\sum_{n=0}^\infty\frac{(\nu,\alpha+\beta+1)_n}{\left(\frac{\alpha+1}{2}+u,\frac{\alpha+1}{2}+u\right)_n(\alpha+\beta+1)_{2n}}P_n^{(\alpha,\beta)}(x)$$
$$\times\lim_{\tau\to\infty}\tau^{2\nu+2n}\,{}_2F_3\left(\begin{matrix}\alpha+1+n,\nu+n\\ \frac{\alpha+1}{2}+t+\nu+n,\frac{\alpha+1}{2}+t+\nu+n,\alpha+\beta+2+2n\end{matrix};-\tau^2\right).$$

The above limit of the ${}_2F_3$ can be computed using the asymptotic expansion for large variables of the generalized hypergeometric function [24, (16.11.8)] assuming $\Re(\alpha+1-\nu)>0$. This completes the proof.

From the expansion formula for $(1-x)^{-\nu}$ in Jacobi polynomials (5.16), one can derive some interesting specialization and limit consequences. We omit the justification for interchange of the sums and limits, and leave it to the interested reader.

Corollary 5.12. *Let $x \in (-1,1)$, $\mu \in (-\frac{1}{2},\infty)\setminus\{0\}$, $\nu \in \mathbb{C}$, such that $\Re(\mu-\nu+\frac{1}{2})>0$. Then*

$$\frac{1}{(1-x)^\nu} = \frac{2^{2\mu-\nu}\Gamma(\mu-\nu+\frac{1}{2})\Gamma(\mu)}{\sqrt{\pi}\,\Gamma(2\mu+1-\nu)} \sum_{n=0}^\infty \frac{(\mu+n)(\nu)_n}{(2\mu+1-\nu)_n} C_n^\mu(x). \tag{5.20}$$

Proof. Specializing (5.16) using the definition of the Gegenbauer polynomials in terms of the Jacobi polynomials (1.1), which completes the proof. □

Corollary 5.13. *Let $x \in (-1,1)$, $\mu \in (-\frac{1}{2},\infty)\setminus\{0\}$, $\nu \in \mathbb{C}$, such that $\Re(\mu-\nu+\frac{1}{2})>0$. Then*

$$\frac{1}{(1-x)^\nu} = \frac{\Gamma(\frac{1}{2}-\nu)}{\sqrt{\pi}\,2^\nu\Gamma(1-\nu)} \sum_{n=0}^\infty \frac{\epsilon_n(\nu)_n}{(1-\nu)_n} T_n(x), \tag{5.21}$$

where $\epsilon_n := 2-\delta_{n,0}$ is the Neumann factor.

Proof. Specializing (5.20) using the limit relation for the Chebyshev polynomials of the first kind $T_n(x)$ with the Gegenbauer polynomials, namely [1, (6.4.13)]

$$\lim_{\mu\to 0} \frac{n+\mu}{\mu} C_n^\mu(x) = \epsilon_n T_n(x),$$

completes the proof. □

The following result generalizes [24, (18.18.19)] for $\nu = -n$, $n \in \mathbb{N}_0$.

Corollary 5.14. *Let $x \in (0,\infty)$, $\alpha > -1$, $\nu \in \mathbb{C}$ such that $\Re(\alpha+1-\nu)>0$. Then*

$$\frac{1}{x^\nu} = \Gamma(\alpha+1-\nu) \sum_{n=0}^\infty \frac{(\nu)_n}{\Gamma(\alpha+1+n)} L_n^\alpha(x). \tag{5.22}$$

Proof. Specializing (5.16) using the limit relation for Laguerre polynomials $L_n^\alpha(x)$ with the Gegenbauer polynomials, namely [20, (9.8.16)]

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^\alpha(x),$$

completes the proof. □

6. Definite integrals

Consider a sequence of orthogonal polynomials $(p_k(x;\boldsymbol{\alpha}))$ (over a domain A, with positive weight $w(x;\boldsymbol{\alpha})$) associated with a linear functional $\mathbf{u}$, where $\boldsymbol{\alpha}$ is a set of fixed parameters. Define s_k, $k \in \mathbb{N}_0$ by

$$s_k^2 := \int_A p_k(x;\boldsymbol{\alpha}) p_k(x;\boldsymbol{\alpha})\, w(x;\boldsymbol{\alpha})\, dx.$$

In order to justify interchange between a generalized generating function via connection relation and an orthogonality relation for p_k, we show that the double sum/integral converges in the L^2-sense with respect to the weight $w(x;\boldsymbol{\alpha})$. This requires

$$\sum_{k=0}^{\infty} d_k^2 s_k^2 < \infty, \tag{6.1}$$

where $d_k = \sum_{n=k}^{\infty} a_n c_{k,n}$.

Here a_n is the coefficient multiplying the orthogonal polynomial in the original generating function, and $c_{k,n}$ is the connection coefficient for p_k (with appropriate set of parameters).

Lemma 6.1. *Let* $\mathbf{u}$ *be a classical linear functional and let* $(p_n(x))$, $n \in \mathbb{N}_0$ *be the sequence of orthogonal polynomials associated with* $\mathbf{u}$. *If* $|p_n(x)| \leq K(n+1)^{\sigma}\gamma^n$, *with* K, σ *and* γ *constants independent of* n, *then* $|s_n| \leq K(n+1)^{\sigma}\gamma^n|s_0|$.

Proof. Let $n \in \mathbb{N}_0$, then

$$s_n^2 = \langle \mathbf{u}, p_n^2 \rangle \leq (K(n+1)^{\sigma}\gamma^n)^2 \langle \mathbf{u}, 1 \rangle = (K(n+1)^{\sigma}\gamma^n)^2 s_0^2.$$

The result follows. □

Given $|p_k(x;\boldsymbol{\alpha})| \leq K(k+1)^{\sigma}\gamma^k$, with K, σ and γ constants independent of k, an orthogonality relation for p_k, and $|t| < 1/\gamma$, one has

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} |c_{k,n} s_k| < \infty,$$

which implies

$$\sum_{k=0}^{\infty} |d_k s_k| < \infty.$$

Therefore one has confirmed (6.1), indicating that we are justified in reversing the order of our generalized sums and the orthogonality relations under the above assumptions.

All polynomial families used throughout this paper fulfill such assumptions. See for instance (5.5). Such inequalities depend entirely on the representation of the linear functional. In this section we derive integral representations from the infinite series expansions presented in the previous sections. In all cases, Lemma 6.1 can be applied and we are justified in interchanging the linear form and the infinite sum.

6.1. Definite integrals for Askey–Wilson and Wilson polynomials

The orthogonality relation for the Askey–Wilson polynomials is given by [20, (14.1.2)]

$$\int_{-1}^{1} p_m(x;\mathbf{a}|q) p_n(x;\mathbf{a}|q) \frac{w(x;\mathbf{a}|q)}{\sqrt{1-x^2}} dx = 2\pi h_n(\mathbf{a}|q)\, \delta_{m,n}, \tag{6.2}$$

where $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$, $w : (-1,1) \to [0,\infty)$ is defined by

$$w(x;\mathbf{a}|q) := \frac{(e^{\pm 2i\theta};q)_\infty}{(a_1e^{\pm i\theta}, a_2e^{\pm i\theta}, a_3e^{\pm i\theta}, a_4e^{\pm i\theta};q)_\infty}, \quad x = \cos\theta,$$

and

$$h_n(\mathbf{a}|q) := \frac{(a_1a_2a_3a_4q^{n-1};q)_n(a_1a_2a_3a_4q^{2n};q)_\infty}{(q^{n+1},a_1a_2q^n,a_1a_3q^n,a_1a_4q^n,a_2a_3q^n,a_2a_4q^n,a_3a_4q^n;q)_\infty}.$$

Corollary 6.2. *Let $n \in \mathbb{N}_0$, $x = \cos\theta \in (-1,1)$, $\beta \in (-1,1)$, $\max\{|a_1|,|a_2|,|a_3|,|a_4|,|t|\} < 1$, $c_n(\beta,t,\mathbf{a};q)$ defined as in (3.1). Then*

$$\int_{-1}^{1} \frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty} p_n(x;\mathbf{a}|q)\frac{w(x;\mathbf{a}|q)}{\sqrt{1-x^2}}dx = 2\pi h_n(\mathbf{a}|q)c_n(\beta,t,\mathbf{a};q).$$

Proof. Multiply (3.1) by $w(x;\mathbf{a}|q)p_n(x;\mathbf{a}|q)/\sqrt{1-x^2}$ and integrate over $(-1,1)$ using (6.2) produces the desired result. $\square$

Remark 6.3. In [19], the Nassrallah–Rahman integral [10, (6.3.2)] is used extensively in relation to the Askey–Wilson expansion given in Theorem 3.1. This integral is given as follows. Note that we temporarily adopt a new notation $a_{12} := a_1a_2$, $a_{13} := a_1a_3$, $a_{14} := a_1a_4$, $a_{123} := a_1a_2a_3$, $a_{1234} := a_1a_2a_3a_4$, etc., and that we define $\{a_{11},\dots,a_{34}\} := \{a_{11},a_{12},a_{13},a_{23},a_{24},a_{34}\}$. Let $\max(|q|,|t|,|a_1|,|a_2|,|a_3|,|a_4|) < 1$. Then the Nassrallah–Rahman integral is given by

$$J(t,u,\mathbf{a}|q) := \int_{-1}^{1} \frac{(ue^{\pm i\theta};q)_\infty}{(te^{\pm i\theta};q)_\infty}\frac{w(x;\mathbf{a}|q)}{\sqrt{1-x^2}}dx = \frac{2\pi(ua_1,ua_2,ua_3,a_{1234},ta_{123};q)_\infty \mathsf{I}(t,u,\mathbf{a}|q)}{(q,ua_{123},a_{12},\dots,a_{34},ta_1,ta_2,ta_3;q)_\infty}, \tag{6.3}$$

where

$$\begin{aligned}\mathsf{I}(t,u,\mathbf{a}|q) &:= {}_8\phi_7\left(\begin{matrix}ua_{123}q^{-1},\pm(ua_{123}q)^{\frac12},a_{12},a_{13},a_{23},u/a_4,u/t\\ \pm(ua_{123}q^{-1})^{\frac12},ua_1,ua_2,ua_3,a_{1234},ta_{123}\end{matrix};q,ta_4\right)\\ &= {}_8W_7(ua_{123}q^{-1};a_{12},a_{13},a_{23},u/a_4,u/t;q,ta_4).\end{aligned}$$

The ${}_8W_7(q,ta_4)$ which appears in the Nassrallah–Rahman integral is very-well poised and exactly matches the requisite parameters for the ${}_8W_7$ used in Theorem 3.1. The connection between the Nassrallah–Rahman integral and the coefficients of the Ismail–Simeonov Askey–Wilson expansion (given in Corollary 6.2) can be seen through the following definite integral identity (a q-analogue of the definite integral identity (3.4) for the Wilson polynomials)

$$\begin{aligned}&\int_{-1}^{1} \frac{(ue^{\pm i\theta},e^{\pm 2i\theta};q)_\infty}{(te^{\pm i\theta},a_1e^{\pm i\theta},\dots,a_4e^{\pm i\theta};q)_\infty}\frac{p_n(x;\mathbf{a}|q)}{\sqrt{1-x^2}}dx\\ &\qquad= t^n(u/t;q)_n\int_{-1}^{1}\frac{(uq^{\frac n2}e^{\pm i\theta},e^{\pm 2i\theta};q)_\infty}{(tq^{-\frac n2}e^{\pm i\theta},a_1q^{\frac n2}e^{\pm i\theta},\dots,a_4q^{\frac n2}e^{\pm i\theta};q)_\infty}\frac{dx}{\sqrt{1-x^2}}\\ &\qquad= t^n(u/t;q)_n\int_{-1}^{1}\frac{(uq^{\frac n2}e^{\pm i\theta};q)_\infty w(x;\mathbf{a}q^{\frac n2}|q)}{(tq^{-\frac n2}e^{\pm i\theta};q)_\infty}\frac{dx}{\sqrt{1-x^2}},\end{aligned} \tag{6.4}$$

where $\mathbf{a}q^{\frac n2} := \{a_1q^{\frac n2},a_2q^{\frac n2},a_3q^{\frac n2},a_4q^{\frac n2}\}$. The proof of the identity (6.4) can be found in Ismail & Simeonov [19]. The proof, which is highly technical, is due to appearance of poles in the integrand with cancelling residues in the unit circle. The proof relies on the Rodrigues-type formula for the Askey–Wilson polynomials [20, (14.1.12)]

$$\tilde{w}(x;\mathbf{a}|q)p_n(x;\mathbf{a}|q)=\left(\frac{q-1}{2}\right)^n q^{\frac{1}{4}n(n-1)}D_q^n\,\tilde{w}(x;\mathbf{a}q^{\frac{n}{2}}|q),$$

where $\tilde{w}(x;\mathbf{a}|q):=w(x;\mathbf{a}|q)/\sqrt{1-x^2}$. The Askey–Wilson operator being defined by

$$D_qf(x):=\frac{\delta_qf(x)}{\delta_qx}=\frac{\breve{f}(q^{\frac{1}{2}}e^{i\theta})-\breve{f}(q^{-\frac{1}{2}}e^{i\theta})}{\frac{1}{2}(q^{\frac{1}{2}}-q^{\frac{1}{2}})(e^{i\theta}-e^{-i\theta})},\tag{6.5}$$

$f(\cos\theta)=\breve{f}(e^{i\theta})$, and the integration by parts formula for the Askey–Wilson operator (6.5) given in [3], [17, Section 16.1].

Now we give a definite integral for the Wilson polynomials which is equivalent to (3.9). This equivalence follows through Lemmas 3.4, 3.8. We will need the weight function for the Wilson polynomials (3.5) and the Wilson square norm (3.12).

Theorem 6.4. *Let $n\in\mathbb{N}_0$, $t,u\in\mathbb{C}$, $\Re(a_1,a_2,a_3,a_4)>0$, and non-real parameters occur in conjugate pairs. Then*

$$\int_0^\infty\frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(u+ix)\Gamma(u-ix)}W_n(x^2;\mathbf{a})\mathsf{W}(x;\mathbf{a})dx=\frac{H_n(\mathbf{a})(a_{123})_u(a_1,a_2,a_3)_t(a_{123}+u)_{2n}(a_{1234}-1)_n}{(a_{123})_t(a_1,a_2,a_3)_u(a_{123}+t)_n(a_{1234}-1)_{2n}n!}$$
$$\times\frac{(u-t)_n}{(a_1+u,a_2+u,a_3+u)_n}{}_7F_6\left(\begin{matrix}a_{123}+u+2n-1,\frac{a_{123}+u+2n+1}{2},a_{12}+n,a_{13}+n,a_{23}+n,u-a_4,u-t+n\\ \frac{a_{123}+u+2n-1}{2},a_1+u+n,a_2+u+n,a_3+u+n,a_{123}+t+n,a_{1234}+2n\end{matrix};1\right).$$

Proof. Multiply both sides of the Wilson polynomial expansion (3.9) by $W_m(x^2;\mathbf{a})\mathsf{W}(x;\mathbf{a})$, integrate over $(0,\infty)$ using orthogonality of the Wilson polynomials. Replace in the resulting expression $m\mapsto n$, and the result follows. □

6.2. Definite integrals for the continuous q-Jacobi and Jacobi polynomials

The orthogonality relation for the continuous q-Jacobi polynomials [20, (14.10.2)], after scaling so that $q^{\alpha+\frac{1}{2}}\mapsto\alpha$ and $q^{\beta+\frac{1}{2}}\mapsto\gamma$ is

$$\int_{-1}^1 P_m^{(\alpha,\gamma)}(x|q)P_n^{(\alpha,\gamma)}(x|q)\frac{w(x;\alpha,\gamma|q)}{\sqrt{1-x^2}}dx=2\pi g_n(\alpha,\gamma;q)\delta_{mn},$$

where

$$w(x;\alpha,\gamma|q):=\left|\frac{(e^{2i\theta};q)_\infty}{(\alpha^{\frac{1}{2}}e^{i\theta},-\gamma^{\frac{1}{2}}e^{i\theta};q^{\frac{1}{2}})_\infty}\right|^2,$$

and

$$g_n(\alpha,\gamma;q):=\frac{\alpha^n(1-\alpha\gamma)(q^{\frac{1}{2}}\alpha,q^{\frac{1}{2}}\gamma,-q(\alpha\gamma)^{\frac{1}{2}};q)_n((\alpha\gamma q)^{\frac{1}{2}},q(\alpha\gamma)^{\frac{1}{2}};q)_\infty}{(1-q^{2n}\alpha\gamma)(q,\alpha\gamma,-(\alpha\gamma)^{\frac{1}{2}};q)_n(q,q^{\frac{1}{2}}\alpha,q^{\frac{1}{2}}\gamma,-(\alpha\gamma)^{\frac{1}{2}},-(\alpha\gamma q)^{\frac{1}{2}};q)_\infty}.$$

Corollary 6.5. *Let $n\in\mathbb{N}_0$, $x=\cos\theta\in(-1,1)$, $\alpha,\gamma\in(-\frac{1}{2},\infty)$, $d_n(\beta,t,\alpha,\gamma;q)$ defined as in (4.3). Then*

$$\int_{-1}^1\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}P_n^{(\alpha,\gamma)}(x|q)\frac{w(x;\alpha,\gamma|q)}{\sqrt{1-x^2}}dx=2\pi g_n(\alpha,\gamma;q)d_n(\beta,t,\alpha,\gamma;q).$$

Proof. Multiply (4.3) by $w(x;\alpha,\gamma|q)P_n^{(\alpha,\gamma)}(x|q)/\sqrt{1-x^2}$ and integrate over $(-1,1)$ produces the result. $\square$

Corollary 6.6. *Let $n \in \mathbb{N}_0$, $\alpha,\beta > -1$, $\nu \in \mathbb{C}$, such that $\Re(\alpha+1-\nu) > 0$. Then*

$$\int_{-1}^{1} (1-x)^{-\nu} P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx = \frac{2^{\alpha+\beta+1-\nu}\Gamma(\alpha+1-\nu)(\nu)_n\Gamma(\beta+1+n)}{n!\Gamma(\alpha+\beta+2-\nu+n)}.$$

Proof. Follows from orthogonality of the Jacobi polynomials [20, (9.8.2)] and (5.16). $\square$

6.3. Definite integrals for the continuous q-ultraspherical/Rogers and Gegenbauer polynomials

The property of orthogonality for the continuous q-ultraspherical/Rogers polynomials found in Koekoek et al. (2010) [20, (3.10.16)] is given by

$$\int_{-1}^{1} C_m(x;\beta|q)C_n(x;\beta|q)\frac{w(x;\beta|q)}{\sqrt{1-x^2}}\,dx = 2\pi\frac{(1-\beta)(\beta,q\beta;q)_\infty(\beta^2;q)_n}{(1-\beta q^n)(\beta^2,q;q)_\infty(q;q)_n}\delta_{mn}, \tag{6.6}$$

where $w:(-1,1)\to[0,\infty)$ is the weight function defined by

$$w(x;\beta|q) := \left|\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty}\right|^2. \tag{6.7}$$

We use this orthogonality relation for proofs of the following definite integrals.

Corollary 6.7. *Let $n \in \mathbb{N}_0$, $x=\cos\theta \in (-1,1)$, $\beta,\gamma \in (-1,1)\setminus\{0\}$, $0<|q|<1$, $|t|<1$. Then*

$$\int_{-1}^{1} \frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}C_n(x;\gamma|q)\frac{w(x;\gamma|q)}{\sqrt{1-x^2}}dx$$
$$= 2\pi\frac{(\gamma,\gamma q;q)_\infty(\beta,\gamma^2;q)_n}{(\gamma^2,q;q)_\infty(q,q\gamma;q)_n}\,{}_2\phi_1\left(\begin{matrix}\beta\gamma^{-1},\beta q^n\\ \gamma q^{n+1}\end{matrix};q,\gamma t^2\right)t^n. \tag{6.8}$$

Proof. We begin with the generalized generating function (5.4), multiply both sides by

$$C_m(x;\gamma|q)\frac{w(x;\gamma|q)}{\sqrt{1-x^2}},$$

where $w(x;\gamma|q)$ is obtained from (6.7), integrating over $(-1,1)$ using the orthogonality relation (6.6) produces the desired result. $\square$

Corollary 6.8. *Let $n \in \mathbb{N}_0$, $\lambda,\mu \in (-\frac{1}{2},\infty)\setminus\{0\}$, $|t|<1$. Then*

$$\int_{-1}^{1} \frac{C_n^{\mu}(x)}{(1-2tx+t^2)^{\lambda}}(1-x^2)^{\mu-\frac{1}{2}}dx = \frac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})(\lambda,2\mu)_n}{\Gamma(\mu+1)(\mu+1)_n n!}\,{}_2F_1\left(\begin{matrix}\lambda-\mu,\lambda+n\\ \mu+n+1\end{matrix};t^2\right)t^n.$$

Proof. Starting from (6.8) and taking the limit $q\uparrow 1^-$, using [20, (14.10.35)]

$$\lim_{q\uparrow 1^-} C_n(x;q^{\lambda}|q) = C_n^{\mu}(x),$$

the result follows. $\square$

Observe that since the Gegenbauer polynomials can be written as [24, (18.5.10)]

$$C_n^\lambda(x) = (2x)^n \frac{(\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -\frac{n}{2}, -\frac{n+1}{2} \\ 1-\lambda-n \end{matrix} ; \frac{1}{x^2} \right),$$

the above integral can be written in terms of a ${}_2F_1$, and we also have a similar ${}_2F_1$ on the right-hand side.

Corollary 6.9. *Let $n \in \mathbb{N}_0$, $\alpha, \beta > -1$, $\nu \in \mathbb{C}$, such that $\Re(\alpha + 1 - \nu) > 0$. Then*

$$\int_{-1}^{1} (1-x)^{-\nu} C_n^\mu(x)(1-x^2)^{\mu-\frac{1}{2}}\, dx = \frac{2^{\alpha+\beta+1-\nu}\Gamma(\alpha+1-\nu)(\nu)_n\Gamma(\beta+1+n)}{n!\Gamma(\alpha+\beta+2-\nu+n)}.$$

Proof. Follows from orthogonality of the Gegenbauer polynomials [20, (9.8.20)] and (5.20). □

Similar definite integrals can be obtained for the Chebyshev polynomials of the first kind multiplied by $(1-x)^{-\nu}$ and for the Laguerre polynomials multiplied by $1/x^\nu$, using (5.21) and (5.22) respectively.

7. Outlook

It has been suggested by a referee that it would be interesting to investigate the transformation properties of the derived definite integrals in this paper since the Rogers generating function is a generalization of the generalized Stieltjes kernel $(z-x)^{-\nu}$. The transformation and transmutation properties of the generalized Stieltjes transformations for the Gauss hypergeometric function has been summarized recently in a paper by Koornwinder [21]. Generalized Stieltjes transforms have evident properties of mapping solutions of the hypergeometric differential equation to other solutions of the same equation, while generalized Stieltjes transforms map solutions of the hypergeometric differential equation to solutions of another differential equation. Unfortunately a similar analysis for our problem is not easily accomplished because the singularities of the Gauss hypergeometric differential equation are 0, 1 and ∞, whereas for instance, for Jacobi-type orthogonal polynomials, the singularities are at ± 1 and ∞. In future research, we would like to apply an analogous result to study the transformation properties for definite integrals of Jacobi-type orthogonal polynomials and also for their q-analogs such as for the continuous q-ultraspherical/Rogers polynomials using the Gegenbauer and Rogers generating functions. This study could have deep consequences.

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