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Erratum: "Developments in determining the gravitational potential using toroidal functions" [Astronomische Nachrichten 321, 363 (2000)]

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We fix several errors in "Developments in determining the gravitational potential using toroidal functions" (Astronomische Nachrichten, Vol. 321, Issue 5/6, p. 363).

1 Introduction

In Cohl et al. (2000), several addition theorems are derived for associated Legendre functions. These addition theorems arise by the study of the Newtonian gravitational potential expressed in axisymmetric coordinate systems that admit separable solutions to Laplace's equation in three-dimensional Euclidean space. There are several errors that appear in this paper in regard to these addition theorems. In this erratum we would like to correct these errors. We use the notation that for $a, a' \in \mathbf{R}$, we define $a_{\lessgtr} := {}^{\min}_{\max}\{a, a'\}$.

In Sect. 5.2 of Cohl et al. (2000), we discuss eigenfunction expansions for the reciprocal distance between two points $|\mathbf{x}-\mathbf{x}'|^{-1}$ in oblate spheroidal coordinates. The expansion formula that is cited in the original reference (Hobson 1931, Sect. 251) is incorrect. This expansion formula is given correctly in (41), p. 218, MacRobert (1947), namely

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mathrm{i}}{a}\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}(-1)^m\epsilon_m\left[\frac{(n-m)!}{(n+m)!}\right]^2 \times P_n^m(\cos\theta)P_n^m(\cos\theta') \times P_n^m(\mathrm{i}\sinh\sigma_<)Q_n^m(\mathrm{i}\sinh\sigma_>)\cos(m(\phi-\phi')),$$

where $\epsilon_m = 2-\delta_{m,0}$ is the Neumann factor commonly occurring in Fourier cosine series, with $\sigma,\sigma' \in [0,\infty)$, $\theta,\theta' \in [0,\pi]$, $\phi,\phi' \in \mathbf{R}$. If we define $\chi > 1$ by

$$\chi = (2\cosh\sigma\cosh\sigma'\sin\theta\sin\theta')^{-1}(\sinh^2\sigma+\sinh^2\sigma' + \sin^2\theta+\sin^2\theta'-2\sinh\sigma\sinh\sigma'\cos\theta\cos\theta'),$$

then for $m = 0, \pm1, \pm2, \ldots$, the correct addition theorems are given by

$$Q_{m-1/2}(\chi) = \mathrm{i}\pi(-1)^m\sqrt{\cosh\sigma\cosh\sigma'\sin\theta\sin\theta'} \times\sum_{n=|m|}^{\infty}(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 P_n^m(\cos\theta)P_n^m(\cos\theta') \times P_n^m(\mathrm{i}\sinh\sigma_<)Q_n^m(\mathrm{i}\sinh\sigma_>),$$

$$P^m_{-1/2}\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) = \mathrm{i}\pi^{-1/2}\Gamma(m+1/2) \times\sqrt{2(\chi^2-1)^{1/2}\cosh\sigma\cosh\sigma'\sin\theta\sin\theta'} \times\sum_{n=|m|}^{\infty}(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 P_n^m(\cos\theta)P_n^m(\cos\theta') \times P_n^m(\mathrm{i}\sinh\sigma_<)Q_n^m(\mathrm{i}\sinh\sigma_>).$$

In Sect. 6.1 of Cohl et al. (2000), we discuss eigenfunction expansions for $|\mathbf{x}-\mathbf{x}'|^{-1}$ in bispherical coordinates. The expansion formula that is cited in the original reference (Morse & Feshbach 1953, 10.3.74) is incorrect. This expansion formula is given correctly in (9), p. 222, MacRobert (1947), namely if we define $s = \cosh\sigma, s' = \cosh\sigma', \tau = \cos\theta, \tau' = \cos\theta'$, then

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{a}\sqrt{(s-\tau)(s'-\tau')}\sum_{n=0}^{\infty}\mathrm{e}^{-(n+1/2)(\sigma_>-\sigma_<)} \times\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos(m(\phi-\phi')),$$

where $\sigma,\sigma' \in [0,\infty)$, $\theta,\theta' \in [0,\pi]$, $\phi,\phi' \in \mathbf{R}$. If we define $\chi > 1$ by

$$\chi = (2\sin\theta\sin\theta'(s-\tau)(s'-\tau'))^{-1} \times(\sin^2\theta(s'-\tau')^2+\sin^2\theta'(s-\tau)^2 +\left[(s'-\tau')\sinh\sigma-(s-\tau)\sinh\sigma'\right]^2),$$

then for $m = 0, \pm1, \pm2, \ldots$, the correct addition theorems are given by

$$Q_{m-1/2}(\chi) = \pi\sqrt{\sin\theta\sin\theta'}\sum_{n=|m|}^{\infty}\frac{(n-m)!}{(n+m)!}$$

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$$\times\, \mathrm{e}^{-(n+1/2)(\sigma_> - \sigma_<)} P_n^m(\cos\theta) P_n^m(\cos\theta'),$$

$$P_{-1/2}^m\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) = (-1)^m \pi^{-1/2}\Gamma\left(m+1/2\right)$$
$$\times \sqrt{2(\chi^2-1)^{1/2}\sin\theta\sin\theta'} \sum_{n=|m|}^{\infty} \frac{(n-m)!}{(n+m)!}$$
$$\times\, \mathrm{e}^{-(n+1/2)(\sigma_> - \sigma_<)} P_n^m(\cos\theta) P_n^m(\cos\theta').$$

In Sect. 6.2 we discuss eigenfunction expansions for $|\mathbf{x}-\mathbf{x}'|^{-1}$ in toroidal coordinates. The expansion formula cited in this section is correct. However, the given addition theorems are incorrect. With $\chi > 1$ defined in (51) of Cohl et al. (2000), then for $m = 0, \pm 1, \pm 2, \ldots$, the correct addition theorems are given by

$$Q_{m-1/2}(\chi) = (-1)^m \sqrt{\sinh\sigma\sinh\sigma'}$$
$$\times \sum_{n=0}^{\infty} \epsilon_n \frac{\Gamma\left(n-m+\frac{1}{2}\right)}{\Gamma\left(n+m+\frac{1}{2}\right)} \cos(n(\psi-\psi'))$$
$$\times P_{n-1/2}^m(\cosh\sigma_<) Q_{n-1/2}^m(\cosh\sigma_<),$$

$$P_{-1/2}^m\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) = \pi^{-3/2}\Gamma\left(m+1/2\right)$$
$$\times \sqrt{2(\chi^2-1)^{1/2}\sinh\sigma\sinh\sigma'}$$
$$\times \sum_{n=0}^{\infty} \epsilon_n \frac{\Gamma\left(n-m+\frac{1}{2}\right)}{\Gamma\left(n+m+\frac{1}{2}\right)} \cos(n(\psi-\psi'))$$
$$\times P_{n-1/2}^m(\cosh\sigma_<) Q_{n-1/2}^m(\cosh\sigma_<).$$

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