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Bilateral discrete and continuous orthogonality relations in the q^{-1} -symmetric Askey scheme

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Abstract In the q^{-1} -symmetric Askey scheme, namely the q^{-1} -Askey–Wilson, continuous dual q^{-1} -Hahn, q^{-1} -Al-Salam–Chihara, continuous big q^{-1} -Hermite and continuous q^{-1} -Hermite polynomials, we compute bilateral discrete and continuous orthogonality relations. We also derive a q -beta integral which comes from the continuous orthogonality relation for the q^{-1} -Askey–Wilson polynomials. In the $q \rightarrow 1^-$ limit, this q -beta integral corresponds to a beta integral of Ramanujan-type which we present and provide two proofs for.

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1 Preliminaries

Recall the notion of a *multiset* which extends the definition of a set where the multiplicity of elements is allowed. This notion becomes important for basic hypergeometric functions, where numerator parameter entries or denominator parameter entries may be identical. We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$, and if \mathbb{C} which represents the set of complex numbers, then $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $\mathbb{C}^\dagger := \{z \in \mathbb{C}^* : |z| < 1\}$. We also need the q -shifted factorial $(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a)$, $n \in \mathbb{N}_0$ and one may define

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), \quad (1)$$

where $|q| < 1$. Let $\mathbf{x} := \{x_1, \dots, x_k\}$, $k \in \mathbb{N}$, be a multiset. Then we will use the product convention

$$(\mathbf{x}; q)_b := (x_1, \dots, x_k; q)_b := (x_1; q)_b \cdots (x_k; q)_b,$$

where $b \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \mathbb{N}$. Define the multisets $\mathbf{a} := \{a_1, \dots, a_r\}$, $\mathbf{b} := \{b_1, \dots, b_s\}$. The *basic hypergeometric series*, which we will often use, is defined for $q, z \in \mathbb{C}^*$ such that $|q| < 1$, $s, r \in \mathbb{N}_0$, $b_j \notin \{q^{-m} : m \in \mathbb{N}_0\}$, $j = 1, \dots, s$, as [12, (1.10.1)]

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$${}_r\phi_s \left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}; q, z \right) := {}_r\phi_s(\mathbf{a}; \mathbf{b}; q, z) := \sum_{k=0}^{\infty} \frac{(\mathbf{a}; q)_k}{(q, \mathbf{b}; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k. \tag{2}$$

For $s + 1 > r$, ${}_r\phi_s$ is an entire function of z , for $s + 1 = r$ then ${}_r\phi_s$ is convergent for $|z| < 1$, and for $s + 1 < r$ the series is divergent unless it is *terminating* (one of the numerator parameters is q^{-n} for $n \in \mathbb{N}_0$).

2 Introduction

The main realization which occurred during the writing of this paper, was that the continuous weight function $w_2(x)$ in [6, Theorem 21.6.4] for continuous q^{-1} -Hermite polynomials exactly matched (by replacing the continuous variable of integration with a discrete variable for summation), the infinite discrete bilateral orthogonality relation [10, (2.13)] for these same polynomials. See the discussion at the top of Sect. 4. It was then expected that a correspondence like this would continue for the other q^{-1} -symmetric families. Having recently been informed of the techniques which were presented in Ismail & Rahman (1995) [9] for obtaining continuous bilateral integrals from infinite discrete bilateral series, the rest of the results were simply the effect of applying these methods to [10, (2.13)] to obtain alternative continuous bilateral orthogonality relations for the q^{-1} -symmetric families.

Herein, we study the following five systems of orthogonal polynomials in the q -symmetric family. The q -Askey–Wilson (or more compactly, Askey–Wilson) polynomials $p_n(x; a, b, c, d|q)$ are defined by [12, (14.1.1)]

$$p_n(x; a, b, c, d|q) := a^{-n} (ab, ac, ad; q)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right), \tag{3}$$

where $x = \cos \theta$. The continuous dual q -Hahn polynomials $p_n(x; a, b, c|q)$ are defined by [12, (14.3.1)]

$$p_n(x; a, b, c|q) := a^{-n} (ab, ac; q)_n {}_3\phi_2 \left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix}; q, q \right). \tag{4}$$

The Al-Salam–Chihara polynomials $Q_n(x; a, b|q)$ are defined by [12, (14.8.1)]

$$Q_n(x; a, b|q) := a^{-n} (ab; q)_n {}_3\phi_2 \left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}; q, q \right). \tag{5}$$

The continuous big q -Hermite polynomials $H_n(x; a|q)$ are defined by [12, (14.18.1)]

$$H_n(x; a|q) := a^{-n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix}; q, q \right), \tag{6}$$

where $x = \frac{1}{2}(z + z^{-1})$. The continuous q -Hermite polynomials $H_n(x|q)$ are defined by [12, (14.26.1)]

$$H_n(x|q) := e^{in\theta} {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix}; q, q^n e^{-2i\theta} \right). \tag{7}$$

The orthogonality relations over $x \in (-1, 1)$ for the above q -symmetric polynomials is given, for instance, in [12, (14.1.2), (14.3.2), (14.8.2), (14.18.1), (14.26.2)]. One can consider the above polynomials as a function of $z = e^{i\theta}$, $\theta \in \mathbb{C}$, even though they are polynomials in $x = \cos \theta$. In this case, we adopt the following notation to view the polynomials as a function of z , namely

$$\begin{aligned} p_n[z; a, b, c, d|q] &:= p_n(x; a, b, c, d|q), \\ p_n[z; a, b, c|q] &:= p_n(x; a, b, c|q), \\ Q_n[z; a, b|q] &:= Q_n(x; a, b|q), \\ H_n[z; a|q] &:= H_n(x; a|q), \\ H_n[z|q] &:= H_n(x|q). \end{aligned}$$

We are interested in the case that $|q| > 1$. To simplify notation, we set

$$\mathbf{p}_n(x; a, b, c, d|q) := \mathbf{p}_n[z; a, b, c, d|q] := i^{-n} p_n[iz; ia, ib, ic, id|q^{-1}], \tag{8}$$

$$\mathbf{p}_n(x; a, b, c|q) := \mathbf{p}_n[z; a, b, c|q] := i^{-n} p_n[iz; ia, ib, ic|q^{-1}], \tag{9}$$

$$\mathbf{Q}_n(x; a, b|q) := \mathbf{Q}_n[z; a, b|q] := i^{-n} Q_n[iz; ia, ib|q^{-1}], \tag{10}$$

$$\mathbf{H}_n(x; a|q) := \mathbf{H}_n[z; a|q] := i^{-n} H_n[iz; ia|q^{-1}], \tag{11}$$

$$\mathbf{H}_n(x|q) := \mathbf{H}_n[z|q] := i^{-n} H_n[iz|q^{-1}], \tag{12}$$

where $|q| < 1$ and $x = \frac{1}{2}(z - z^{-1})$. We refer to the polynomials (8)–(12), as the q^{-1} -symmetric family of polynomials in the q -Askey scheme. In particular, we refer to them as the q^{-1} -Askey–Wilson, continuous dual q^{-1} -Hahn, q^{-1} -Al-Salam–Chihara, continuous big q^{-1} -Hermite and continuous q^{-1} -Hermite polynomials respectively. These versions of the polynomials have the nice property that they remain symmetric in their parameters and are very easy to compute directly from terminating basic hypergeometric representations of the symmetric subfamilies of the Askey–Wilson polynomials using [12, (1.8.7)] $(a; q^{-1})_n = q^{-\binom{n}{2}} (-a)^n (\frac{1}{a}; q)_n$, $a \neq 0$. Furthermore in both the q -symmetric Askey scheme or in the q^{-1} -symmetric Askey scheme, these polynomials can be obtained simply by taking limits as $d \rightarrow c \rightarrow b \rightarrow a \rightarrow 0$. Note that it is easily verified that the q^{-1} -Askey–Wilson polynomials $\mathbf{p}_n(x; a, b, c, d|q)$ which are simply renormalized Askey–Wilson polynomials with parameters given by their reciprocals, are given by

$$\mathbf{p}_n(x; a, b, c, d|q) = q^{-3\binom{n}{2}} (abcd)^n p_n[iz; -\frac{i}{a}, -\frac{i}{b}, -\frac{i}{c}, -\frac{i}{d}|q], \tag{13}$$

where $x = \frac{1}{2}(z - z^{-1})$.

For the q^{-1} -Askey–Wilson polynomials, two terminating basic hypergeometric series representations are given by

$$\mathbf{p}_n(x; a, b, c, d|q) = q^{-3\binom{n}{2}} (-a^2bcd)^n \left(-\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}; q\right)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, \frac{q^{n-1}}{abcd}, \frac{z}{a}, -\frac{1}{az} \\ -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad} \end{matrix}; q, q \right) \tag{14}$$

$$= q^{-3\binom{n}{2}} (-abcdz)^n \left(-\frac{1}{ab}, -\frac{1}{cz}, -\frac{1}{dz}; q\right)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, \frac{z}{a}, \frac{z}{b}, -q^{1-n}cd \\ -\frac{1}{ab}, -q^{1-n}cz, -q^{1-n}dz \end{matrix}; q, q \right). \tag{15}$$

Two basic hypergeometric series representations of the continuous dual q^{-1} -Hahn polynomials are given by

$$\mathbf{p}_n(x; a, b, c|q) = q^{-\binom{n}{2}} (-a)^n \left(\frac{z}{a}, -\frac{1}{az}; q\right)_n {}_3\phi_2 \left(\begin{matrix} q^{-n}, -q^{1-n}ab, -q^{1-n}ac \\ -q^{1-n}az, \frac{q^{1-n}a}{z} \end{matrix}; q, q \right) \tag{16}$$

$$= q^{-2\binom{n}{2}} (-abc)^n \left(-\frac{1}{ab}, -\frac{1}{ac}; q\right)_n {}_3\phi_2 \left(\begin{matrix} q^{-n}, \frac{z}{a}, -\frac{1}{az} \\ -\frac{1}{ab}, -\frac{1}{ac} \end{matrix}; q, -\frac{q^n}{bc} \right). \tag{17}$$

A terminating basic hypergeometric representation of the q^{-1} -Al-Salam–Chihara polynomials is given by

$$\mathbf{Q}_n(x; a, b|q) = q^{-\binom{n}{2}} (-b)^n \left(-\frac{1}{ab}; q\right)_n {}_3\phi_1 \left(\begin{matrix} q^{-n}, \frac{z}{a}, -\frac{1}{az} \\ -\frac{1}{ab} \end{matrix}; q, q^n \frac{a}{b} \right). \tag{18}$$

A terminating basic hypergeometric representation of the continuous big q^{-1} -Hermite polynomials is given by

$$\mathbf{H}_n(x; a|q) = \left(-\frac{1}{a}\right)^n {}_3\phi_0 \left(\begin{matrix} q^{-n}, \frac{z}{a}, -\frac{1}{az} \\ - \end{matrix}; q, -q^n a^2 \right). \tag{19}$$

A terminating basic hypergeometric representation of the continuous q^{-1} -Hermite polynomials is given by

$$\mathbf{H}_n(x|q) = z^n {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q, -\frac{q}{z^2} \right). \tag{20}$$

For comparison with similar polynomials which have been previously used in the literature, we note that

$$\mathbf{p}_n(x; a, b, c, d|q) = q^{-3\binom{n}{2}} (-abcd)^n p_n(x, qa, qb, qc, qd), \tag{21}$$

where p_n is Ismail–Zhang–Zhou’s q^{-1} -Askey–Wilson polynomials [11, (2.7)] defined by

$$p_n(x, \mathbf{a}) := \left(\frac{a}{q}\right)^n \left(-\frac{q^2}{ab}, -\frac{q^2}{ac}, -\frac{q^2}{ad}; q\right)_n {}_4\phi_3 \left(q^{-n}, \frac{q^{n+3}}{abcd}, \frac{qz}{a}, -\frac{q}{az}; q, q \right). \tag{22}$$

Furthermore,

$$\mathbf{p}_n(x; a, b, c|q) = q^{-2\binom{n}{2}} (-bc)^n \left(-\frac{1}{ab}, -\frac{1}{bc}; q\right)_n V_n(x; qa, qb, qc|q), \tag{23}$$

where V_n is Ismail–Zhang–Zhou’s continuous dual q^{-1} -Hahn polynomials [11, (5.2)] defined by

$$V_n(x; a, b, c|q) := \left(\frac{a}{q}\right)^n \frac{(-\frac{q^2}{ac}; q)_n}{(-\frac{q^2}{bc}; q)_n} {}_3\phi_2 \left(q^{-n}, \frac{qz}{a}, -\frac{q}{az}; q, -\frac{q^{n+2}}{bc} \right). \tag{24}$$

Also,

$$\mathbf{Q}_n(x; a, b|q) = q^{-\binom{n}{2}} (-1)^n (q; q)_n Q_n(x; a, b), \tag{25}$$

where Q_n is Ismail’s q^{-1} -Al-Salam–Chihara polynomials [7, (3.5)] defined by

$$Q_n(x; a, b) := a^n \frac{(\frac{z}{a}; q)_n}{(q; q)_n} {}_2\phi_1 \left(q^{-n}, -\frac{1}{bz}; q, \frac{qb}{z} \right). \tag{26}$$

Observe that Askey’s continuous q^{-1} -Hermite polynomials are identical to ours

$$\mathbf{H}_n(x|q) = h_n(x|q), \tag{27}$$

which was the motivation of our notation.

It will be helpful to discuss the state of the art in regards to orthogonality relations for the q^{-1} -symmetric families, namely the q^{-1} -Askey–Wilson polynomials and their symmetric subfamilies. The orthogonality (and biorthogonality) relations we discuss are all for a finite-family, namely there exists an $N \in \mathbb{N}_0$ such that the orthogonality relation is only valid for all $m, n \leq N$. However when one considers the limit as $d \rightarrow 0$ of the q^{-1} -Askey–Wilson polynomials to the continuous dual q^{-1} -Hahn polynomials, then the orthogonality relations lead to an infinite family. However, for the remainder of this section, we restrict ourselves to the four parameter finite family of orthogonal polynomials which correspond to the q^{-1} -Askey–Wilson polynomials (see [11, §2]). These polynomials represent a finite family of orthogonal polynomials because there always exists some $N \in \mathbb{N}_0$ (which depends on the values of the parameters) over which the orthogonality relation is valid only for degrees $0 \leq n \leq N$.

The first orthogonality relation found for this finite family is that which comes from the weight function which Askey found for the continuous q^{-1} -Hermite polynomials in [2]. According to Ismail & Masson [8, (7.31)], Askey proved that these polynomials satisfy the following orthogonality relation in [1], which is given as follows

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathbf{p}_m[q^x; \mathbf{a}|q] \mathbf{p}_n[q^x; \mathbf{a}|q] \frac{(-q^{1+x} \mathbf{a}, q^{1-x} \mathbf{a}; q)_{\infty}}{(-q^{2x+1}, -q^{1-2x}; q)_{\infty}} dx \\ &= \frac{(q, -qab, -qac, -qad, -qbc, -qbd, -qcd; q)_{\infty}}{(qabcd; q)_{\infty}} \\ & \times q^{-6\binom{n}{2}} (-a^2 b^2 c^2 d^2)^n \frac{(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}, -\frac{1}{bc}, -\frac{1}{bd}, -\frac{1}{cd}; q)_n (\frac{1}{qabcd}; q)_{2n}}{(\frac{1}{abcd}; q)_n (\frac{1}{abcd}; q)_{2n}} \delta_{m,n}. \end{aligned} \tag{28}$$

This orthogonality relation is explicitly given in [11, (2.9)]. The total mass of this orthogonality relation corresponding to the $m = n = 0$ case is the famous Askey q -beta integral [1]. This orthogonality relation, by taking limits $d \rightarrow c \rightarrow b \rightarrow a \rightarrow 0$, leads to orthogonality relations for the symmetric subfamilies of the q^{-1} -Askey–Wilson polynomials. There also exists, in the literature, an infinite discrete bilateral orthogonality

relation for the q^{-1} -Askey–Wilson polynomials. This orthogonality relation is given in [11, (2.13)] with typographical errors (in the original version submitted on Wed, May 11, 2022) and the corrected version is given in Theorem 3.1 below.

There exists a q -beta integral due to Ismail–Masson [8, (7.30)] (see also cf. [9, (3.13)]). It is given by

$$\int_0^\infty \frac{(iaz, -\frac{ia}{z}, ibz, -\frac{ib}{z}, icz, -\frac{ic}{z}, idz, -\frac{id}{z}; q)_\infty}{(fz, gz, -\frac{qz}{f}, -\frac{qz}{g}, -\frac{f}{z}, -\frac{g}{z}, \frac{q}{fz}, \frac{q}{gz}; q)_\infty} \left(1 + \frac{1}{z^2}\right) dz = \frac{2\pi i (\frac{ab}{q}, \frac{ac}{q}, \frac{ad}{q}, \frac{bc}{q}, \frac{bd}{q}, \frac{cd}{q}; q)_\infty}{(q, \frac{g}{f}, \frac{qf}{g}, -fg, -\frac{q}{fg}, \frac{abcd}{q^3}; q)_\infty}, \tag{29}$$

where $\Im f, \Im g$ and $\Im(f/g)$ are not $0 \pmod{2\pi}$ and $\Im(fg) \neq \pi \pmod{2\pi}$. This q -beta integral leads to an orthogonality relation for continuous q^{-1} -Hermite polynomials, but it leads to a biorthogonality relation in the 4-parameter case, see Ismail & Masson [8].

In this paper, we obtain a new continuous orthogonality relation in the 4-parameter case by starting with the infinite discrete bilateral orthogonality relation for the q^{-1} -Askey–Wilson polynomials given in Theorem 3.1 below and using the methods described in [9] in order to obtain a continuous analogue. It is given in Theorem 4.4 below. The weight function for this orthogonality relation in the $d \rightarrow c \rightarrow b \rightarrow a \rightarrow 0$ case of the continuous q^{-1} -Hermite polynomials is given in [6, Theorem 21.6.4] for $w_2(x)$. We derive orthogonality relations for the q^{-1} -symmetric subfamilies which can be obtained by taking the $d \rightarrow c \rightarrow b \rightarrow a \rightarrow 0$ limits of the 4-parameter q^{-1} -Askey–Wilson polynomials.

This paper is organized as follows. In Sect. 3 we present in Theorem 3.1, the infinite discrete bilateral orthogonality relation for q^{-1} -Askey–Wilson polynomials given in [11, (2.13)]. We then obtained the corresponding infinite discrete orthogonality relations for all q^{-1} -symmetric subfamilies. In Sect. 4, we show how one can obtain infinite continuous bilateral orthogonality relations for the q^{-1} -symmetric families by starting with Theorem 3.1 and exploiting the methods presented in Ismail & Rahman (1995) [9]. One interesting consequence of the results presented in Sect. 4 is the derivation of a q -beta integral given in Theorem 4.5 representing the total mass of the infinite continuous bilateral orthogonality relation and its $q \rightarrow 1$ limit in Theorem 4.6 which is of Ramanujan type. Both of these integrals appear to be new. See the companion paper [3] for a powerful extension of these results.

3 Infinite discrete bilateral orthogonality relations

The q^{-1} -symmetric families in the q -Askey scheme satisfy the following infinite discrete bilateral orthogonality relations.

Theorem 3.1 *Let $m, n, N \in \mathbb{N}_0, q \in \mathbb{C}^\dagger, \alpha, a, b, c, d \in \mathbb{C}^*, m, n \leq N$ such that $|qabcd| < |q|^{2N}$. Then the q^{-1} -Askey–Wilson polynomials satisfy the following infinite discrete bilateral orthogonality relation:*

$$\begin{aligned} & \sum_{k=-\infty}^\infty (1 + q^{2k}\alpha^2) \mathbf{p}_m[q^k\alpha; \mathbf{a}|q] \mathbf{p}_n[q^k\alpha; \mathbf{a}|q] \frac{(\frac{\alpha}{\mathbf{a}}; q)_k}{(-q\alpha\mathbf{a}; q)_k} (qabcd)^k \\ &= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qad, -qbc, -qbd, -qcd; q)_\infty}{(-q\alpha\mathbf{a}, \frac{q\mathbf{a}}{\alpha}, qabcd; q)_\infty} \\ & \times q^{-6\binom{n}{2}} (a^2b^2c^2d^2)^n \frac{(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}, -\frac{1}{bc}, -\frac{1}{bd}, -\frac{1}{cd}; q)_n (\frac{1}{qabcd}; q)_{2n}}{(\frac{1}{abcd}; q)_n (\frac{1}{abcd}; q)_{2n}} \delta_{m,n}. \end{aligned} \tag{30}$$

Proof Using [11, (2.13)] and applying (21) derives the infinite discrete bilateral orthogonality relation. Now define the left-hand side of (30) as

$$\sum_{k=-\infty}^\infty A_{k,m,n}(\alpha; a, b, c, d|q). \tag{31}$$

Then using (15), one can see that as $k \rightarrow \pm\infty$, one has

$$A_{k,n,n}(\alpha; \mathbf{a}|q) \sim B_n^\pm(\alpha; \mathbf{a}|q) (q^{1-2n}abcd)^{|k|}, \tag{32}$$

where the constants $B_n^\pm(\alpha; \mathbf{a}|q)$ given by

$$B_n^+(\alpha; \mathbf{a}|q) := \vartheta\left(\frac{\alpha}{\mathbf{a}}; q\right)q^{-2\binom{n}{2}} \left(\frac{q^{1-n}abcd}{\alpha}\right)^{2n} \frac{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_n}, \tag{33}$$

$$B_n^-(\alpha; \mathbf{a}|q) := \alpha^2 \vartheta(-q\alpha\mathbf{a}; q)q^{-6\binom{n}{2}} (abcd\alpha)^{2n} \frac{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_n}, \tag{34}$$

are independent of k . Therefore the bilateral series is convergent for some $N \in \mathbb{N}_0$ such that $|qabcd| < |q|^{2N}$. This completes the proof. \square

Choosing $(m, n) = (0, 0)$ provides the following infinite discrete summation which provides the total mass for the above infinite discrete orthogonality relation.

Theorem 3.2 *Let $q \in \mathbb{C}^\dagger$, $\alpha, a, b, c, d \in \mathbb{C}^*$. Then one has the following infinite discrete bilateral summation which is equivalent to Bailey’s bilateral ${}_6\psi_6$ summation [14, (17.8.7)], namely*

$$\sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \frac{(\frac{\alpha}{\mathbf{a}}; q)_k}{(-q\alpha\mathbf{a}; q)_k} (qabcd)^k = \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qad, -qbc, -qbd, -qcd; q)_\infty}{(-q\alpha\mathbf{a}, \frac{q\mathbf{a}}{\alpha}, qabcd; q)_\infty}. \tag{35}$$

Proof Setting $(m, n) = (0, 0)$ in Theorem 3.1 completes the proof. \square

Theorem 3.3 *Let $m, n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a, b, c \in \mathbb{C}^*$. Then the continuous dual q^{-1} -Hahn polynomials satisfy the following infinite discrete bilateral orthogonality relation:*

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{p}_m[q^k\alpha; \mathbf{a}|q] \mathbf{p}_n[q^k\alpha; \mathbf{a}|q] \frac{(\frac{\alpha}{\mathbf{a}}; q)_k}{(-q\alpha\mathbf{a}; q)_k} q^{\binom{k}{2}} (-q\alpha abc)^k \\ &= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qbc; q)_\infty}{(-q\alpha\mathbf{a}, \frac{q\mathbf{a}}{\alpha}; q)_\infty} q^{-4\binom{n}{2}} \left(\frac{a^2b^2c^2}{q}\right)^n \left(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{bc}; q\right)_n \delta_{m,n}. \end{aligned} \tag{36}$$

Proof Start with (30) and taking the limit as $d \rightarrow 0$ completes the proof. \square

Theorem 3.4 *Let $m, n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a, b \in \mathbb{C}^*$. Then the q^{-1} -Al-Salam–Chihara polynomials satisfy the following infinite discrete bilateral orthogonality relation:*

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{Q}_m[q^k\alpha; \mathbf{a}|q] \mathbf{Q}_n[q^k\alpha; \mathbf{a}|q] \frac{(\frac{\alpha}{\mathbf{a}}; q)_k}{(-q\alpha\mathbf{a}; q)_k} q^{2\binom{k}{2}} (q\alpha^2 ab)^k \\ &= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}, -qab; q)_\infty}{(-q\alpha\mathbf{a}, \frac{q\mathbf{a}}{\alpha}; q)_\infty} q^{-2\binom{n}{2}} \left(\frac{ab}{q}\right)^n \left(q, -\frac{1}{ab}; q\right)_n \delta_{m,n}. \end{aligned} \tag{37}$$

Proof Start with (36) and taking the limit as $c \rightarrow 0$ completes the proof. \square

Theorem 3.5 *Let $m, n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a \in \mathbb{C}^*$. Then the continuous big q^{-1} -Hermite polynomials satisfy the following infinite discrete bilateral orthogonality relation:*

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{H}_m[q^k\alpha; a|q] \mathbf{H}_n[q^k\alpha; a|q] \frac{(\frac{\alpha}{a}; q)_k}{(-q\alpha a; q)_k} q^{3\binom{k}{2}} (-q\alpha^3 a)^k \\ &= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}; q)_\infty}{(-q\alpha a, \frac{q\mathbf{a}}{\alpha}; q)_\infty} q^{-\binom{n}{2}} (q; q)_n \delta_{m,n}. \end{aligned} \tag{38}$$

Proof Start with (37) and taking the limit as $b \rightarrow 0$ completes the proof. \square

Theorem 3.6 *Let $m, n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha \in \mathbb{C}^*$. Then the continuous q^{-1} -Hermite polynomials satisfy the following infinite discrete bilateral orthogonality relation:*

$$\sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{H}_m[q^k\alpha|q] \mathbf{H}_n[q^k\alpha|q] q^{4\binom{k}{2}} (q\alpha^4)^k = (q, -\alpha^2, -\frac{q}{\alpha^2}; q)_\infty \frac{q^{-\binom{n}{2}} (q; q)_n}{q^n} \delta_{m,n}. \tag{39}$$

Proof Start with (38) and taking the limit as $a \rightarrow 0$ completes the proof. \square

4 Infinite continuous bilateral orthogonality relations

In this section our aim is to extend [6, Theorem 21.6.4] for $w_2(x)$ to all the q^{-1} -symmetric families with parameters a, b, c, d . Recall the definition of the orthogonality measure for continuous q^{-1} -Hermite polynomials $w_2(x)$ [6, (21.6.4)], namely

$$w_2(x) = \exp\left(-\frac{2}{\log q^{-1}} \left[\log(x + \sqrt{x^2 + 1})\right]^2\right). \tag{40}$$

If you choose $x = \frac{1}{2}(z - z^{-1})$, then the above definition reduces to

$$w_2(x) = \exp\left(-\frac{2(\log z)^2}{\log q^{-1}}\right), \tag{41}$$

for the polynomials $h_n(x|q)$ which are defined in [6, (21.2.5)]. These polynomials are related to the standard continuous q^{-1} -Hermite polynomials using [6, (21.2.1)]

$$h_n(x|q) = \mathbf{H}_n(x|q) = i^{-n} H_n[iz|q^{-1}], \tag{42}$$

where $x = \frac{1}{2}(z - 1/z)$. In terms of the polynomials $\mathbf{H}_n[z|q]$, the equivalent orthogonality relation is [6, (21.7.7)]

$$\int_{-\infty}^{\infty} (q^x + q^{-x}) \mathbf{H}_m[q^x|q] \mathbf{H}_n[q^x|q] \exp(-2x^2 \log q^{-1}) dx = q^{-\binom{n}{2}} \frac{(q; q)_n}{q^{n+\frac{1}{8}}} \sqrt{\frac{2\pi}{\log q^{-1}}} \delta_{m,n}, \tag{43}$$

which has a positive definite measure of orthogonality for $q \in (0, 1)$. Define

$$\omega_2(x) := \exp(-2x^2 \log q^{-1}) = q^{2x^2}, \tag{44}$$

which follows using the laws of logarithms. Pulling out a factor of q^{-x} in the prefactor of (43), one obtains the following equivalent expression for the continuous orthogonality relation for continuous q^{-1} -Hermite polynomials, namely

$$\int_{-\infty}^{\infty} (1 + q^{2x}) \mathbf{H}_m[q^x|q] \mathbf{H}_n[q^x|q] q^{2x^2-x} dx = q^{-\binom{n}{2}} \frac{(q; q)_n}{q^{n+\frac{1}{8}}} \sqrt{\frac{2\pi}{\log q^{-1}}} \delta_{m,n}. \tag{45}$$

Cursory observation identifies the corresponding infinite discrete bilateral orthogonality relation given by

$$\sum_{k=-\infty}^{\infty} (1 - q^{2k} \alpha^2) H_m[q^k \alpha|q^{-1}] H_n[q^k \alpha|q^{-1}] q^{2k^2-k} \alpha^4 = (q, \alpha^2, \frac{q}{\alpha^2}; q)_{\infty} \frac{q^{-\binom{n}{2}} (q; q)_n}{(-q)^n} \delta_{m,n}, \tag{46}$$

which after the replacement $\alpha \mapsto i\alpha$, one obtains

$$\sum_{k=-\infty}^{\infty} (1 + q^{2k} \alpha^2) \mathbf{H}_m[q^k \alpha|q] \mathbf{H}_n[q^k \alpha|q] q^{2k^2-k} \alpha^4 = (q, -\alpha^2, -\frac{q}{\alpha^2}; q)_{\infty} \frac{q^{-\binom{n}{2}} (q; q)_n}{q^n} \delta_{m,n}. \tag{47}$$

This is the infinite discrete bilateral analogue of (45) for $\alpha = 1$ which corresponds to a positive definite orthogonality measure for $q \in (0, 1)$. This can be seen after replacing $\alpha \mapsto 1, k \mapsto x$ and replacing the infinite bilateral sum over $k \in \mathbb{Z}$ with a continuous integral over $x \in \mathbb{R}$, or otherwise, using the methods described in [9].

Now we can use the methods in [9] to obtain precisely the correspondence between the infinite discrete bilateral orthogonality relation (47) and the infinite continuous orthogonality relation given in (45). Both orthogonality relations having already been established in [6] and [11]. This leads us to the following theorem.

Theorem 4.1 For $m, n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha \in \mathbb{C}^*$, we have the discrete orthogonality relation

$$\begin{aligned} \Psi_{m,n}(\alpha|q) &:= \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2)\mathbf{H}_n[q^k\alpha|q]\mathbf{H}_m[q^k\alpha|q]q^{2k^2-k} \alpha^{4k} \\ &= (q, -\alpha^2, -\frac{q}{\alpha^2}; q)_\infty \frac{q^{-\binom{n}{2}}(q; q)_n}{q^n} \delta_{m,n}. \end{aligned} \tag{48}$$

This relation implies the continuous orthogonality relation

$$\begin{aligned} \mathbf{K}_{m,n}(\alpha|q) &:= \int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2)\mathbf{H}_n[q^x\alpha|q]\mathbf{H}_m[q^x\alpha|q]q^{2x^2-x} \alpha^{4x} dx \\ &= \int_0^1 \Psi_{m,n}(q^x\alpha|q)q^{2x^2-x} \alpha^{4x} dx \\ &= (q; q)_\infty \frac{q^{-\binom{n}{2}}(q; q)_n \delta_{m,n}}{q^n} \mathbf{J}(\alpha|q), \end{aligned} \tag{49}$$

where

$$\begin{aligned} \mathbf{J}(\alpha|q) &= \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x} \alpha^{4x} dx \\ &= \frac{1}{(q; q)_\infty} \int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2)q^{2x^2-x} \alpha^{4x} dx = \frac{\sqrt{2\pi} \alpha \exp\left(\frac{2(\log \alpha)^2}{\log q^{-1}}\right)}{q^{\frac{1}{8}} \sqrt{\log q^{-1}}(q; q)_\infty}. \end{aligned} \tag{50}$$

Proof The infinite discrete bilateral orthogonality relation for continuous q^{-1} -Hermite polynomials (48) is known, namely (39), with the finite q -shifted factorials rewritten as infinite q -shifted factorials. Using the methods in [9] one has

$$\begin{aligned} &\int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2)\mathbf{H}_n[q^x\alpha|q]\mathbf{H}_m[q^x\alpha|q]q^{2x^2-x} \alpha^{4x} \omega(x; q) dx \\ &= \int_0^1 \Psi_{m,n}(q^x\alpha|q)q^{2x^2-x} \alpha^{4x} \omega(x; q) dx \\ &= (q; q)_\infty \frac{q^{-\binom{n}{2}}(q; q)_n}{q^n} \delta_{m,n} \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x} \alpha^{4x} \omega(x; q) dx, \end{aligned} \tag{51}$$

where $\omega(x; q)$ is unit-periodic on $x \in \mathbb{R}$. Choosing $\omega(x; q) \equiv 1$ provides correspondence (49). The integral $\mathbf{J}(\alpha|q)$ and its relation with $\mathbf{K}_{0,0}(\alpha|q)$, namely

$$\mathbf{K}_{0,0}(\alpha|q) = (q; q)_\infty \mathbf{J}(\alpha|q), \tag{52}$$

is clear by setting $(m, n) = (0, 0)$ in (49). This gives

$$(q; q)_\infty \mathbf{J}(\alpha|q) = \int_{-\infty}^{\infty} q^{2x^2-x} \alpha^{4x} dx + \int_{-\infty}^{\infty} q^{2t^2+t} \alpha^{4t+2} dt = 2 \int_{-\infty}^{\infty} q^{2x^2-x} \alpha^{4x} dx$$

setting $t = x - \frac{1}{2}$. Now the substitution $y = (x - \frac{1}{4})\sqrt{2 \log q^{-1}}$ leads to

$$(q; q)_\infty \mathbf{J}(\alpha|q) = \frac{\sqrt{2}\alpha}{q^{\frac{1}{8}} \sqrt{\log q^{-1}}} \int_{-\infty}^{\infty} e^{-y^2} \alpha^{cy} dy \quad \text{with } c := \frac{4}{\sqrt{2 \log q^{-1}}}.$$

Using the integral

$$\int_{-\infty}^{\infty} e^{-x^2} \alpha^{cx} dx = \sqrt{\pi} \exp(\frac{1}{4}c^2(\log \alpha)^2), \tag{53}$$

if $\Re c \log \alpha < 0$, which is just a re-written form of the Gaussian integral, completes the proof. □

An important special case of this integral is given by

$$J(1|q) = \int_0^1 (-q^{2x}, -q^{1-2x}; q)_\infty q^{2x^2-x} dx = \frac{1}{(q; q)_\infty} \int_{-\infty}^\infty (1 + q^{2x})q^{2x^2-x} dx = \frac{q^{-\frac{1}{8}}}{(q; q)_\infty} \sqrt{\frac{2\pi}{\log q^{-1}}}. \tag{54}$$

However, we already know the continuous orthogonality relation corresponding to the left-hand side of (49) with $\alpha = 1$, namely (45). Therefore we are able to solve for the $\alpha = 1$ special case of the following integrals

$$K_{0,0}(1|q) = \int_{-\infty}^\infty (1 + q^{2x})q^{2x^2-x} dx = q^{-\frac{1}{8}} \sqrt{\frac{2\pi}{\log q^{-1}}}, \tag{55}$$

$$J(1|q) = \int_0^1 (-q^{2x}, -q^{1-2x}; q)_\infty q^{2x^2-x} dx = \frac{q^{-\frac{1}{8}}}{(q; q)_\infty} \sqrt{\frac{2\pi}{\log q^{-1}}}. \tag{56}$$

Since we know $J(1|q)$, we can use it to derive several items for the continuous big q^{-1} -Hermite polynomials.

Theorem 4.2 For $m, n \in \mathbb{N}_0, q \in \mathbb{C}^\dagger, \alpha, a \in \mathbb{C}^*$, we have the discrete orthogonality relation

$$\begin{aligned} \Psi_{m,n}(\alpha; a|q) &= \sum_{k=-\infty}^\infty (1 + q^{2k}\alpha^2) \mathbf{H}_n[q^k\alpha; a|q] \mathbf{H}_m[q^k\alpha; a|q] (-q^{k+1}\alpha a, \frac{q^{1-k}a}{\alpha}; q)_\infty q^{2k^2-k} \alpha^{4k} \\ &= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}; q)_\infty q^{-\binom{n}{2}} (q; q)_n}{(-q\alpha a, \frac{qa}{\alpha}; q)_\infty q^n} \delta_{m,n}. \end{aligned} \tag{57}$$

This relation implies the continuous orthogonality relation

$$\begin{aligned} K_{m,n}(\alpha; a|q) &:= \int_{-\infty}^\infty (1 + q^{2x}\alpha^2) \mathbf{H}_n[q^x\alpha; a|q] \mathbf{H}_m[q^x\alpha; a|q] (-q^{x+1}\alpha a, \frac{q^{1-x}a}{\alpha}; q)_\infty q^{2x^2-x} \alpha^{4x} dx \\ &= \int_0^1 \Psi_{m,n}(q^x\alpha; a|q) q^{2x^2-x} \alpha^{4x} dx \\ &= (q; q)_\infty \frac{q^{-\binom{n}{2}} (q; q)_n \delta_{m,n}}{q^n} \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x} \alpha^{4x} dx \\ &= \frac{\sqrt{2\pi} \alpha \exp\left(\frac{2(\log \alpha)^2}{\log q^{-1}}\right)^{\frac{1}{8}}}{q} \sqrt{\log q^{-1}} \frac{q^{-\binom{n}{2}} (q; q)_n \delta_{m,n}}{q^n}. \end{aligned} \tag{58}$$

Proof The infinite discrete bilateral orthogonality relation for continuous big q^{-1} -Hermite polynomials (57) is known, namely (38) with the finite q -shifted factorials rewritten as infinite q -shifted factorials. Using the methods in [9] one has

$$\begin{aligned} &\int_{-\infty}^\infty (1 + q^{2x}\alpha^2) \mathbf{H}_n[q^x\alpha; a|q] \mathbf{H}_m[q^x\alpha; a|q] (-q^{x+1}\alpha a, \frac{q^{1-x}a}{\alpha}; q)_\infty q^{2x^2-x} \alpha^{4x} \omega(x; q) dx \\ &= \int_0^1 \Psi_{m,n}(q^x\alpha; a|q) q^{2x^2-x} \alpha^{4x} \omega(x; q) dx \\ &= (q; q)_\infty \frac{q^{-\binom{n}{2}} (q; q)_n \delta_{m,n}}{q^n} \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x} \alpha^{4x} \omega(x; q) dx, \end{aligned} \tag{59}$$

where $\omega(x; q)$ is unit-periodic on $x \in \mathbb{R}$. Choosing $\omega(x; q) \equiv 1$ and evaluating the integral (50) completes the proof. \square

Second is a continuous analogue of the infinite discrete bilateral orthogonality relation for continuous big q^{-1} -Hermite polynomials. Note that the reason we are able to obtain this is because the norm of orthogonality is independent of the parameter a .

Theorem 4.3 For $m, n \in \mathbb{N}_0, q \in \mathbb{C}^\dagger, a \in \mathbb{C}^\dagger$, we have the continuous orthogonality relation

$$\int_{-\infty}^{\infty} (1 + q^{2x}) \mathbf{H}_n[q^x; a|q] \mathbf{H}_m[q^x; a|q] (-q^{x+1} a, q^{1-x} a; q)_{\infty} q^{2x^2-x} dx = \frac{q^{-\binom{n}{2}} (q; q)_n}{q^{n+\frac{1}{8}}} \sqrt{\frac{2\pi}{\log q^{-1}}} \delta_{m,n}. \tag{60}$$

Proof Set $\alpha = 1$ in Theorem 4.2. □

Now we proceed directly to the correspondence theorem for the q^{-1} -Askey–Wilson polynomials.

Theorem 4.4 Let $m, n, N \in \mathbb{N}_0, q \in \mathbb{C}^\dagger, \alpha, a, b, c, d \in \mathbb{C}^*, \mathbf{a}$ be the multiset given by $\{a, b, c, d\}, m, n \leq N$ such that $|abcd| < |q|^{2N-1}$. Then we have the discrete orthogonality relation for q^{-1} -Askey–Wilson polynomials:

$$\begin{aligned} \Psi_{m,n}(\alpha; \mathbf{a}|q) &:= \sum_{k=-\infty}^{\infty} (1 + q^{2k} \alpha^2) \mathbf{p}_n[q^k \alpha; \mathbf{a}|q] \mathbf{p}_m[q^k \alpha; \mathbf{a}|q] (-q^{k+1} \alpha \mathbf{a}, \frac{q^{1-k}}{\alpha} \mathbf{a}; q)_{\infty} q^{2k^2-k} \alpha^{4k} \\ &= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qad, -qbc, -qbd, -qcd; q)_{\infty}}{(qabcd; q)_{\infty}} \\ &\quad \times \frac{q^{-6\binom{n}{2}} (-a^2 b^2 c^2 d^2)^n (q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}, -\frac{1}{bc}, -\frac{1}{bd}, -\frac{1}{cd}; q)_n (\frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}; q)_n (\frac{1}{abcd}; q)_{2n}} \delta_{m,n}. \end{aligned} \tag{61}$$

This relation implies the continuous orthogonality relation

$$\begin{aligned} \mathbf{K}_{m,n}(\alpha; \mathbf{a}|q) &:= \int_{-\infty}^{\infty} (1 + q^{2x} \alpha^2) \mathbf{p}_n[q^x \alpha; \mathbf{a}|q] \mathbf{p}_m[q^x \alpha; \mathbf{a}|q] (-q^{x+1} \alpha \mathbf{a}, \frac{q^{1-x}}{\alpha} \mathbf{a}; q)_{\infty} q^{2x^2-x} \alpha^{4x} dx \\ &= \int_0^1 \Psi_{m,n}(q^x \alpha; \mathbf{a}|q) q^{2x^2-x} \alpha^{4x} dx \\ &= \frac{\sqrt{2\pi} \alpha \exp\left(\frac{2(\log \alpha)^2}{\log q^{-1}}\right) (-qab, -qac, -qad, -qbc, -qbd, -qcd; q)_{\infty}}{q^{\frac{1}{8}} \sqrt{\log q^{-1}} (qabcd; q)_{\infty}} \\ &\quad \times \frac{q^{-6\binom{n}{2}} (-a^2 b^2 c^2 d^2)^n (q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}, -\frac{1}{bc}, -\frac{1}{bd}, -\frac{1}{cd}; q)_n (\frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}; q)_n (\frac{1}{abcd}; q)_{2n}} \delta_{m,n}. \end{aligned} \tag{62}$$

Proof The infinite discrete bilateral orthogonality relation for q^{-1} -Askey–Wilson polynomials (61) is known, namely (30) with the finite q -shifted factorials rewritten as infinite q -shifted factorials. Then using the methods in [9] one has

$$\begin{aligned} &\int_{-\infty}^{\infty} (1 + q^{2x} \alpha^2) \mathbf{p}_n[q^x \alpha; \mathbf{a}|q] \mathbf{p}_m[q^x \alpha; \mathbf{a}|q] (-q^{x+1} \alpha \mathbf{a}, \frac{q^{1-x}}{\alpha} \mathbf{a}; q)_{\infty} q^{2x^2-x} \alpha^{4x} \omega(x; q) dx \\ &= \int_0^1 \Psi_{m,n}(q^x \alpha; \mathbf{a}|q) q^{2x^2-x} \alpha^{4x} \omega(x; q) dx \\ &= q^{-\binom{n}{2}} \frac{(q; q)_n}{q^n} \frac{(q, -q^{1-n} ab, -q^{1-n} ac, -q^{1-n} ad, -q^{1-n} bc, -q^{1-n} bd, -q^{1-n} cd, q^{2-2n} abcd; q)_{\infty}}{(q^{1-2n} abcd, q^{2-n} abcd; q)_{\infty}} \\ &\quad \times \int_0^1 (-q^{2x} \alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_{\infty} q^{2x^2-x} \alpha^{4x} \omega(x; q) dx, \end{aligned} \tag{63}$$

where $\omega(x; q)$ is unit-periodic on $x \in \mathbb{R}$. Choosing $\omega(x; q) \equiv 1$ and evaluating the integral (50) provides the result. Now define the first integral on the left-hand side of (62) as

$$\int_{-\infty}^{\infty} \mathbf{C}_{m,n}(x; \alpha; \mathbf{a}|q) dx. \tag{64}$$

Then using (15), one can see that as $x \rightarrow \pm\infty$, one has

$$\mathbf{C}_{n,n}(x; \alpha; \mathbf{a}|q) \sim \begin{cases} \mathbf{D}_n^+(\alpha; \mathbf{a}|q) q^{2x^2-(2n+1)x} \alpha^{4x} \left(\frac{q^{1-x} \mathbf{a}}{\alpha}; q\right)_{\infty}, & \text{as } x \rightarrow \infty \\ \mathbf{D}_n^-(\alpha; \mathbf{a}|q) q^{2x^2+(2n+1)x} \alpha^{4x} \left(-q^{x+1} \alpha \mathbf{a}; q\right)_{\infty}, & \text{as } x \rightarrow -\infty, \end{cases}$$

where the constants $\mathbf{D}_n^{\pm}(\alpha; \mathbf{a}|q)$ given by

$$\mathbf{D}_n^+(\alpha; \mathbf{a}|q) := q^{-6\binom{n}{2}} \left(\frac{abcd}{\alpha}\right)^{2n} \frac{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_n}, \tag{65}$$

$$\mathbf{D}_n^-(\alpha; \mathbf{a}|q) := \alpha^2 q^{-6\binom{n}{2}} (abcd\alpha)^{2n} \frac{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}, \frac{1}{qabcd}; q)_n}, \tag{66}$$

are independent of x . Now for $x \rightarrow \pm\infty$, set $x = \pm(m + \epsilon)$ with $|\epsilon| < 1, m \in \mathbb{N}_0, m \rightarrow \infty$. After simplification we see that convergence as $x \rightarrow \infty$ requires $|q^{2n-1}abcd| < 1$ and convergence as $x \rightarrow -\infty$ requires that $|q^{1-2n}abcd| < 1$. Therefore convergence of the integral is guaranteed for $m, n \leq N$ where $|qabcd| < |q|^{2N}$. This completes the proof. \square

This leads us to a new q -beta integral which arises by selecting the $K_{0,0}(\alpha; \mathbf{a}|q)$ term in (62).

Theorem 4.5 *Let $q \in \mathbb{C}^{\dagger}, \alpha, a, b, c, d \in \mathbb{C}^*, \mathbf{a}$ be the multiset given by $\{a, b, c, d\}$. Then one has the following q -beta integral and $|abcd| < |q|^{-1}$, namely*

$$\begin{aligned} & \int_{-\infty}^{\infty} (1 + q^{2x} \alpha^2) (-q^{x+1} \alpha \mathbf{a}, \frac{q^{1-x}}{\alpha} \mathbf{a}; q)_{\infty} q^{2x^2-x} \alpha^{4x} dx \\ &= \frac{\sqrt{2\pi} \alpha \exp\left(\frac{2(\log \alpha)^2}{\log q^{-1}}\right) (-qab, -qac, -qad, -qbc, -qbd, -qcd; q)_{\infty}}{q^{\frac{1}{8}} \sqrt{\log q^{-1}} (qabcd; q)_{\infty}}. \end{aligned} \tag{67}$$

Proof Setting $(m, n) = (0, 0)$ in (62) and the requirement for convergence corresponds to Theorem 4.4 with $N = 0$. This completes the proof. \square

We are also able to compute the $q \rightarrow 1^-$ limit of the above q -beta integral. This is accomplished by converting all the infinite q -shifted factorials in Theorem 4.5 into q -gamma functions which are defined by [5, (I.35)]

$$\Gamma_q(x) := \frac{(q; q)_{\infty} (1 - q)^{1-x}}{(q^x; q)_{\infty}}, \tag{68}$$

and then computing the limit as $q \rightarrow 1^-$ since $\lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x)$.

Theorem 4.6 *Let $a, b, c, d \in \mathbb{C}^*, \Re(a + b + c + d) > -1$. Then one has the following symmetric beta integral*

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{dx}{\Gamma(2x, -2x, 1+a+x, 1+a-x, 1+b+x, 1+b-x, 1+c+x, 1+c-x, 1+d+x, 1+d-x)} \\ &= -\frac{1}{2\pi^2} \frac{\Gamma(a + b + c + d + 1)}{\Gamma(a + b + 1, a + c + 1, a + d + 1, b + c + 1, b + d + 1, c + d + 1)}, \end{aligned} \tag{69}$$

where we are using the convention that a comma delineated list of argument to the gamma function represents multiplication by separate gamma functions with their corresponding arguments.

Proof We obtain this result in two different ways. The first proof is a direct proof of the result by expressing it as bilateral hypergeometric series and then summing it. The second proof is by starting with Theorem 4.5 and taking the $q \rightarrow 1^-$ limit.

First Proof: Let $\mathbf{a} := \{a_1, a_2, a_3, a_4\} := \{a, b, c, d\}$, $a_j > 0$, $j \in \{1, 2, 3, 4\}$. We define functions

$$f_j(x; a_j) := \frac{1}{\Gamma(1 + a_j + x)\Gamma(1 + a_j - x)}.$$

We adopt the definition for the Fourier transform given in Erdélyi et al. [4, Chapter III]. The Fourier transform

$$F_j(t; a_j) := \mathcal{F}(f_j)(t) := \int_{-\infty}^{\infty} e^{-ixt} f_j(x; a_j) dx,$$

is known [15, (1.2)] (see also [4, (3.3.5)])

$$F_j(t; a_j) = \begin{cases} \frac{(2 \cos(\frac{1}{2}t))^{2a_j}}{\Gamma(2a_j+1)} & \text{if } -\pi < t < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\mathcal{F}(fg) = \frac{1}{2\pi} \mathcal{F}(f) * \mathcal{F}(g),$$

where $F * G$ denotes convolution

$$(F * G)(t) = \int_{-\infty}^{\infty} F(s)G(t - s) ds.$$

Set

$$f(x) := f_1(x)f_2(x)f_3(x)f_4(x).$$

Then

$$F := \mathcal{F}(f) = \frac{1}{8\pi^3} F_1 * F_2 * F_3 * F_4.$$

It follows from the definition of the convolution that $F_1 * F_2$ and $F_3 * F_4$ vanish outside $[-2\pi, 2\pi]$. Therefore, the even function $F(t)$ vanishes outside $[-4\pi, 4\pi]$. We expand $F(t)$ in a Fourier cosine series

$$F(t) = \frac{1}{2} \sum_{k=0}^{\infty} \epsilon_k w_k \cos(\frac{1}{4}kt),$$

where $\epsilon_0 = 1$, $\epsilon_k = 2$ for $k \in \mathbb{N}$ is the Neumann factor, and

$$w_k := \frac{1}{4\pi} \int_{-4\pi}^{4\pi} F(s) \cos(\frac{1}{4}ks) ds = \frac{1}{4\pi} \int_{-\infty}^{\infty} F(s) \cos(\frac{1}{4}ks) ds.$$

For every real ω we have

$$\int_{-\infty}^{\infty} F(s) e^{i\omega s} ds = 2\pi f(\omega).$$

Therefore,

$$w_k = \frac{1}{2} f(\frac{1}{4}k) = \frac{1}{2} \prod_{j=1}^4 \frac{1}{\Gamma(1 + a_j + \frac{1}{4}k)\Gamma(1 + a_j - \frac{1}{4}k)}.$$

Consider the integral

$$\int_{-\infty}^{\infty} f(x) \cos(xt) dx = F(t) = \frac{1}{2} \sum_{k=0}^{\infty} \epsilon_k w_k \cos(\frac{1}{4}kt). \tag{70}$$



In particular, if $t = 0$ then

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{2} \sum_{k=0}^{\infty} \epsilon_k w_k.$$

If we differentiate (70) with respect to t we find

$$\int_{-\infty}^{\infty} f(x)x \sin(xt) dx = -F'(t) = \frac{1}{4} \sum_{k=0}^{\infty} \epsilon_k k w_k \sin(\frac{1}{4}kt).$$

If $t = 2\pi$ one obtains

$$\int_{-\infty}^{\infty} \frac{f(x)}{\Gamma(2x)\Gamma(-2x)} dx = -\frac{2}{\pi} \int_{-\infty}^{\infty} x \sin(2\pi x) f(x) dx = J := -\frac{1}{2\pi} \sum_{k=0}^{\infty} \epsilon_k k w_k \sin(\frac{1}{2}k\pi).$$

Setting $k = 4n + 1, k = 4m + 3$ for $m, n \in \mathbb{N}_0$ we find

$$J = -\frac{1}{4\pi} \sum_{n=0}^{\infty} (4n + 1) f(n + \frac{1}{4}) + \frac{1}{4\pi} \sum_{m=0}^{\infty} (4m + 3) f(m + \frac{3}{4}).$$

If we replace m by $-n - 1$ and note that f is an even function, then this represents J as a bilateral sum

$$J = -\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} (4n + 1) f(n + \frac{1}{4}).$$

Using the bilateral hypergeometric function ${}_5H_5$ [14, (16.4.16)] this becomes

$$J = -\frac{1}{4\pi} {}_5H_5 \left(\begin{matrix} \frac{5}{4}, \frac{1}{4} - a_1, \frac{1}{4} - a_2, \frac{1}{4} - a_3, \frac{1}{4} - a_4 \\ \frac{1}{4}, a_1 + \frac{5}{4}, a_2 + \frac{5}{4}, a_3 + \frac{5}{4}, a_4 + \frac{5}{4} \end{matrix}; 1 \right) \prod_{j=1}^4 \frac{1}{\Gamma(a_j + \frac{5}{4})\Gamma(a_j + \frac{3}{4})}.$$

The bilateral hypergeometric function can be evaluated using Dougall’s formula [16, (6.1.2.5)], and we obtain

$$J = -\frac{1}{2\pi^2} \frac{\Gamma(a_1 + a_2 + a_3 + a_4 + 1)}{\prod_{1 \leq i < j \leq 4} \Gamma(1 + a_i + a_j)}.$$

This completes the first proof.

Second proof: First start with Theorem 4.5 and replace $(\alpha, a, b, c, d) \mapsto i(q^\alpha, q^a, q^b, q^c, q^d)$. Then re-writing it in terms of q -gamma functions using (68) leads to

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\Gamma_q(2\alpha + 2x + 1) q^{2x^2 + (4\alpha - 1)x} e^{2i\pi x} dx}{\Gamma_q(2\alpha + 2x, 1 + a \pm (x + \alpha), 1 + b \pm (x + \alpha), 1 + c \pm (x + \alpha), 1 + d \pm (x + \alpha))} \\ &= \frac{i q^{\alpha - \frac{1}{8}} \sqrt{2\pi} \exp\left(\frac{2(\log(iq^\alpha))^2}{\log q^{-1}}\right)}{(q; q)_\infty^3 (1 - q) \sqrt{\log q^{-1}}} \frac{\Gamma_q(a + b + c + d + 1)}{\Gamma_q(a + b + 1, a + c + 1, a + d + 1, b + c + 1, b + d + 1, c + d + 1)}. \end{aligned}$$

Taking the $q \rightarrow 1^-$ with justification of interchanging the limit and the integral being provided by the first proof above. In the limit as $q \rightarrow 1^-$ of the above expression, we see that all the q -gamma functions become gamma functions, powers of q become unity and one has

$$\lim_{q \rightarrow 1^-} \frac{i q^{\alpha - \frac{1}{8}} \sqrt{2\pi} \exp\left(\frac{2(\log(iq^\alpha))^2}{\log q^{-1}}\right)}{(q; q)_\infty^3 (1 - q) \sqrt{\log q^{-1}}} = \sqrt{2\pi} i e^{-2i\pi\alpha} \mathsf{T}.$$

since

$$\begin{aligned} & \sqrt{\log q^{-1}} \sim \sqrt{1 - q}, \\ & \exp\left(\frac{2(\log(iq^\alpha))^2}{\log q^{-1}}\right) \sim e^{-2\pi i\alpha} \exp\left(\frac{\pi^2}{2 \log q}\right), \quad \text{where } \mathsf{T} := \lim_{q \rightarrow 1^-} \frac{\exp\left(-\frac{\pi^2}{2 \log q^{-1}}\right)}{(q; q)_\infty^3 (1 - q)^{\frac{3}{2}}}. \end{aligned}$$



Then one obtains

$$\int_{-\infty}^{\infty} \frac{(\alpha+x) e^{2\pi i x} dx}{\Gamma(1+a\pm(\alpha+x), 1+b\pm(\alpha+x), 1+c\pm(\alpha+x), 1+d\pm(\alpha+x))} = \frac{i\sqrt{\frac{\pi}{2}} e^{-2i\pi\alpha} \Gamma(a+b+c+d+1)}{\Gamma(a+b+1, a+c+1, a+d+1, b+c+1, b+d+1, c+d+1)}.$$

Then making the substitution $y = x + \alpha$ and then replacing $y \mapsto x$ obtains the expression in terms of Γ . If one takes the limit as $a, b, c, d, \alpha \rightarrow 0$ of the beta integral, then using the reflection formula for the gamma function [14, (5.5.3)], the beta integral becomes

$$\int_{-\infty}^{\infty} \frac{e^{2i\pi x} \sin^4(\pi x)}{x^3} dx = \frac{i\pi^3}{4}.$$

In fact, the similar case where $a, b, c, d \rightarrow 0$ ($\alpha \neq 0$) is equivalent to the same integral. We now see that the value of Γ is given by $\Gamma = 1/(2\pi)^{\frac{3}{2}}$. Note that one can also see the value of the limit Γ by examining [13, (3.13)]. Then we write the complex exponential using Euler’s formula and only the sin term contributes. Finally we use the reflection formula for gamma functions to write the sine function as a product of two gamma functions. The limit of the constraint $|qabcd| \mapsto |q^{a+b+c+d+1}| < 1$ provides the updated constraint. This completes the second proof. \square

Remark 4.7 Since the beta integral (69) is negative, one cannot use it to build a set of orthogonal polynomials with a positive measure.

Now we present the correspondence theorem for the continuous dual q^{-1} -Hahn polynomials.

Theorem 4.8 *Let $m, n \in \mathbb{N}_0, q \in \mathbb{C}^\dagger, \alpha, a, b, c \in \mathbb{C}^*, \mathbf{a}$ be the multiset given by $\{a, b, c\}$. Then we have the discrete orthogonality relation for continuous dual q^{-1} -Hahn polynomials:*

$$\Psi_{m,n}(\alpha; \mathbf{a}|q) := \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{p}_n[q^k\alpha; \mathbf{a}|q] \mathbf{p}_m[q^k\alpha; \mathbf{a}|q] (-q^{k+1}\alpha \mathbf{a}, \frac{q^{1-k}}{\alpha} \mathbf{a}; q)_\infty q^{2k^2-k} \alpha^{4k} = (q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qbc; q)_\infty q^{-4\binom{n}{2}} \left(\frac{a^2b^2c^2}{q}\right)^n \left(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{bc}; q\right)_n \delta_{m,n}. \tag{71}$$

This relation implies the continuous orthogonality relation

$$\mathbf{K}_{m,n}(\alpha; \mathbf{a}|q) := \int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2) \mathbf{p}_n[q^x\alpha; \mathbf{a}|q] \mathbf{p}_m[q^x\alpha; \mathbf{a}|q] (-q^{x+1}\alpha \mathbf{a}, \frac{q^{1-x}}{\alpha} \mathbf{a}; q)_\infty q^{2x^2-x} \alpha^{4x} dx = \int_0^1 \Psi_{m,n}(q^x\alpha; \mathbf{a}|q) q^{2x^2-x} \alpha^{4x} dx = \frac{\sqrt{2\pi} \alpha \exp\left(\frac{2(\log \alpha)^2}{\log q^{-1}}\right) (-qab, -qac, -qbc; q)_\infty q^{-4\binom{n}{2}} \left(\frac{a^2b^2c^2}{q}\right)^n \left(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{bc}; q\right)_n \delta_{m,n}}{q^{\frac{1}{8}} \sqrt{\log q^{-1}}} \tag{72}$$

Proof The infinite discrete bilateral orthogonality relation for continuous dual q^{-1} -Hahn polynomials (71) is known, namely (36) with the finite q -shifted factorials rewritten as infinite q -shifted factorials. Then using the methods in [9] one has

$$\int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2) \mathbf{p}_n[q^x\alpha; \mathbf{a}|q] \mathbf{p}_m[q^x\alpha; \mathbf{a}|q] (-q^{x+1}\alpha \mathbf{a}, \frac{q^{1-x}}{\alpha} \mathbf{a}; q)_\infty q^{2x^2-x} \alpha^{4x} \omega(x; q) dx = \int_0^1 \Psi_{m,n}(q^x\alpha; \mathbf{a}|q) q^{2x^2-x} \alpha^{4x} \omega(x; q) dx = q^{-\binom{n}{2}} \frac{(q; q)_n}{q^n} (q, -q^{1-n}ab, -q^{1-n}ac, -q^{1-n}bc; q)_\infty \delta_{m,n} \times \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x} \alpha^{4x} \omega(\alpha; q) dx, \tag{73}$$



where $\omega(x; q)$ is unit-periodic on $x \in \mathbb{R}$. Choosing $\omega(x; q) \equiv 1$ and evaluating the integral (50) completes the proof. \square

Theorem 4.9 Let $m, n \in \mathbb{N}_0, q \in \mathbb{C}^\dagger, \alpha, a, b \in \mathbb{C}^*, \mathbf{a}$ be the multiset given by $\{a, b\}$. Then we have the discrete orthogonality relation

$$\begin{aligned} \Psi_{m,n}(\alpha; \mathbf{a}|q) &:= \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{Q}_n[q^k\alpha; \mathbf{a}|q] \mathbf{Q}_m[q^k\alpha; \mathbf{a}|q] (-q^{k+1}\alpha\mathbf{a}, \frac{q^{1-k}}{\alpha}\mathbf{a}; q)_\infty q^{2k^2-k}\alpha^{4k} \\ &= (q, -\alpha^2, -\frac{q}{\alpha^2}, -qab; q)_\infty q^{-2\binom{n}{2}} \left(\frac{ab}{q}\right)^n \left(q, -\frac{1}{ab}; q\right)_n \delta_{m,n}. \end{aligned} \tag{74}$$

This relation implies the continuous orthogonality relation

$$\begin{aligned} \mathbf{K}_{m,n}(\alpha; \mathbf{a}|q) &:= \int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2) \mathbf{Q}_n[q^x\alpha; \mathbf{a}|q] \mathbf{Q}_m[q^x\alpha; \mathbf{a}|q] (-q^{x+1}\alpha\mathbf{a}, \frac{q^{1-x}}{\alpha}\mathbf{a}; q)_\infty q^{2x^2-x}\alpha^{4x} dx \\ &= \int_0^1 \Psi_{m,n}(q^x\alpha; \mathbf{a}|q) q^{2x^2-x}\alpha^{4x} dx \\ &= \frac{\sqrt{2\pi} \alpha \exp\left(\frac{2(\log \omega)^2}{\log q^{-1}}\right) (-qab; q)_\infty}{q^{\frac{1}{8}} \sqrt{\log q^{-1}}} q^{-\binom{n}{2}} \left(\frac{ab}{q}\right)^n \left(q, -\frac{1}{ab}; q\right)_n \delta_{m,n}. \end{aligned} \tag{75}$$

Proof The infinite discrete bilateral orthogonality relation for q^{-1} -Al-Salam–Chihara polynomials (74) is known, namely (37) with the finite q -shifted factorials rewritten as infinite q -shifted factorials. Then using the methods in [9] one has

$$\begin{aligned} &\int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2) \mathbf{Q}_n[q^x\alpha; \mathbf{a}|q] \mathbf{Q}_m[q^x\alpha; \mathbf{a}|q] (-q^{x+1}\alpha\mathbf{a}, \frac{q^{1-x}}{\alpha}\mathbf{a}; q)_\infty q^{2x^2-x}\alpha^{4x} \omega(x; q) dx \\ &= \int_0^1 \Psi_{m,n}(q^x\alpha; \mathbf{a}|q) q^{2x^2-x}\alpha^{4x} \omega(x; q) dx \\ &= q^{-\binom{n}{2}} \frac{(q; q)_n}{q^n} (q, -q^{1-n}ab; q)_\infty \delta_{m,n} \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x}\alpha^{4x} \omega(\alpha; q) dx, \end{aligned} \tag{76}$$

where $\omega(x; q)$ is unit-periodic on $x \in \mathbb{R}$. Choosing $\omega(x; q) \equiv 1$ and evaluating the integral (50) completes the proof. \square

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References

1. Askey, R.: Beta integrals and q -extensions. In: Proceedings of the Ramanujan Centennial International Conference (Annamalainagar, 1987), volume 1 of RMS Publ., pages 85–102. Ramanujan Math. Soc., Annamalainagar (1988)
2. Askey, R.: Continuous q -Hermite polynomials when $q > 1$. In: q -series and partitions (Minneapolis, MN, 1988), volume 18 of IMA Vol. Math. Appl., pages 151–158. Springer, New York (1989)
3. Cohl, H.S.; Volkmer, H.: Evaluation of beta integrals of Ramanujan type and integral representations for bilateral hypergeometric series. *Ramanujan J.* **67**(28), 32 (2025)
4. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G.: Tables of Integral Transforms, vol. I. McGraw-Hill Book Company Inc, New York-Toronto-London (1954)
5. Gasper, G.; Rahman, M.: *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*, second edition. With a foreword by Richard Askey. Cambridge University Press, Cambridge (2004)
6. Ismail, M. E. H.: Classical and Quantum Orthogonal Polynomials in One Variable, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge. With two chapters by Walter Van Assche, With a foreword by Richard A. Askey, Corrected reprint of the 2005 original (2009)
7. Ismail, M.E.H.: Solutions of the Al-Salam—Chihara and allied moment problems. *Anal. Appl.* **18**(2), 185–210 (2020)
8. Ismail, M.E.H.; Masson, D.R.: q -Hermite polynomials, biorthogonal rational functions, and q -beta integrals. *Trans. Am. Math. Soc.* **346**(1), 63–116 (1994)
9. Ismail, M.E.H.; Rahman, M.: Some basic bilateral sums and integrals. *Pacific J. Math.* **170**(2), 497–515 (1995)
10. Ismail, M. E. H.; Zhang, R.; Zhou, K.: Orthogonal polynomials of Askey—Wilson type (2022) [arXiv: 2205.05280](https://arxiv.org/abs/2205.05280)
11. Ismail, M.E.H.; Zhang, R.; Zhou, K.: q -fractional Askey—Wilson integrals and related semigroups of operators. *Physica D: Nonlinear Phenomena* **442**, 133534 (2022)
12. Koekoek, R.; Lesky, P.A.; Swarttouw, R.F.: *Hypergeometric orthogonal polynomials and their q -analogues*. Springer Monographs in Mathematics. Springer-Verlag, Berlin. With a foreword by Tom H. Koornwinder (2010)
13. Olde Daalhuis, A. B.: Asymptotic expansions for q -gamma, q -exponential, and q -Bessel functions. *J. Math. Anal. Appl.* **186**(3), 896–913 (1994)
14. Olver, F. W. J., Olde Daalhuis, A. B., Lozier, D. W., Schneider, B. I., Boisvert, R. F., Clark, C. W., Miller, B. R., Saunders, B. V., Cohl, H. S., McClain, M. A. (eds): NIST Digital Library of Mathematical Functions. <https://dlmf.nist.gov/>. Release 1.2.4 of 2025-03-15
15. Ramanujan, S.: A class of definite integrals (@art. *J. Math.* **48** (1920), 294–310). In: *Collected papers of Srinivasa Ramanujan*, pages 216–229. AMS Chelsea Publ., Providence, RI (2000)
16. Slater, L.J.: *Generalized Hypergeometric Functions*. Cambridge University Press, Cambridge (1966)

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