Fourier and Gegenbauer expansions for fundamental solutions of the Laplacian and powers in R^d and H^d

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Abstract

We compute fundamental solutions and associated Fourier cosine series for the Laplacian (and its powers) in Euclidean space $\mathbf{R}^d$ and in hyperbolic space $\mathbf{H}^d$ by introducing natural coordinate systems. More specific, this is done as follows.

We prove parameter derivative formulae for certain associated Legendre functions. We derive closed-form expressions of normalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$ and compute Fourier expansions for these fundamental solutions in terms of natural angles in axisymmetric subgroup type coordinate systems. We give azimuthal and separation angle Fourier expansions for pure hyperspherical coordinate systems, as well as Fourier expansions in mixed Euclidean-hyperspherical coordinate systems. Using azimuthal Fourier expansions compared with Gegenbauer polynomial expansions of fundamental solutions for powers of the Laplacian, we construct multi-summation addition theorems in pure hyperspherical subgroup type coordinate systems. We give some examples of multi-summation addition theorems for a certain sub-class of pure hyperspherical coordinate systems in $\mathbf{R}^d$ for $d \in \{3, 4, \ldots\}$. We also give an example of a logarithmic multi-summation addition theorem, namely that for an unnormalized fundamental solution for powers of the Laplacian in $\mathbf{R}^4$. In the d-dimensional hyperboloid model of hyperbolic geometry $\mathbf{H}^d$, we compute spherically symmetric normalized fundamental solutions for the Laplace-Beltrami operator. Finally, we also compute Fourier expansions for unnormalized fundamental solutions for this space in two and three dimensions.

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Contents

Glossary

$\mathbf{N}$	$\{1, 2, \ldots\}$ the set of natural numbers
$\mathbf{N}_0$	$\{0, 1, 2, \ldots\} = \mathbf{N} \cup \{0\}$
$\mathbf{Z}$	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the set of integers
$\mathbf{R}$	$(-\infty, \infty)$ the set of real numbers
(a, b)	$\{x \in \mathbf{R} : a < x < b\}$ open interval on the real line
$[a, b]$	$\{x \in \mathbf{R} : a \le x \le b\}$ closed interval on the real line
$(a, b]$, $[a, b)$	half-open intervals on the real line with $a < b$
$\mathbf{C}$	$\{a + bi : a, b \in \mathbf{R}\}$ the set of complex numbers
$\mathbf{x}$	position vector on a d-dimensional Riemannian manifold
$\widehat{\mathbf{x}}$	unit vector on a d-dimensional Riemannian manifold
$\mathbf{R}^d$	d-dimensional Euclidean space
$\mathbf{H}^d$	d-dimensional unit hyperboloid model of hyperbolic geometry
$\mathbf{S}^d$	d-dimensional unit hyper-sphere
$\mathbf{R}^{p,q}$	$(p + q)$-dimensional pseudo-Riemannian manifold of signature (p, q)
$\mathbf{R}^{d,1}$	$(d + 1)$-dimensional Minkowski space (special case of $\mathbf{R}^{p,q}$)
$C_c^k(\mathbf{R}^d)$	space of k times continuously differentiable functions with compact support in $\mathbf{R}^d$
$\mathcal{D}(\mathbf{R}^d)$	space of test functions in $\mathbf{R}^d$
$(\mathcal{D}(\mathbf{R}^d))'$	dual of space of test functions in $\mathbf{R}^d$
$(\mathbf{x}, \mathbf{y})$	Euclidean inner-product
$\Vert \cdot \Vert$	Euclidean norm
$d(\mathbf{x}, \mathbf{x}')$	geodesic distance between two points on a Riemannian manifold
$[\mathbf{x}, \mathbf{y}]$	Minkowski inner-product
Δ	Laplacian, the Laplace-Beltrami operator on a Riemannian manifold
$\mathcal{G}_k^d, \mathcal{H}_k^d$	normalized fundamental solution for kth power of Laplacian

	in $\mathbf{R}^d$ and $\mathbf{H}^d$
$\mathcal{G}^d, \mathcal{H}^d$	normalized fundamental solution for the Laplacian in $\mathbf{R}^d$ and $\mathbf{H}^d$ (i.e. $\mathcal{G}_1^d, \mathcal{H}_1^d$)
$\mathfrak{g}^d, \mathfrak{h}^d$	unnormalized fundamental solution for the Laplacian in $\mathbf{R}^d$ and $\mathbf{H}^d$
$\mathfrak{g}_k^d, \mathfrak{l}_k^d$	unnormalized fundamental solutions for the polyharmonic equation in $\mathbf{R}^d$
Θ_d^p	logarithmic p-sequence of functions in $\mathbf{R}^d$ which give an unnormalized fundamental solution for powers $k \geq d/2$ of the Laplacian (d even)
R_p^k	"logarithmic polynomial" related to the Fourier series of logarithmic fundamental solution for the polyharmonic equation in $\mathbf{R}^d$
$\mathfrak{R}_{n,p}, \mathfrak{r}_{n,p}^{k_1,k_2}$	functions related to the Fourier coefficient of logarithmic fundamental solution for the polyharmonic equation in $\mathbf{R}^d$
$D_{n,p}, E_{n,p}$	functions related to the Fourier coefficient of logarithmic fundamental solution for the polyharmonic equation in $\mathbf{R}^d$
$\mathfrak{P}_{n,p}, \mathfrak{Q}_{n,p}$	Fourier coefficients for logarithmic fundamental solution for the polyharmonic equation in $\mathbf{R}^d$
χ	toroidal parameter
$Y_{\lambda,\mu}$	normalized hyperspherical harmonic
$\Theta(l_j, l_{j+1}; \theta)$	product hyperspherical harmonic in standard hyperspherical coordinates
$\boldsymbol{\lambda}$	the set of all angular quantum numbers for hyperspherical harmonics
$E(d), O(d)$	Euclidean and orthogonal Lie groups
$SO(d), O(d,1)$	special orthogonal and pseudo-orthogonal Lie groups
δ_{ij}	Kronecker delta
δ	Dirac delta distribution on a d-dimensional manifold
$\lfloor\cdot\rfloor, \lceil\cdot\rceil$	floor and ceiling functions
$\mathrm{Re}\, z, \mathrm{Im}\, z$	the real and imaginary part of a complex number z
Γ, ψ	gamma and digamma functions
γ	Euler's constant $\approx 0.57721566490153286060651209008$2
$\binom{z}{n}$	binomial coefficient
$(z)_n, [z]_n$	Pochhammer symbol for rising and falling factorial
$n!, n!!$	factorial and double factorial

I_ν, K_ν	modified Bessel function of the first and second kind
P_ν^μ, Q_ν^μ	associated Legendre function of the first and second kind
${}_2F_1(a, b; c; \cdot)$	Gauss hypergeometric function
C_n^a	Gegenbauer polynomial
$P_n^{(a,b)}$	Jacobi polynomial
$F(\varphi, \cdot), E(\varphi, \cdot)$	Legendre's incomplete elliptic integral of the first and second kind
K, E	Legendre's complete elliptic integral of the first and second kind
$\Pi(\varphi, \alpha, \cdot), \Pi(\alpha, \cdot)$	Legendre's incomplete and complete elliptic integral of the third kind
$r_{\lessgtr}$	${}^{\min}_{\max}\{r, r'\}$ for $r, r' \in (0, \infty)$

1 Introduction

In this thesis, we are primarily interested in fundamental solutions for linear partial differential equations on oriented Riemannian or pseudo-Riemannian manifolds in connection with the theory of special functions. The special functions which we study arise in terms of linear partial differential equations on these manifolds either as closed-form expressions for fundamental solutions, or in terms of eigenfunction expansions for fundamental solutions in terms of a basis for their separated solution space. Through the theory of separation of variables we know that the separated solutions for these linear partial differential equations are given in terms of commuting sets of symmetry operators for these equations (see Kalnins (1986) [62] and Miller (1977) [69]) which satisfy eigenvalue problems whose eigenvalues are separation constants.

Fundamental solutions to the classical linear partial differential operators of Mathematical Physics (i.e. the Laplace, Helmholtz, wave, and heat operators) and their eigenfunction expansions are given in terms of some of the most frequently appearing special functions such as gamma, exponential, logarithmic, modified Bessel and associated Legendre functions (Abramowitz & Stegun (1972) [1]). These special functions are often referred to as higher transcendental functions (see Erdélyi et al. (1981) [35]). We are extremely interested in the properties of special functions, especially the special functions that one encounters through separation of variables of the Laplace equation in those separable coordinate systems.

The Riemannian and pseudo-Riemannian manifolds we are interested in are smooth manifolds equipped with a Riemannian metric with signature (see Lee (1997) [65]). The Riemannian or pseudo-Riemannian metric allows one to measure geometric quantities such as distances and angles which parametrize the points of each manifold through specific curvilinear coordinate systems. The separated solutions that we study have arguments which are given in terms of the natural distances and angles which parametrize these manifolds through specific coordinate systems. Certain coordinate systems are natural for specific operators on these manifolds through the study of separation of variables. Separation of variables has a long and interesting history, which has been recently studied in terms of Lie group theoretic methods in connection with the theory of simple-separation or R-separation of variables for these linear equations. Henceforth we use the term separation to refer to both types. It however should be mentioned that the appearance of R-separation is intrinsic to conformal symmetries for linear operators on these manifolds (see Boyer, Kalnins & Miller (1976) [13]), such as that exhibited by the Laplacian in Euclidean space, and corresponding conformally invariant Laplacians in curved spaces.

The partial differential equations that we focus upon are given in terms of natural powers of the Laplacian, the Laplace-Beltrami operator, on these manifolds. The Laplace-Beltrami operator, a generalization of the Laplacian in Euclidean space, provides a tool for expressing Laplace's equation and its harmonic solutions on a Riemannian or pseudo-Riemannian manifold. By introducing the notion of distributions on Riemannian or pseudo-Riemannian manifolds (see §32.2 in Triebel (1986) [97]), we can study fundamental solutions for Laplace's and polyharmonic equations on these spaces. The manifolds which we study are isotropic (and non-compact) and therefore there exists fundamental solutions for polyharmonic operators with pure radial argument, and spherically symmetric dependence. Our aim is to work on manifolds of arbitrary dimension. We study the symmetric nature of rotationally invariant curvilinear coordinate systems which yield separable solutions for polyharmonic equations.

The study of separation of variables, fundamental solutions and special functions for the Laplace equation in Euclidean space and on curved manifolds for higher dimensions is classical, but yet remains a field which still has many aspects unexplored. We demonstrate in this thesis some concrete aspects of the theory of linear partial differential operators and special functions which are easily accessible, and lead to intriguing results which remain to be fully explained in term of more abstract methods. Also, we would like to emphasize that our work is focused on yielding closed-form useful special function realizations which can be directly implemented with any procedures in Mathematical Physics which might rely on these results for future computations.

We are primarily interested in spaces of constant sectional curvature and would like to study fundamental solutions on these spaces. We construct spherically symmetric coordinate systems, solve Laplace's equation, and compute normalized fundamental solutions on these spaces. Normalization of fundamental solutions is performed by solving the inhomogeneous Laplace equation with right-hand side given by the Dirac delta distribution (often referred to as a Dirac delta function), such that the integral over the entire space obtains unity. In general, the constant sectional curvature spaces include Euclidean space, Hyperbolic space, and hyperspherical space. In this thesis we restrict our attention to fundamental solutions of the Laplacian and the polyharmonic operator in Euclidean and hyperbolic spaces. We exclude the hyperspherical space from our study, due to it being a compact manifold.

The fact that this classical study is *recently* relevant is due to several reasons. First, fundamental solutions for Laplace's and polyharmonic equations are ubiquitous in Pure and Applied Mathematics, Physics, and Engineering. Second, the fact that Fourier expansions encapsulate rotationally-invariant symmetries of geometrical shapes makes it an ideal model case for critical study above and beyond the purely spherically symmetric shape, and therefore it provides a powerful tool when implemented numerically in a variety of important problems. Third, Fourier expansions for fundamental solutions gives rise to a hardly-studied aspect of Special Function Theory, one which allows further exploration of the properties of higher transcendental functions. Let us now focus on the specific details which are presented in this thesis.

In Chapter 2 we introduce the special functions which are encountered in the main body of the text. The specific material on derivatives of associated Legendre functions with respect their parameters appearing in §2.6.5 will not be used in the main body of this thesis, but regardless, it is new and interesting. On the other hand, the concept of derivatives of associated Legendre functions with respect to their parameters is heavily relied upon in §6.3.

In Chapter 3 we give closed-form expressions, in Euclidean space $\mathbf{R}^d$, for normalized fundamental solutions of the Laplacian and compute normalized fundamental solutions for powers of the Laplacian. The material appearing in this chapter is not new, but will be heavily relied upon in future chapters.

In Chapter 4, we introduce the rotationally invariant subgroup type coordinates in Euclidean space which yield solutions to Laplace's equation through separation of variables. The subgroup type coordinates that we describe are general mixed Euclidean-hyperspherical coordinates systems. These subgroup type coordinate systems are described using a powerful graphical method called a "method of trees." We describe in a detailed fashion, all specific combinatoric and topological aspects of this method. Using the "method of trees," we construct examples of mixed Euclidean-hyperspherical coordinates in $\mathbf{R}^d$. Mixed Euclidean-hyperspherical coordinates are those coordinate systems which correspond to a hyperspherical coordinate system of dimension $p \in \mathbf{N}$ embedded in a d-dimensional Euclidean space $\mathbf{R}^d$ ($p \le d$) with Cartesian coordinates. In the case where $p = d$, we call these pure hyperspherical coordinates. We refer to the set of all pure hyperspherical coordinate systems as the general hyperspherical coordinate systems. Each mixed Euclidean-hyperspherical coordinate system provides a natural angle $\psi \in [0, 2\pi)$ for performing Fourier expansions about. This is a detail which will be fully exploited in the following chapter.

In Chapter 5 we perform Fourier expansions of fundamental solutions in Euclidean space in all dimensions for all natural powers of the Laplacian. Most of the results in this chapter are new. Fourier expansions describe the general non-axisymmetric structure of scalar fields. We derive and give examples of a generalized Fourier series for complex binomials of the form $1/[z-\cos\psi]^\mu$, where $\psi \in \mathbf{R}$, $\mu \in \mathbf{C}$ and $z \in \mathbf{C}\setminus(-\infty, 1]$ with $|z| < 1$. It should be mentioned that the derivations appearing in §5.2 is joint work with Diego Dominici (see Cohl & Dominici (2010) [24]). We show that the Fourier series is given compactly with coefficients given in terms of odd-half-integer degree, general complex-order associated Legendre functions of the second kind. These algebraic functions naturally arise in classical physics through the theory of fundamental solutions of Laplace's equation, in the odd dimensions and in the even dimensions for $1 \le k < d/2$. These represent powers of distances between two points in a Euclidean geometry. Fourier expansions for algebraic distance functions have a rich history, and our expansion makes an appearance in the theory of arbitrarily-shaped charge distributions in electrostatics [9, 66, 80, 82, 83, 99, 103], magnetostatics [10, 89], quantum direct and exchange Coulomb interactions [8, 25, 34, 45, 78], Newtonian gravity [12, 20, 37, 43, 55, 56, 68, 75, 85, 86, 88], the Laplace coefficients for planetary disturbing function [32, 33], and potential fluid flow around actuator discs [15, 54], just to name a few direct physical applications. Fourier expansions give a precise $e^{im\phi}$ description for these applications. We also compute Fourier expansions in even dimensions for powers $k \ge d/2$ where a fundamental solution is given in terms of a logarithmic contribution.

In Chapter 6 we construct normalized hyperspherical harmonics, in pure hyperspherical coordinate systems, in terms of Gegenbauer, Jacobi and Chebyshev polynomials and their limits such as integer-order, integer-degree associated Legendre functions of the first kind. We generalize a method originated by Sack (1964) [84] to expand unnormalized fundamental solutions of the Laplacian in $\mathbf{R}^d$ in terms of Gegenbauer polynomials with superscript given by $d/2-1$ and the argument is given by the separation angle in pure hyperspherical coordinates. We then use the addition theorem for hyperspherical harmonics to expand Gegenbauer polynomials in terms of pure hyperspherical harmonics over a set of natural quantum numbers given by the separation constants in these coordinates. We use the Fourier expansions along with these hyperspherical expansions to construct multi-summation addition theorems in these subgroup type coordinate systems.

In Chapter 7, we derive closed-form expressions for normalized fundamental solutions of the Laplacian in the hyperboloid model of hyperbolic geometry $\mathbf{H}^d$. This model of hyperbolic space is a d-dimensional hyperboloid

$$x_0^2 - x_1^2 - \cdots - x_d^2 = 1,$$

embedded in a $d+1$-dimensional Minkowski space. We examine only the case for $d \in \{2, 3, \ldots\}$. This hyperboloid then represents a non-compact Riemannian manifold with negative-constant sectional curvature. We introduce aspects of this space, including a discussion of general subgroup type coordinate systems on it. We solve for the radial harmonics in the set of all general hyperbolic hyperspherical coordinate systems on the hyperboloid. In joint work with Ernie Kalnins, we prove that the spherically symmetric harmonics are given in terms of associated Legendre functions of the first and second kinds with argument $\cosh r$, degree $d/2-1$ and order given by $\pm(d/2-1)$. Due to the isotropy of the hyperbolic space, we expect there to exist a spherically symmetric fundamental solution on the hyperboloid. We obtain spherically symmetric harmonic solutions on the hyperboloid by solving the Laplace equation in a spherically symmetric curvilinear coordinate system. This reduces the problem of computing a spherically symmetric fundamental solution of the Laplacian in this space, to solving an ordinary differential equation in terms of the radius r on the hyperboloid. We solve this ordinary differential equation, and solutions are seen to be given in terms of a definite integral with integrand given by powers of a reciprocal hyperbolic sine function. We compute several exact matching expressions for a normalized fundamental solution of the Laplace-Beltrami operator in this d-dimensional hyperbolic space. This result, which is summarized in Theorem 7.5.1, expresses a normalized fundamental solution in terms of a finite summation expression, Gauss hypergeometric functions, and in terms of associated Legendre functions of the second kind. These associated Legendre functions, as opposed to the Legendre functions of the first kind, exactly match the behaviour of an expected fundamental solutions in both the singular and far-field regimes. We then conclude this chapter with a computation of Fourier expansions for fundamental solutions on the hyperboloid for $d = 2$ and $d = 3$.

2
Special functions

We review some basic information concerning special functions (Abramowitz & Stegun (1972) [1], Magnus, Oberhettinger & Soni (1966) [67], Miller (1977) [69], Moon & Spencer (1988) [70]). Most of this material will be necessary information for the rest of this thesis. The material in §2.6.5, however, is new material (see Cohl (2010) [23]) concerning certain derivatives with respect to parameters of associated Legendre functions.

The special functions that we use in this thesis include elementary functions (such as trigonometric and hyperbolic functions), the gamma function (and the functions and symbols which are related to it such as the factorial and double factorial, the Pochhammer symbols and the binomial coefficients), the digamma function, the Gauss hypergeometric function, associated Legendre functions (both the first and second kind), and some important orthogonal polynomials such as Jacobi, Gegenbauer, Chebyshev, and associated Legendre.

Associated Legendre functions of both kinds have 2 complex parameters as well as a complex argument. The material presented in this chapter on associated Legendre functions, namely that appearing in §2.6, is more extensive than the other sections in this chapter. This is because associated Legendre functions are some of the most commonly-encountered functions appearing in this thesis. We will introduce some of the most important properties of associated Legendre functions, such as recurrence relations, negative-degree and order conditions, Whipple formulae, etc. A generalization of associated Legendre functions, the Jacobi functions (see §7.4.3 in Vilenkin & Klimyk (1991) [101]), have 3 complex parameters as well as a complex argument. It is important to realize that associated Legendre functions can be written in terms of Jacobi functions. Even more important is the fact that both of these functions are special cases of the Gauss hypergeometric function. In this thesis, we will restrict our attention to associated Legendre functions. However Jacobi functions, and in particular Jacobi polynomials, play an important role the study of the study of hyperspherical harmonics in higher dimensions (see Chapter 6).

Throughout this thesis we rely on the following definitions. For $a_1, a_2, \ldots \in \mathbf{C}$, if $i, j \in \mathbf{Z}$ and $j < i$ then $\displaystyle\sum_{n=i}^{j} a_n = 0$ and $\displaystyle\prod_{n=i}^{j} a_n = 1$.

2.1 Elementary functions

For complex numbers z, we routinely use the complex exponential, logarithmic, trigonometric, and hyperbolic functions (see Chapter 3 in Churchill & Brown (1984) [21]).

The exponential function can be defined over the entire complex plane using the power series definition

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The logarithmic function is defined for points $z \neq 0$ in the complex plane as

$$\log z := \log r + i \arg z,$$

where $\arg z \in (-\pi, \pi]$. When $z \neq 0$ and the exponent w is any complex number, then z^w is defined by the equation

$$z^w := \exp(w \log z).$$

2.1.1 Trigonometric functions

Now we list some of the basic facts about trigonometric functions. These are functions whose arguments are angles measured in radians. These functions are the cosine, sine, tangent, cotangent, secant, and cosecant functions defined over the complex plane (see §23 in Churchill & Brown (1984) [21]). The definitions of the sine and cosine functions are given in terms of the exponential function as

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \tag{2.1}$$

and

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i},$$

($z \in \mathbf{C}$) and all of the other properties of the trigonometric functions follow from these. For instance, we therefore have Euler's formula

$$e^{\pm iz} = \cos z \pm i \sin z.$$

One basic relationship between the sine and the cosine is the Pythagorean trigonometric identity

$$\cos^2 z + \sin^2 z = 1.$$

The double-angle formulae are

$$\sin 2z = 2 \sin z \cos z,$$

and

$$\cos 2z = 2\cos^2 z - 1 = 1 - 2\sin^2 z = \cos^2 z - \sin^2 z.$$

One can expand the square of the cosine and sine functions using the double-angle identities

$$\cos^2 z = \frac{1}{2}(1 + \cos 2z),$$

and

$$\sin^2 z = \frac{1}{2}(1 - \cos 2z).$$

The sine, cosine and tangent addition formulae are

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2,$$

and

$$\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}, \tag{2.2}$$

respectively. The cosine product to sum identity is

$$\cos z_1 \cos z_2 = \frac{1}{2}\left\{\cos(z_1 + z_2) + \cos(z_1 - z_2)\right\}. \tag{2.3}$$

2.1.2 Hyperbolic functions

We list some of the basic facts about hyperbolic functions. These functions are the hyperbolic cosine, sine, tangent, cotangent, secant, and cosecant functions defined over the complex plane (see §24 in Churchill & Brown (1984) [21]). The arguments of the hyperbolic functions are referred to as hyperangles. The definitions of the hyperbolic sine and cosine functions are given as

$$\cosh z := \frac{e^z + e^{-z}}{2},$$

and

$$\sinh z := \frac{e^z - e^{-z}}{2},$$

($z \in \mathbf{C}$) and all of the other properties of the hyperbolic functions follow from these. For instance, we also have

$$e^{\pm z} = \cosh z \pm \sinh z.$$

A basic relationship between the hyperbolic sine and the hyperbolic cosine is the analogue of the Pythagorean trigonometric identity

$$\cosh^2 z - \sinh^2 z = 1,$$

and consequently

$$\frac{1}{\sinh^2 z} = \coth^2 z - 1.$$

The double-hyperangle formulae are

$$\sinh 2z = 2 \sinh z \cosh z,$$

and

$$\cosh 2z = 2\cosh^2 z - 1 = 2\sinh^2 z + 1 = \cosh^2 z + \sinh^2 z.$$

One can expand the square of the hyperbolic cosine and hyperbolic sine functions using the double-hyperangle identities

$$\cosh^2 z = \frac{1}{2}(\cosh 2z + 1),$$

and

$$\sinh^2 z = \frac{1}{2}(\cosh 2z - 1).$$

The hyperangle sum and difference formulae are

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2,$$

and

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.$$

For $x \in (1, \infty)$, the inverse hyperbolic cosine is given by

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right). \tag{2.4}$$

2.2 Gamma function and some related functions

2.2.1 Gamma function, factorial and double factorial

The gamma function (see Chapter 6 in Abramowitz & Stegun (1972) [1], Chapter 1 in Andrews, Askey & Roy (1999) [3]) is an important combinatoric function and is ubiquitous in special function theory. It is naturally defined over the right-half complex plane through Euler's integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

$\operatorname{Re} z > 0$. The Euler reflection formula allows one to obtain values in the left-half complex plane ((6.1.17) in Abramowitz & Stegun (1972) [1]), namely

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \tag{2.5}$$

$0 < \operatorname{Re} z < 1$, for $\operatorname{Re} z = 0, z \neq 0$, and then for z shifted by integers using the following recurrence relation

$$\Gamma(z + 1) = z\Gamma(z).$$

The gamma function has poles of order 1 for $z \in -\mathbf{N}_0$ with residues given by $(-1)^n/n!$ (see p. 370 of Amman & Escher (2008) [2]). The gamma function is a natural generalization of the factorial function over the natural numbers, $n \in \mathbf{N}$ where

$$\Gamma(n + 1) = n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n.$$

Through our connection with the gamma function, we see that $0! = 1$.

The double factorial function (Abramowitz & Stegun (1972) [1]) is related to the gamma function [105]

$$z!! = 2^{(1+2z-\cos \pi z)/4} \pi^{(-1+\cos \pi z)/4} \Gamma\left(1 + \frac{1}{2} z\right).$$

Using this definition, we can see that $0!! = 1$. The following properties hold for $n \in \mathbf{N}$

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1),$$

and

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!. \tag{2.6}$$

The double factorial function naturally generalizes for the negative odd integers (Arfken & Weber (1995) [5], p. 600)

$$(-2n-1)!! = \frac{(-1)^n \, 2^n \, n!}{(2n)!}, \tag{2.7}$$

$n \in \mathbf{N}_0$, i.e. $(-1)!! = 1$ and $(-3)!! = -1$. All negative even double factorials are undefined. For $n \in \mathbf{N}_0$, gamma functions with arguments given by odd-half-integers (see (6.1.12) in Abramowitz & Stegun (1972) [1]) are given by

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}, \tag{2.8}$$

and using Euler's reflection formula (2.5) yields

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{(2n-1)!!}. \tag{2.9}$$

2.2.2 Pochhammer symbols and binomial coefficients

Pochhammer symbols represent either rising or the falling factorials. The Pochhammer symbols are well-defined over the entire complex plane, i.e. for $z \in \mathbf{C}$. Unfortunately there is no standard convention used for Pochhammer symbols, so we utilize the following convention which is consistent with usage in Special Function theory, i.e. with hypergeometric functions. The Pochhammer symbol for rising factorial is given by

$$(z)_n = \begin{cases} 1 & \text{if } n = 0, \\ (z) \cdot (z+1) \cdot (z+2) \cdots (z+n-1) & \text{if } n \geq 1, \end{cases} \tag{2.10}$$

where $n \in \mathbf{N}_0$ and $z \in \mathbf{C}$. The Pochhammer symbol for the rising factorial is expressible in terms of gamma functions as

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \tag{2.11}$$

for all $z \in \mathbf{C} \setminus (-\mathbf{N}_0)$. Similarly the Pochhammer symbol for the falling factorial, with the same quantities, is given by

$$[z]_n = \begin{cases} 1 & \text{if } n = 0, \\ (z) \cdot (z-1) \cdot (z-2) \cdots (z-n+1) & \text{if } n \geq 1. \end{cases}$$

The Pochhammer symbol for the falling factorial is also expressible in terms of gamma functions as

$$[z]_n = \frac{\Gamma(z+1)}{\Gamma(z-n+1)},$$

for all $z \in \mathbf{C} \setminus \{\ldots, n-3, n-2, n-1\}$. One can clearly relate the two symbols by

$$[z]_n = (-1)^n (-z)_n. \tag{2.12}$$

The binomial coefficient (Abramowitz & Stegun (1972) [1]) can be defined through the Pochhammer symbol for the falling factorial

$$\binom{z}{n} = \frac{[z]_n}{n!}, \tag{2.13}$$

where $z \in \mathbf{C}$ and $n \in \mathbf{N}_0$. The binomial coefficient (for $n, k \in \mathbf{Z}$) is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

if $0 \leq k \leq n$ and zero otherwise. The binomial coefficient satisfies the following identity

$$\binom{n}{k} = \binom{n}{n-k}, \tag{2.14}$$

where $n, k \in \mathbf{Z}$, except where $k < 0$ or $n - k < 0$.

2.3 Digamma function

The digamma function is defined in terms of the derivative of the gamma function,

$$\frac{d}{dz}\Gamma(z) = \psi(z)\Gamma(z), \tag{2.15}$$

for $z \in \mathbf{C} \setminus (-\mathbf{N}_0)$. Like the gamma function, the digamma function is singular for $z \in (-\mathbf{N}_0)$. A good reference for the basic properties of the digamma function is §1.2 of Magnus, Oberhettinger & Soni (1966) [67].

Two important properties of the digamma function are the recurrence relation for digamma functions

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{2.16}$$

and the reflection formula for digamma functions

$$\psi(-z) = \psi(z+1) + \pi \cot \pi z. \tag{2.17}$$

One useful special value of the digamma function is

$$\psi(1) = -\gamma, \tag{2.18}$$

where γ is Euler's constant $\approx 0.57721566490153286060651209008$2.

2.4 Gauss hypergeometric function

The Gauss hypergeometric function (Chapter 2 in Andrews, Askey & Roy (1999) [3], Chapter 15 in Abramowitz & Stegun (1972) [1], Chapter 2 in Magnus, Oberhettinger & Soni (1966) [67]) can be defined for $|z| < 1$ in terms of Pochhammer symbols (see (15.1.1) in Abramowitz & Stegun (1972) [1]) through the series

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{2.19}$$

The series on the unit circle $|z| = 1$ converges absolutely if $\mathrm{Re}(c-a-b) \in (0, \infty)$, converges conditionally if $z \neq 1$ and $\mathrm{Re}(c-a-b) \in (-1, 0]$, and diverges if $\mathrm{Re}(c-a-b) \in (-\infty, -1]$.

Now we discuss analytic continuation of the Gauss hypergeometric function. Newton's binomial theorem is given by

$$(1+w)^{\mu} = \sum_{k=0}^{\infty} \binom{\mu}{k} w^k, \tag{2.20}$$

where $w, \mu \in \mathbf{C}$, $|w| < 1$. If we make the replacement $w \mapsto -w$ and $\mu \mapsto -\mu$ then we have

$$(1-w)^{-\mu} = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} w^k. \tag{2.21}$$

The beta integral, which is defined for $\mathrm{Re}\, x, \mathrm{Re}\, y > 0$, is given by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

One may also speak of the beta function $B(x, y)$, which is obtained from the beta integral by analytic continuation (see §1.1 in Andrews, Askey & Roy (1999) [3]). The beta function is given by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},$$

where $\mathrm{Re}\, x, \mathrm{Re}\, y > 0$. If we consider the following integral where $|z| < 1$, replace $(1 - zt)^{-a}$ with Newton's binomial series (2.21) and evaluate the subsequent beta integral, we are left with

$$\int_0^1 (1 - zt)^{-a} t^{b-1} (1 - t)^{c-b-1} dt = \frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{2.22}$$

Through (2.19) and (2.22) we have what is called Euler's integral representation for the Gauss hypergeometric function, namely

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 (1 - zt)^{-a} t^{b-1} (1 - t)^{c-b-1} dt, \tag{2.23}$$

where $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$. This is for all $|z| < 1$. If we take the principal value of $(1 - zt)^{-a}$, then this integral represents a one valued analytic function in the z-plane cut along the real axis from 1 to ∞, and therefore gives the analytic continuation of (2.19). The function $(1 - zt)^{-a}$ is in general multivalued and one may study the multivalued nature of ${}_2F_1(a, b; c; z)$ using this integral. Analytic continuation may also be applied to the parameters a, b and c.

One consequence of Euler's integral (2.23) is the Gauss summation formula for ${}_2F_1(a, b; c; 1)$ (see (1.2.11) in Gasper & Rahman (2004) [44]). This is a consequence of the resulting beta integral, namely

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \tag{2.24}$$

where $\mathrm{Re}(c - a - b) > 0$ and $c \notin -\mathbf{N}_0$.

If you make a substitution $t = 1 - u$ in Euler's integral (2.23) and re-arrange the integrand, what results is a very useful linear transformation of the Gauss hypergeometric function, Pfaff's transformation (see (2.2.6) in Andrews, Askey & Roy (1999) [3]). If this transformation is applied to a and b separately then we have the two Pfaff transformations

$${}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1\left(a, c - b; c; \frac{z}{z - 1}\right), \tag{2.25}$$

and

$${}_2F_1(a, b; c; z) = (1 - z)^{-b} {}_2F_1\left(c - a, b; c; \frac{z}{z - 1}\right). \tag{2.26}$$

Though the symmetry is not apparent from Pfaff's formula or Euler's integral, the Gauss hypergeometric function is always symmetric in a and b, namely

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z), $$

furthermore

$$ {}_2F_1(0,b;c;z) = {}_2F_1(a,0;c;z) = 1. \tag{2.27} $$

Since the Gauss hypergeometric function is symmetric in a and b, we may apply Pfaff's transformation to itself resulting in what is called Euler's formula, namely

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z). \tag{2.28} $$

The Gauss hypergeometric function satisfies the Gauss hypergeometric differential equation

$$ z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0. $$

The Gauss hypergeometric differential equation (2.39) has 3 regular singular points at $z=0$, $z=+1$, and $z=\infty$. The method of Frobenius (see for instance Ince (1944) [57], Chapter XVI) can be used to find series solutions in a neighbourhood of the regular singular points. Since the Gauss hypergeometric equation is a second order ordinary differential equation, there will be two linearly independent solutions about each singular point which gives 6 solutions. Any three solutions must be linearly related. By applying Pfaff's and Euler's transformations (2.25), (2.26) and (2.28), these 6 solutions are then expanded into 24 solutions. These are often referred to as Kummer's 24 solutions (see for instance §1.2 in Gray (2008)).

As a directly relevant example, relations between solutions of the Gauss hypergeometric equation can be obtained through comparison of solutions in a neighbourhood of the regular singular point at infinity with solutions about the regular singular points 0 and 1. For example, one can obtain for $a-b \notin \mathbf{Z}$ (see p. 48 in Magnus, Oberhettinger & Soni (1966) [67])

$$ \begin{aligned} {}_2F_1(a,b;c;z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1\left(b,b-c+1;b-a+1;\frac{1}{z}\right), \end{aligned} \tag{2.29} $$

where $\arg(-z) \in (-\pi, \pi]$, and

$$
\begin{aligned}
{}_2F_1(a,b;c;z) = {}& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-z)^{-a}{}_2F_1\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \\
& + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-z)^{-b}{}_2F_1\left(b, c-a; b-a+1; \frac{1}{1-z}\right), \qquad (2.30)
\end{aligned}
$$

where $\arg(1-z) \in (-\pi, \pi]$.

The derivative of the Gauss hypergeometric function (see (15.2.1) in Abramowitz & Stegun (1972) [1]) is given by

$$\frac{d}{dz}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1, b+1; c+1; z). \qquad (2.31)$$

One can simplify the Gauss hypergeometric functions using Gauss' relations for contiguous hypergeometric functions. Some of those that will be useful for us are

$$z\,{}_2F_1(a+1,b+1;c+1;z) = \frac{c}{a-b}\big[{}_2F_1(a,b+1;c;z) - {}_2F_1(a+1,b;c;z)\big], \qquad (2.32)$$

(see p. 58 in Erdélyi et al. (1981) [35]) and

$${}_2F_1(a,b+1;c;z) = \frac{b-a}{b}{}_2F_1(a,b;c;z) + \frac{a}{b}{}_2F_1(a+1,b;c;z) \qquad (2.33)$$

(see (15.2.14) in Abramowitz & Stegun (1972) [1]).

One important quadratic transformation of the Gauss hypergeometric function which we will use is

$${}_2F_1(a,b;a-b+1;z) = (1+z)^{-a}{}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; a-b+1; \frac{4z}{(z+1)^2}\right) \qquad (2.34)$$

(see (3.1.9) in Andrews, Askey & Roy (1999) [3]). Also we will regularly use the expression of a binomial as a Gauss hypergeometric function, namely (see (15.1.8) in Abramowitz & Stegun (1972) [1])

$${}_2F_1(a,b;b;z) = (1-z)^{-a}. \qquad (2.35)$$

Another useful Gauss hypergeometric function is given in (15.1.3) Abramowitz & Stegun (1972) [1] as

$${}_2F_1\left(1, 1; 2; \frac{1-z}{2}\right) = \frac{2}{z-1}\log\left(\frac{z+1}{2}\right). \qquad (2.36)$$

Legendre's complete elliptic integrals of the first and second kind (see §2.5) can be given in terms of the Gauss hypergeometric function as follows

$$K(k) = \frac{\pi}{2} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{2.37}$$

and

$$E(k) = \frac{\pi}{2} \, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \tag{2.38}$$

2.5 Elliptic integrals and Jacobi elliptic functions

Here we discuss the Legendre forms of elliptic integrals (see Byrd & Friedman (1954) [19] and Chapter 17 in Abramowitz & Stegun (1972) [1]). The general definition of an elliptic integral is an integral of the form

$$\int R(x,y)dx,$$

where R is a rational function and y^2 is a polynomial in x, of degree 3 or 4. One may always express an elliptic integral, through a change of variables, in terms of Legendre's first, second, or third incomplete forms, or to a limiting case, such as for complete elliptic integrals. For an extensive tabulation of more than a thousand elliptic integrals and a detailed discussion on their explicit reduction in terms of Legendre's incomplete forms, see Byrd & Friedman (1954) [19].

The incomplete elliptic integral of the first kind $F : (0, \pi/2] \times [0, 1) \to \mathbf{R}$ is defined as

$$F(\varphi, k) := \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

The incomplete elliptic integral of the second kind $F : (0, \pi/2] \times [0, 1) \to \mathbf{R}$ is defined as

$$E(\varphi, k) := \int_0^\varphi \sqrt{1 - k^2 \sin^2\theta}\, d\theta.$$

The incomplete elliptic integral of the third kind $\Pi : (0, \pi/2] \times (0, 1)^2 \to \mathbf{R}$ is defined as

$$\Pi(\varphi, \alpha, k) := \int_0^\varphi \frac{d\theta}{(1 - \alpha^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}.$$

The variable φ is the called the argument and the variable k is called the modulus. Note that we are using Legendre's notation for the modulus of an elliptic integral, namely that it is given by k whereas another standard notation, namely Milne-Thompson's notation in Abramowitz & Stegun (1972) [1] is to use modulus given by $m = k^2$. The complete elliptic integrals of the first, second and third kinds, K, E, and Π are given as the incomplete elliptic integrals of the first, second, and third kind respectively with argument $\varphi = \frac{\pi}{2}$.

Now we introduce Jacobi elliptic functions (see §120 in Byrd & Friedman (1954) [19] and Chapter 16 in Abramowitz & Stegun (1972) [1]). The Jacobi elliptic functions like the trigonometric functions have a real period, and like the hyperbolic functions, they have an imaginary period. They are thus doubly periodic in the complex plane. One can define Jacobi elliptic functions in terms of the inverse of the incomplete elliptic integral of the first kind. If we take

$$u = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

then the Jacobi elliptic function, $\mathrm{sn}\, u$ (sine amplitude) is given by

$$\mathrm{sn}\, u := \sin \varphi,$$

$\mathrm{cn}\, u$ (cosine amplitude) is given by

$$\mathrm{cn}\, u := \cos \varphi,$$

and the delta amplitude is given by

$$\mathrm{dn}\, u := \sqrt{1 - k^2 \sin^2 \varphi}.$$

There are a total of twelve Jacobi elliptic functions given in terms of quotients and reciprocals of sn, cn and dn.

2.6 Associated Legendre functions

Associated Legendre functions (Chapter 8 in Abramowitz & Stegun (1972) [1], Chapter 4 and §5.4 in Magnus, Oberhettinger & Soni (1966) [67]) satisfy the associated Legendre differential equation

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1 - z^2}\right] w = 0, \tag{2.39}$$

where $\nu, \mu \in \mathbf{C}$ and $z \in \mathbf{C}\backslash(-\infty, 1]$ are referred to as degree, order and argument respectively. The associated Legendre differential equation (2.39) has 3 regular singular points at $z = +1$, $z = -1$, and $z = \infty$. The associated Legendre functions are special cases of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ where the parameters a, b, c are such that a quadratic transformation of the Gauss hypergeometric function exists. Because of this property it is possible to express a solution of the associated Legendre differential equation in terms of the Gauss hypergeometric function with a choice of 18 different arguments in 72 different ways (see §4.1.1 in Magnus, Oberhettinger & Soni (1966) [67]).

Solutions of the associated Legendre differential equation are given by the associated Legendre functions of the first kind P_ν^μ and the associated Legendre functions of the second kind Q_ν^μ respectively. Associated Legendre functions of the first kind are regular in the neighbourhood of $z = 1$ and are singular at infinity. Associated Legendre functions of the second kind have a singularity at $z = 1$ and vanish at infinity. Note that there is an important convention for associated Legendre functions such that when the order μ in P_ν^μ or Q_ν^μ is left off, i.e. P_ν or Q_ν, this indicates that $\mu = 0$.

For $|z| > 1$ (and by analytic continuation elsewhere), the associated Legendre function of the first kind can be defined by

$$P_\nu^\mu(z) = \frac{\Gamma\left(-\frac{1}{2}-\nu\right) z^{\mu-\nu-1}}{2^{\nu+1}\sqrt{\pi}(z^2-1)^{\mu/2}\Gamma(-\nu-\mu)}\,{}_2F_1\left(\frac{\nu-\mu+1}{2},\frac{\nu-\mu+2}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right)$$
$$+\frac{2^\nu\Gamma\left(\frac{1}{2}+\nu\right) z^{\mu+\nu}}{(z^2-1)^{\mu/2}\Gamma(\nu-\mu+1)}\,{}_2F_1\left(\frac{-\nu-\mu}{2},\frac{-\nu-\mu+1}{2};\frac{1}{2}-\nu;\frac{1}{z^2}\right)$$

(see (8.1.5) in Abramowitz & Stegun (1972) [1]). For any expression of the form $(z^2-1)^\alpha$, read this as

$$(z^2-1)^\alpha := (z+1)^\alpha(z-1)^\alpha, \tag{2.40}$$

for any fixed $\alpha \in \mathbf{C}$ and $z \in \mathbf{C}\backslash\{-1,1\}$. For $|z| > 1$ (and by analytic continuation elsewhere), the associated Legendre function of the second kind can be defined by the following relation with the Gauss hypergeometric function

$$Q_\nu^\mu(z) = \frac{\sqrt{\pi}e^{i\pi\mu}\Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})z^{\nu+\mu+1}}\,{}_2F_1\left(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right) \tag{2.41}$$

(see (8.1.3) in Abramowitz & Stegun (1972) [1]). The associated Legendre functions of the second kind, $Q_\nu^\mu(z)$ are defined for all $\nu, \mu \in \mathbf{C}$ except for

$$(\nu,\mu) \notin \bigcup_{n=0}^{\infty}\bigcup_{k=-n}^{n}\left(-\frac{1}{2}+k,-\frac{1}{2}-n\right).$$

This is due to the fact that the associated Legendre function of the second kind $Q_\nu^\mu(z)$ has simple poles at these values of (ν,μ) which originate from the factor $\Gamma(\nu+\mu+1)$ in (2.41).

If the variable z is real and lies in the open interval $(-1,1)$, then we relabel the argument as x and then the corresponding associated Legendre functions are defined as follows (§8.3

in Abramowitz & Stegun (1972) [1], §4.3.1 in Magnus, Oberhettinger & Soni (1966) [67])

$$P_\nu^\mu(x) = e^{i\pi\mu/2} P_\nu^\mu(x+i0) = e^{-i\pi\mu/2} P_\nu^\mu(x-i0) = \frac{1}{2}\left[e^{i\pi\mu/2} P_\nu^\mu(x+i0) + e^{i\pi\mu/2} P_\nu^\mu(x-i0)\right],$$

where

$$f(x \pm i0) := \lim_{\epsilon\to 0} f(x \pm i\epsilon).$$

Associated Legendre functions of the first kind in this interval can then be defined for $x \in (-1,1)$ in terms of the Gauss hypergeometric function as

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+x}{1-x}\right]^{\mu/2} {}_2F_1\left(-\nu, 1+\nu; 1-\mu; \frac{1-x}{2}\right). \tag{2.42}$$

One interesting consequence of (2.42) is the closed-form expression for a particular associated Legendre function

$$P_0^{-n}(x) = \frac{1}{n!}\left(\frac{1-x}{1+x}\right)^{n/2}, \tag{2.43}$$

where $n \in \mathbf{N}_0$. An analogous expression to (2.42) for $Q_\nu^\mu(x)$ in terms of the Gauss hypergeometric function exists (§4.3.1 in Magnus, Oberhettinger and Soni (1966)), but since this particular associated Legendre function is not used in this thesis, we will not list it here.

2.6.1 General properties of associated Legendre functions

As mentioned above, associated Legendre functions are functions which satisfy quadratic transformations of the hypergeometric function (see §2.4.3 of Magnus, Oberhettinger & Soni (1966) [67]). One such example of a quadratic transformations of the hypergeometric function is given by (cf. (2.42))

$${}_2F_1\left(a, b; a+b-\frac{1}{2}; x\right) = 2^{2+b-3/2}\Gamma\left(a+b-\frac{1}{2}\right)\frac{x^{(3-2a-2b)/4}}{\sqrt{1-x}} P_{b-a-1/2}^{3/2-a-b}\left(\sqrt{1-x}\right), \tag{2.44}$$

where $x \in (0,1)$ (see for instance Magnus, Oberhettinger & Soni (1966) [67], p. 53).

In Magnus, Oberhettinger & Soni (1966) pp. 156–163 [67] there is a table that expresses associated Legendre functions of the first and second kind for $z \in \mathbf{C} \setminus (-\infty, 1]$ in terms of hypergeometric functions with 36 entries. Some of these, that we will use, are entry (27)

$$e^{-i\pi\mu}Q_\nu^\mu(z) = \frac{\sqrt{\pi}2^{\mu-1}\Gamma\left(\frac{\nu+\mu+1}{2}\right)e^{\pm i\pi(\mu-\nu-1)/2)}(z^2-1)^{-\mu/2}}{\Gamma\left(\frac{\nu-\mu+2}{2}\right)} {}_2F_1\left(\frac{-\nu-\mu}{2}, \frac{\nu-\mu+1}{2}; \frac{1}{2}; z^2\right)$$

$$+ \frac{\sqrt{\pi}2^{\mu}\Gamma\left(\frac{\nu+\mu+2}{2}\right)e^{\pm i\pi(\mu-\nu)/2)}z(z^2-1)^{-\mu/2}}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} {}_2F_1\left(\frac{-\nu-\mu+1}{2}, \frac{\nu-\mu+2}{2}; \frac{3}{2}; z^2\right). \quad (2.45)$$

and entry (30)

$$e^{-i\pi\mu}Q_\nu^\mu(z) = \frac{\sqrt{\pi}2^{\mu-1}\Gamma\left(\frac{\nu+\mu+1}{2}\right)e^{\mp i\pi(\nu+1/2)}(z^2-1)^{\nu/2}}{\Gamma\left(\frac{\nu-\mu+2}{2}\right)} {}_2F_1\left(\frac{-\nu-\mu}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}; \frac{z^2}{z^2-1}\right)$$

$$+ \frac{\sqrt{\pi}2^{\mu}\Gamma\left(\frac{\nu+\mu+2}{2}\right)e^{\mp i\pi(\nu-1/2)}z}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)(z^2-1)^{(1-\nu)/2}} {}_2F_1\left(\frac{-\nu-\mu+1}{2}, \frac{-\nu+\mu+1}{2}; \frac{3}{2}; \frac{z^2}{z^2-1}\right). \quad (2.46)$$

In both of these formulae the upper or lower sign is used accordingly whether $\operatorname{Im} z \gtrless 0$.

The associated Legendre functions (of both kinds) satisfy three-term recurrence relations in terms of the order (Gradshteyn & Ryzhik (2007) [48] (8.732.3)) are

$$Q_\nu^{\mu+1}(z) = \frac{1}{\sqrt{z^2-1}}\left[(\nu-\mu)zQ_\nu^\mu(z) - (\nu+\mu)Q_{\nu-1}^\mu(z)\right], \quad (2.47)$$

$$Q_\nu^{\mu+2}(z) = -2(\mu+1)\frac{z}{\sqrt{z^2-1}}Q_\nu^{\mu+1}(z) + (\nu-\mu)(\nu+\mu+1)Q_\nu^\mu(z), \quad (2.48)$$

and

$$Q_\nu^{\mu+1}(\cosh\eta) = -\mu\coth\eta\, Q_\nu^\mu(\cosh\eta) + \frac{dQ_\nu^\mu}{d\eta}. \quad (2.49)$$

The recurrence relation for associated Legendre functions in terms of the degree is

$$(2\nu+1)zQ_\nu^\mu(z) = (\nu-\mu+1)Q_{\nu+1}^\mu(z) + (\nu+\mu)Q_{\nu-1}^\mu(z). \quad (2.50)$$

Some other properties of associated Legendre functions are as follows. The negative-degree condition for associated Legendre functions of the first kind, cf. (8.2.1) in Abramowitz & Stegun (1972) [1],

$$P^\mu_{-\nu-\frac{1}{2}}(z) = P^\mu_{\nu-\frac{1}{2}}(z); \quad (2.51)$$

the negative-degree condition for associated Legendre functions of the second kind, cf. (8.2.2) in Abramowitz & Stegun (1972) [1],

$$Q^{\mu}_{-\nu-\frac{1}{2}}(z) = \frac{1}{\cos\pi(\nu-\mu)}\left[\cos\pi(\nu+\mu)\; Q^{\mu}_{\nu-\frac{1}{2}}(z) + \pi \mathrm{e}^{i\mu\pi}\sin\nu\pi\; P^{\mu}_{\nu-\frac{1}{2}}(z)\right]; \tag{2.52}$$

the negative-order condition for associated Legendre functions of the first kind, cf. (8.2.5) in Abramowitz & Stegun (1972) [1],

$$P^{-\mu}_{\nu-\frac{1}{2}}(z) = \frac{\Gamma(\nu-\mu+\frac{1}{2})}{\Gamma(\nu+\mu+\frac{1}{2})}\left[P^{\mu}_{\nu-\frac{1}{2}}(z) - \frac{2}{\pi}\mathrm{e}^{-i\mu\pi}\sin\mu\pi\; Q^{\mu}_{\nu-\frac{1}{2}}(z)\right]; \tag{2.53}$$

and, finally, the negative-order condition for associated Legendre functions of the second kind, cf. (8.2.6) in Abramowitz & Stegun (1972) [1],

$$Q^{-\mu}_{\nu-\frac{1}{2}}(z) = \mathrm{e}^{-2i\mu\pi}\frac{\Gamma(\nu-\mu+\frac{1}{2})}{\Gamma(\nu+\mu+\frac{1}{2})}Q^{\mu}_{\nu-\frac{1}{2}}(z). \tag{2.54}$$

2.6.2 The Whipple formulae for associated Legendre functions

There is a transformation over an open subset of the complex plane which is particularly useful in studying associated Legendre functions (see Abramowitz & Stegun (1972) [1] and Hobson (1955) [51]). This transformation, which is valid on a certain domain of the complex numbers, accomplishes the following

$$\left.\begin{array}{ccc} \cosh z & \leftrightarrow & \coth w \\ \coth z & \leftrightarrow & \cosh w \\ \sinh z & \leftrightarrow & (\sinh w)^{-1} \end{array}\right\}. \tag{2.55}$$

This transformation is accomplished using the map $w : \mathfrak{D} \to \mathbf{C}$, with

$$\mathfrak{D} := \mathbf{C} \setminus \left\{z \in \mathbf{C} : \mathrm{Re}\, z \le 0 \text{ and } \mathrm{Im}\, z = 2\pi n,\ n \in \mathbf{Z}\right\},$$

and w defined by

$$w(z) := \log\coth\frac{z}{2}. \tag{2.56}$$

The map w is periodic with period $2\pi i$ and is locally injective. The map w restricted to $\mathfrak{D} \cap \{z \in \mathbf{C} : -\pi < \mathrm{Im}\, z < \pi\}$ is verified to be an involution. The transformation (2.55) is the restriction of the mapping w to this restricted domain.

This transformation is particularly useful for certain associated Legendre functions which have natural domain given by the real interval $(1,\infty)$, such as toroidal harmonics (see Cohl et al. (2001) [25], Cohl & Tohline (1999) [26]) (and for other associated Legendre functions which one might encounter in potential theory), associated Legendre functions of the first and second kind with odd-half-integer degree and integer-order. The real argument of these associated Legendre functions naturally occur in $[1,\infty)$, and these are the simultaneous ranges of both the real hyperbolic cosine and cotangent functions. One application of this map occurs with the Whipple formulae for associated Legendre functions (Cohl & al. (2000) [27], Whipple (1917) [107]) under index (degree and order) interchange. See for instance, (8.2.7) and (8.2.8) in Abramowitz & Stegun (1972) [1], namely

$$P_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right)=\sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{1/4}\mathrm{e}^{-i\mu\pi}}{\Gamma(\nu+\mu+1)}Q_\nu^\mu(z), \tag{2.57}$$

and

$$Q_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right)=-i(\pi/2)^{1/2}\Gamma(-\nu-\mu)(z^2-1)^{1/4}\mathrm{e}^{-i\nu\pi}P_\nu^\mu(z),$$

which are valid for $\mathrm{Re}\, z>0$ and for all complex ν and μ, except where the functions are not defined.

2.6.3 Explicit forms for associated Legendre functions of the second kind

There are some specific closed-form expressions for $Q_\nu^\mu(z)$ such that $\nu,\mu\in\mathbf{C}$ and $z\in\mathbf{C}\setminus(-\infty,1]$. For instance, the 1/2-order associated Legendre functions of the second kind ((8.6.10) in Abramowitz & Stegun (1972) [1]) are given by

$$Q_\nu^{1/2}(z)=i\sqrt{\frac{\pi}{2}}(z^2-1)^{-1/4}\left[z+\sqrt{z^2-1}\right]^{-\nu-1/2}, \tag{2.58}$$

One special case of (2.58) is

$$Q_{-1/2}^{1/2}(z)=i\sqrt{\frac{\pi}{2}}(z^2-1)^{-1/4}. \tag{2.59}$$

Similarly the $-1/2$-order associated Legendre functions of the second kind ((8.6.11) in Abramowitz & Stegun (1972) [1]) are

$$Q_\nu^{-1/2}(z)=-i\frac{\sqrt{2\pi}}{2\nu+1}(z^2-1)^{-1/4}\left[z+\sqrt{z^2-1}\right]^{-\nu-1/2}. \tag{2.60}$$

Using (2.48), (2.60) and (2.58) we can compute the 3/2-order associated Legendre functions of the second kind

$$Q_\nu^{3/2}(z) = -i\sqrt{\frac{\pi}{2}}(z^2-1)^{-3/4}\left(z+\sqrt{z^2-1}\right)^{-(\nu+1/2)}\left[z+\left(\nu+\frac{1}{2}\right)\sqrt{z^2-1}\right]. \quad (2.61)$$

Some special cases of (2.61) are given by

$$Q_{-1/2}^{3/2}(z) = \frac{1}{i}\sqrt{\frac{\pi}{2}}z(z^2-1)^{-3/4}, \quad (2.62)$$

$$Q_{1/2}^{3/2}(z) = \frac{1}{i}\sqrt{\frac{\pi}{2}}(z^2-1)^{-3/4}. \quad (2.63)$$

Some useful odd-half-integer degree, integer-order associated Legendre functions of the second kind (see Cohl (2003) [22]) are given by

$$Q_{-1/2}(z) = \sqrt{\frac{2}{z+1}}K\left(\sqrt{\frac{2}{z+1}}\right), \quad (2.64)$$

$$Q_{1/2}(z) = z\sqrt{\frac{2}{z+1}}K\left(\sqrt{\frac{2}{z+1}}\right) - \sqrt{2(z+1)}E\left(\sqrt{\frac{2}{z+1}}\right),$$

$$Q_{-1/2}^1(z) = \frac{-1}{\sqrt{2(z-1)}}E\left(\sqrt{\frac{2}{z+1}}\right), \quad (2.65)$$

and

$$Q_{1/2}^1(z) = \frac{-z}{\sqrt{2(z-1)}}E\left(\sqrt{\frac{2}{z+1}}\right) + \sqrt{\frac{z-1}{2}}K\left(\sqrt{\frac{2}{z+1}}\right), \quad (2.66)$$

where K and E are Legendre's complete elliptic integrals of the first and second kind respectively (see §2.5). The rest of the functions $Q_{n-1/2}^m$, for $n, m \in \mathbf{Z}$ can be generated using (2.50), and they are called toroidal harmonics (see §5.2).

The associated Legendre functions of second kind Q_n for $n \in \{0,1,2\}$ (see (8.4.2), (8.4.4) and (8.4.6) in Abramowitz & Stegun (1972) [1]) are given by

$$Q_0(z) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right), \quad (2.67)$$

$$Q_1(z) = \frac{z}{2}\log\left(\frac{z+1}{z-1}\right) - 1, \quad \text{and} \quad (2.68)$$

$$Q_2(z) = \frac{3z^2-1}{4}\log\left(\frac{z+1}{z-1}\right) - \frac{3z}{2}. \quad (2.69)$$

For $n > 2$ one can use (2.50) to generate the rest of these.

2.6.4 Derivatives with respect to the degree of certain integer-order associated Legendre functions of the first kind

The derivative with respect to its degree for the associated Legendre function of the first kind evaluated at the zero degree is given in §4.4.3 of Magnus, Oberhettinger & Soni (1966) [67] as

$$\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=0} = \frac{z-1}{2}\,{}_2F_1\left(1,1;2;\frac{1-z}{2}\right). \tag{2.70}$$

An important generalization of this formula has recently been derived (see Szmytkowski (2009) [95]) and is given by

$$\begin{aligned}\left[\frac{\partial}{\partial\nu}P_\nu^m(z)\right]_{\nu=p} &= P_p^m(z)\log\frac{z+1}{2} \\ &\quad + \left[2\psi(2p+1)-\psi(p+1)-\psi(p-m+1)\right]P_p^m(z) \\ &\quad + (-1)^{p+m}\sum_{k=0}^{p-m-1}(-1)^k\frac{2k+2m+1}{(p-m-k)(p+m+k+1)} \\ &\quad \times\left[1+\frac{k!(p+m)!}{(k+2m)!(p-m)!}\right]P_{k+m}^m(z) \\ &\quad + (-1)^p\frac{(p+m)!}{(p-m)!}\sum_{k=0}^{m-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k^{-m}(z),\end{aligned} \tag{2.71}$$

where $p, m \in \mathbf{N}_0$ and $0 \le m \le p$. Some special cases of (2.71) include for $m = 0$

$$\begin{aligned}\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=p} &= P_p(z)\log\frac{z+1}{2} + 2\left[\psi(2p+1)-\psi(p+1)\right]P_p(z) \\ &\quad + 2(-1)^p\sum_{k=0}^{p-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k(z),\end{aligned} \tag{2.72}$$

for $m = p$

$$\begin{aligned}\left[\frac{\partial}{\partial\nu}P_\nu^p(z)\right]_{\nu=p} &= P_p^p(z)\log\frac{z+1}{2} + \left[2\psi(2p+1)-\psi(p+1)+\gamma\right]P_p^p(z) \\ &\quad + (-1)^p(2p)!\sum_{k=0}^{p-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k^{-p}(z),\end{aligned} \tag{2.73}$$

where γ is Euler's constant defined in (2.18). Of course we also have for $m = p = 0$

$$\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=0} = \log\frac{z+1}{2},$$

which exactly matches (2.70).

2.6.5 Derivatives with respect to the degree and order of associated Legendre functions for $|z| > 1$ using modified Bessel functions

We present results for parameter derivatives of associated Legendre functions which have been recently published in Cohl (2010) [23]. We should note that these results are not going to be used in the rest of this thesis. However we present them nonetheless, since they represent interesting and new research by the author. In this section, we present and derive formulae for parameter derivatives of associated Legendre functions of the first kind P_ν^μ and the second kind Q_ν^μ, with respect to their parameters, namely the degree ν and the order μ. Some formulae relating to these derivatives have been previously noted (see §4.4.3 in Magnus, Oberhettinger & Soni (1966) [67]) and also there has been recent work in this area [16, 92, 93, 94, 95] with Brychkov (2009) [17] giving a recent reference covering the regime for argument $z \in (-1, 1)$. We cover parameter derivatives of associated Legendre functions for argument $z \in \mathbf{C} \setminus (-\infty, 1]$.

We incorporate derivatives with respect to order evaluated at integer-orders for modified Bessel functions (see Abramowitz & Stegun (1972) [1], Brychkov & Geddes (2005) [18], Magnus, Oberhettinger & Soni (1966) [67]) to compute derivatives with respect to the degree and the order of associated Legendre functions. Below we apply these results through certain integral representations of associated Legendre functions in terms of modified Bessel functions. Modified Bessel functions of the first and second kind respectively can be defined for unrestricted values of ν (see for instance §3.7 in Watson (1944) [104]) by

$$I_\nu(z) := \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)},$$

and

$$K_\nu(z) := \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}.$$

For $\nu = n \in \mathbf{N}_0$, the first equation yields

$$I_n(z) = I_{-n}(z).$$

It may be verified that

$$K_n(z) = \lim_{\nu \to n} K_\nu(z)$$

is well defined. The modified Bessel function of the second kind is commonly referred to as a Macdonald function.

The strategy applied in this section is to use integral representations of associated Legendre functions, expressed in terms of modified Bessel functions, and differentiate with respect to the relevant parameters.

In the following proposition, we have used the convention (2.40).

Proposition 2.6.1. *Define the function* $f : \mathbf{C} \setminus \{-1, 1\} \to \mathbf{C}$ *by*

$$f(z) = \frac{z}{\sqrt{z^2 - 1}} := \frac{z}{\sqrt{z + 1}\sqrt{z - 1}}.$$

This function f has the following properties

1. $f\big|_{\mathbf{C}\setminus[-1,1]}$ *is even and* $f\big|_{(-1,1)}$ *is odd.*
2. *The sets* $(0, 1)$ *and* $(-1, 0)$ *are mapped onto* $i(-\infty, 0)$ *and* $i(0, \infty)$ *respectively.*
3. *The sets* $i(-\infty, 0)$ *and* $i(0, \infty)$ *are both mapped to* $(0, 1)$.
4. $f(0) = 0$.
5. *If* $z \in \mathbf{C} \setminus [-1, 1]$ *then* $\operatorname{Re} \dfrac{z}{\sqrt{z^2 - 1}} > 0$.

Proof. Note that

$$\arg(\sqrt{w}) = \frac{1}{2} \arg w,$$

for $w \in \mathbf{C} \setminus \{0\}$. If $z \in \mathbf{C}$ and $\operatorname{Im} z > 0$ then

$$\arg(-(z \pm 1)) = -\pi + \arg(z \pm 1),$$

so

$$\arg\left(\sqrt{-(z \pm 1)}\right) = -\frac{\pi}{2} + \arg\left(\sqrt{z \pm 1}\right),$$

and we have

$$\sqrt{-(z \pm 1)} = -i\sqrt{z \pm 1}.$$

Hence

$$f(-z) = \frac{-z}{i^2\sqrt{z + 1}\sqrt{z - 1}} = f(z).$$

Similarly if $\operatorname{Im} z < 0$ then

$$\sqrt{-(z \pm 1)} = i\sqrt{z \pm 1},$$

and we have the same result.

Let $x > 1$. Then

$$\arg \sqrt{-(x \pm 1)} = \frac{\pi}{2},$$

so

$$f(-x) = \frac{-x}{\sqrt{-(x+1)}\sqrt{-(x+1)}} = \frac{x}{\sqrt{x+1}\sqrt{x-1}} = f(x).$$

Therefore $f\big|_{\mathbf{C}\setminus[-1,1]}$ is even.

If $x \in (0,1)$ then

$$f(x) = \frac{-ix}{\sqrt{1+x}\sqrt{1-x}},$$

and

$$f(-x) = \frac{ix}{\sqrt{1+x}\sqrt{1-x}} = -f(x).$$

Moreover, $f(0) = 0$. Therefore $f\big|_{(-1,1)}$ maps to the imaginary axis and is odd.

If $x \in (0,\infty)$ then

$$f(ix) = \frac{ix}{\sqrt{ix+1}\sqrt{ix-1}} = \frac{x}{\sqrt{1+x^2}},$$

and

$$f(-ix) = \frac{-ix}{\sqrt{-ix+1}\sqrt{-ix-1}} = \frac{x}{\sqrt{1+x^2}},$$

so f maps both the positive and negative imaginary axes to the real interval $(0,1)$. Clearly $f(0) = 0$. This completes the proof of 1, 2, 3 and 4.

Before we prove 5 we first show that f maps quadrant I into quadrant IV. This is non-trivial. Let $r \in (0,\infty)$ and $\theta \in (0,\pi/2)$. Then

$$f(re^{i\theta}) = \frac{r\exp\left[i(\theta - \frac{1}{2}\phi - \frac{1}{2}\psi)\right]}{\left(r^4 - 2r^2\cos(2\theta) + 1\right)^{1/4}},$$

where

$$\phi := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right),$$

and

$$\psi := \begin{cases} \pi + \tan^{-1}\left(\dfrac{r\sin\theta}{r\cos\theta - 1}\right) & \text{if } r\cos\theta < 1, \\ \frac{\pi}{2} & \text{if } r\cos\theta = 1, \\ \tan^{-1}\left(\dfrac{r\sin\theta}{r\cos\theta - 1}\right) & \text{if } r\cos\theta > 1. \end{cases}$$

Firstly we would like to prove that $\theta - \frac{1}{2}\phi - \frac{1}{2}\psi < 0$, or equivalently

$$\phi + \psi > 2\theta. \tag{2.74}$$

We will break the problem into nine main cases:

$$\left[\begin{array}{ll} \text{I.} & \theta \in \left(0, \dfrac{\pi}{4}\right). \\ \text{II.} & \theta = \dfrac{\pi}{4}. \\ \text{III.} & \theta \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right). \end{array}\right.$$

$$\left[\begin{array}{ll} \text{A.} & r\cos\theta < 1. \\ \text{B.} & r\cos\theta = 1. \\ \text{C.} & r\cos\theta > 1. \end{array}\right.$$

Case IA. We need to show that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \pi + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) > 2\theta,$$

for all $r < 1/\cos\theta$ and $\theta \in (0, \pi/4)$. First note that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) \in \left(0, \frac{\pi}{2}\right),$$

and

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) \in \left(-\frac{\pi}{2}, 0\right),$$

so

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \pi + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

and since $2\theta \in (0, \pi/2)$, we have the desired result.
Case IB. We need to show that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \frac{\pi}{2} > 2\theta,$$

for $r = 1/\cos\theta$ and $\theta \in (0, \pi/4)$. Since $r = 1/\cos\theta$ this reduces to

$$\tan^{-1}\left(\frac{1}{2}\tan\theta\right) + \frac{\pi}{2} > 2\theta.$$

Which is true since $2\theta \in (0, \pi/2)$ and $\tan^{-1}\left(\frac{1}{2}\tan\theta\right) > 0$.

Case IC. Define $g : \{(\theta, r) : \theta \in (0, \pi/4), r \in (1/\cos\theta, \infty)\} \to \mathbf{R}$ by

$$g(\theta, r) := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) - 2\theta.$$

We need to show that $g(\theta, r) > 0$ for all $\theta \in (0, \pi/4)$ and $r > 1/\cos\theta$. Fix $\theta \in (0, \pi/4)$. Then

$$\frac{\partial g}{\partial r}(\theta, r) = -\frac{4r\cos\theta\sin\theta}{r^4 - 2r^2\cos(2\theta) + 1} < 0.$$

Therefore $r \mapsto g(\theta, r)$ is strictly decreasing. Moreover

$$\lim_{r\to\infty} g(\theta, r) = \tan^{-1}(\tan\theta) + \tan^{-1}(\tan\theta) - 2\theta = 0.$$

Hence $g(\theta, r) > 0$ for all $r \in (1/\cos\theta, \infty)$.

Case IIA. This follows as in Case IA.

Case IIB. Trivial.

Case IIC. In this case $\theta = \pi/4$ and $r > \sqrt{2}$. Consider the function $g : (\sqrt{2}, \infty) \to \mathbf{R}$ defined by

$$g(r) := \tan^{-1}\left(\frac{1}{1 + \frac{\sqrt{2}}{r}}\right) + \tan^{-1}\left(\frac{1}{1 - \frac{\sqrt{2}}{r}}\right).$$

We need to show that $g > \pi/2$. The derivative of g is given by

$$\frac{dg(r)}{dr} = -\frac{2r}{1 + r^4} < 0.$$

This implies that g is a strictly decreasing function. Taking the limit

$$\lim_{r\to\infty} g(r) = \tan^{-1}(1) + \tan^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Since g is a strictly decreasing function of r, we have the desired result.

Case IIIA. Define $g : \{(\theta, r) : \theta \in (\pi/4, \pi/2), r \in (0, 1/\cos\theta)\} \to \mathbf{R}$ by

$$g(\theta, r) := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) - 2\theta.$$

We need to show that $g(\theta, r) > -\pi$ for all $\theta \in (\pi/4, \pi/2)$ and $r < 1/\cos\theta$. Fix $r \in (0, \infty)$. Then if $\theta > \cos^{-1}(1/r)$ and $\theta \in (\pi/4, \pi/2)$,

$$\frac{\partial g}{\partial r}(\theta, r) = \frac{2(r^2\cos(2\theta) - 1)}{r^4 - 2r^2\cos(2\theta) + 1} < 0,$$

since $\cos(2\theta) \in (-1, 0)$. Therefore $\theta \mapsto g(\theta, r)$ is strictly decreasing. Since

$$\lim_{\theta \to \frac{\pi}{2}-} g(\theta, r) = \tan^{-1}(r) + \tan^{-1}(-r) - \pi = -\pi,$$

the required inequality follows.

Case IIIB. For $\theta \in (\pi/4, \pi/2)$, would like to prove the inequality

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \frac{\pi}{2} > 2\theta,$$

with $r = 1/\cos\theta$, or equivalently,

$$\tan^{-1}\left(\frac{1}{2}\tan\theta\right) + \frac{\pi}{2} > 2\theta.$$

Consider $g : (\pi/4, \pi/2) \to \mathbf{R}$ defined by

$$g(\theta) := \tan^{-1}\left(\frac{1}{2}\tan\theta\right) - 2\theta + \frac{\pi}{2}.$$

We need to show that $g > 0$. Then

$$\frac{\partial g}{\partial \theta}(\theta) = -\frac{6}{4 + \tan^2\theta} < 0$$

and

$$\lim_{\theta \to \frac{\pi}{2}-} g(\theta) = \lim_{\theta \to \frac{\pi}{2}-} \tan^{-1}\left(\frac{1}{2}\tan\theta\right) - \pi + \frac{\pi}{2} = 0.$$

The required estimate follows.

Case IIIC. Define $g : \{(\theta, r) : \theta \in (0, \pi/2), r > 1/\cos\theta\} \to \mathbf{R}$ by

$$g(\theta, r) := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right).$$

We would like to prove the inequality $g(\theta, r) > 2\theta$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. We first show that $g(\theta, r) > \pi/2$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. Then

$$\frac{\partial g}{\partial \theta}(\theta, r) = \frac{2r^2(r^2 - \cos(2\theta))}{(r^2 + 2r\cos\theta + 1)(r^2 - 2r\cos\theta + 1)} > 0,$$

for all $\theta \in (\pi/4, \cos^{-1}(1/r))$ since $\cos(2\theta) < 0$ and all factors are positive. Hence $g(\theta, r) > g(\pi/4, r)$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. Next

$$\frac{dg}{dr}\left(\frac{\pi}{4}, r\right) = \frac{-2r}{(r^2 + \sqrt{2}r + 1)(r^2 - \sqrt{2}r + 1)} < 0,$$

for all $r \in (\sqrt{2}, \infty)$ and

$$\lim_{r\to\infty} g\left(\frac{\pi}{4}, r\right) = \frac{\pi}{2}.$$

Therefore $g\left(\frac{\pi}{4}, r\right) > \frac{\pi}{2}$ for all $r \in (\sqrt{2}, \infty)$ and hence $g(\theta, r) > \frac{\pi}{2}$ for all $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $r > 1/\cos\theta$. We have shown that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) > \frac{\pi}{2},$$

for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. Since also $2\theta > \pi/2$, the inequality

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) > 2\theta,$$

is equivalent to the inequality

$$\tan(2\theta) < \tan\left[\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right)\right].$$

Using the addition formula for the tangent function (2.2), the above inequality reduces to

$$\tan(2\theta) < \frac{r^2\sin(2\theta)}{r^2\cos(2\theta) - 1},$$

which is trivially verified. Thus (2.74) is valid for all $\theta \in \left(0, \frac{\pi}{2}\right)$.

Secondly we would like to show that $\theta - \frac{1}{2}\phi - \frac{1}{2}\psi > -\frac{\pi}{2}$ or equivalently

$$\phi + \psi - 2\theta < \pi. \tag{2.75}$$

This inequality if clear for cases B, C, IIA and IIIA. All that remains is to prove (2.75) for case IA. Define $g : \{(\theta, r) : \theta \in (0, \pi/4), r \in (0, 1/\cos\theta)\} \to \mathbf{R}$ by

$$g(\theta, r) := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right) - 2\theta.$$

We need to show that $g < 0$. Fix $\theta \in (0, \pi/4)$. Then

$$\frac{\partial g}{\partial r}(\theta, r) = -\frac{4r\cos\theta\sin\theta}{r^4 - 2r^2\cos(2\theta) + 1} < 0.$$

Therefore $r \mapsto g(\theta, r)$ is strictly decreasing. Moreover

$$\lim_{r \to \frac{1}{\cos\theta}-} g(\theta, r) = \tan^{-1}\left(\frac{1}{2}\tan\theta\right) - \frac{\pi}{2} - 2\theta.$$

However $2\theta + \frac{\pi}{2} \in (\pi/2, \pi)$. It follows that $g(\theta, r) < 0$ for all $\theta \in (0, \pi/4)$ and $r \in (0, 1/\cos\theta)$.

Thus f maps quadrant I into quadrant IV.

Due to the evenness of the f, quadrants I & III are mapped to quadrants IV, and quadrants II & IV are mapped to quadrant I. Therefore if $z \in \mathbf{C} \setminus [-1, 1]$ then $\operatorname{Re}\dfrac{z}{\sqrt{z^2-1}} > 0$. This completes the proof of 5. The range of f is $\{z \in \mathbf{C} : \operatorname{Re} z \geq 0 \text{ and } z \neq 1\}$. Every complex number in the range of the function is taken twice except for elements in $(0, 1)$ and on the imaginary axis. These complex numbers are taken only once.

Parameter derivative formulas from $K_\nu(t)$

By starting with Gradshteyn & Ryzhik (2007) (6.628.7) [48] (see also Prudnikov et al. (1988) (2.16.6.3) [81]) and using the Whipple formulae (2.57), we have for $\operatorname{Re} z > -1$ and $\operatorname{Re}\mu > |\operatorname{Re}\nu| - \frac{1}{2}$,

$$\begin{aligned}\int_0^\infty e^{-zt} K_\nu(t) t^{\mu-1/2} dt &= \sqrt{\frac{\pi}{2}}\Gamma\left(\mu - \nu + \frac{1}{2}\right)\Gamma\left(\mu + \nu + \frac{1}{2}\right)\left(z^2-1\right)^{-\mu/2} P_{\nu-1/2}^{-\mu}(z) \\ &= \Gamma\left(\mu - \nu + \frac{1}{2}\right)\left(z^2-1\right)^{-\mu/2-1/4} e^{-i\pi\nu} Q_{\mu-1/2}^{\nu}\left(\frac{z}{\sqrt{z^2-1}}\right), \quad (2.76)\end{aligned}$$

where K_ν is a modified Bessel function of the second kind with order ν. We would like to generate an analytical expression for the derivative of the associated Legendre function of the second kind with respect to its order, evaluated at integer-orders. In order to do this our strategy is to solve the above integral expression for the associated Legendre function of the second kind, differentiate with respect to the order, evaluate at integer-orders, and take advantage of the corresponding formula for differentiation with respect to order for modified

Bessel functions of the second kind (see Abramowitz & Stegun (1972) [1], Brychkov (2010) [17], Brychkov & Geddes (2005) [18], Magnus, Oberhettinger & Soni (1966) [67]). Using the expression for the associated Legendre function of the second kind in (2.76), we solve for $Q^{\mu}_{\nu-1/2}(z)$ and re-express using the map in (2.56). This gives us the following expression

$$Q^{\mu}_{\nu-1/2}(z) = \frac{\left(z^2-1\right)^{-\nu/2-1/4} e^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)} \int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) K_\mu(t) t^{\nu-1/2} dt. \tag{2.77}$$

In order to justify differentiation under the integral sign we use the following well-known corollary of the bounded convergence theorem (cf. §8.2 in Lang (1993) [64]).

Proposition 2.6.2. *Let (X,μ) be a measure space, $U \subset \mathbf{R}$ open and $f : X \times U \to \mathbf{R}$ a function. Suppose*

1. *for all $y \in U$ the function $x \mapsto f(x,y)$ is measurable,*
2. *$\frac{\partial f}{\partial y}(x,y)$ exists for all $(x,y) \in X \times U$,*
3. *there exists $g \in \mathcal{L}^1(X)$ such that $\left|\frac{\partial f}{\partial y}(x,y)\right| \le g(x)$ for all $(x,y) \in X \times U$.*

Then the function $y \mapsto \int_X f(x,y) d\mu(x)$ is differentiable on U and

$$\frac{d}{dy}\left(\int_X f(x,y) d\mu(x)\right) = \int_X \frac{\partial f}{\partial y}(x,y) d\mu(x).$$

We call g a $\mathcal{L}^1$-majorant.

We wish to differentiate (2.77) with respect to the order μ and evaluate at $\mu_0 = \pm m$, where $m \in \mathbf{N}_0$. The derivative of the modified Bessel function of the second kind with respect to its order (see Abramowitz & Stegun (1972) [1], Brychkov (2010) [17], Brychkov & Geddes (2005) [18], Magnus, Oberhettinger & Soni (1966) [67]) is given by

$$\left[\frac{\partial}{\partial \mu} K_\mu(t)\right]_{\mu=\pm m} = \pm m! \sum_{k=0}^{m-1} \frac{2^{m-1-k}}{k!(m-k)} t^{k-m} K_k(t) \tag{2.78}$$

(see for instance (1.14.2.2) in Brychkov (2008) [16]). For a fixed t, $K_\mu(t)$ is an even function of $\mu \in \mathbf{R}$ (see (9.6.6) in Abramowitz & Stegun (1972) [1]), i.e.

$$K_{-\mu}(t) = K_\mu(t),$$

and for $\mu \in [0,\infty)$, $K_\mu(t)$ is a strictly increasing function of μ. Also, for a fixed t, $\frac{\partial K_\mu(t)}{\partial \mu}$ is an odd function of $\mu \in \mathbf{R}$ and for $\mu \in [0,\infty)$, $\frac{\partial K_\mu(t)}{\partial \mu}$ is also a strictly increasing function of

μ. Using (2.78) we can make the following estimate

$$\left|\frac{\partial}{\partial\mu}K_\mu(t)\right| < \left.\frac{\partial K_\tau}{\partial\tau}\right|_{\tau=\pm(m+1)}, \tag{2.79}$$

for all $\mu \in (\mu_0 - 1, \mu_0 + 1)$.

To justify differentiation under the integral sign in (2.77), with respect to μ, evaluated at μ_0, we use Proposition 2.6.2. If we fix z and ν, the integrand of (2.77) can be given by the function $f : \mathbf{R} \times (0, \infty) \to \mathbf{C}$ defined by

$$f(\mu, t) := \exp\left(\frac{-zt}{\sqrt{z^2 - 1}}\right) t^{\nu-1/2} K_\mu(t).$$

Since $\dfrac{\partial K_\mu(t)}{\partial\mu}$ is a strictly increasing function of $\mu \in [0, \infty)$, we have for all $\mu \in (\mu_0 - 1, \mu_0 + 1)$

$$\begin{aligned}
\left|\frac{\partial f}{\partial\mu}(\mu, t)\right| &= \exp\left(\frac{-zt}{\sqrt{z^2 - 1}}\right) t^{\nu-1/2} \left|\frac{\partial}{\partial\mu}K_\mu(t)\right| \\
&< \exp\left(\frac{-zt}{\sqrt{z^2 - 1}}\right) t^{\nu-1/2} \left|\left[\frac{\partial}{\partial\tau}K_\tau(t)\right]_{\tau=\pm(m+1)}\right| \\
&= \exp\left(\frac{-zt}{\sqrt{z^2 - 1}}\right) t^{\nu-1/2} \left|\left[\frac{\partial}{\partial\tau}K_\tau(t)\right]_{\tau=m+1}\right|, \\
&\le \exp\left(\frac{-zt}{\sqrt{z^2 - 1}}\right) t^{\nu-1/2} (m+1)! \sum_{k=0}^{m} \frac{2^{m-k}}{k!(m+1-k)} t^{k-m-1} K_k(t), \\
&\le \exp\left(\frac{-zt}{\sqrt{z^2 - 1}}\right) t^{\nu-1/2} (m+1)! 2^m t^{-1} K_m(t) =: g(t),
\end{aligned}$$

where we used (2.78) and the fact that $K_k(t) \le K_m(t)$ for all $k \in \{0, \ldots, m-1\}$. Then g is a $\mathcal{L}^1$-majorant for the derivative of the integrand, since the integral (2.77) converges for $\operatorname{Re} \dfrac{z}{\sqrt{z^2 - 1}} > -1$ and $\operatorname{Re}\nu > m - \frac{1}{2}$.

The conditions for differentiating under the integral sign have been satisfied and we can re-write (2.77) as

$$\left[\frac{\partial}{\partial\mu}Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = \left(z^2-1\right)^{-\nu/2-1/4}\left[\frac{\partial}{\partial\mu}\frac{e^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\right]_{\mu=\pm m} \times\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)K_{\pm m}(t)t^{\nu-1/2}dt + \frac{\left(z^2-1\right)^{-\nu/2-1/4}(-1)^m}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)} \times\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)t^{\nu-1/2}\left[\frac{\partial}{\partial\mu}K_\mu(t)\right]_{\mu=\pm m}dt. \tag{2.80}$$

The derivative from the first term is given as

$$\left[\frac{\partial}{\partial\mu}\frac{e^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\right]_{\mu=\pm m} = \frac{(-1)^m}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)}\left[i\pi+\psi\left(\nu\mp m+\frac{1}{2}\right)\right],$$

where the ψ is the digamma function, (2.15).

Substituting these expressions for the derivatives into the two integrals and using the map in (2.56) to re-evaluate these integrals in terms of associated Legendre functions gives the following general expression for the derivative of the associated Legendre function of the second kind with respect to its order evaluated at integer-orders as

$$\frac{\Gamma(\nu\mp m+\frac{1}{2})}{\Gamma(\nu-m+\frac{1}{2})}\left[\frac{\partial}{\partial\mu}Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = \left[i\pi+\psi\left(\nu\mp m+\frac{1}{2}\right)\right]Q^m_{\nu-1/2}(z) \pm m!\sum_{k=0}^{m-1}\frac{(-1)^{k-m}\left(z^2-1\right)^{(k-m)/2}}{k!(m-k)2^{k-m+1}}Q^k_{\nu+k-m-1/2}(z).$$

We are now able to obtain formulas for integer values of μ. For $\mu=0$, the sum gives no contribution and therefore

$$\left[\frac{\partial}{\partial\mu}Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=0} = \left[i\pi+\psi\left(\nu+\frac{1}{2}\right)\right]Q_{\nu-1/2}(z).$$

This agrees with that given in §4.4.3 of Magnus, Oberhettinger & Soni (1966) [67]. For $\mu=-1$ we have

$$\left(\nu^2-\frac{1}{4}\right)\left[\frac{\partial}{\partial\mu}Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=-1} = \left[i\pi+\psi\left(\nu+\frac{3}{2}\right)\right]Q^1_{\nu-1/2}(z)+\left(z^2-1\right)^{-1/2}Q_{\nu-3/2}(z),$$

and for $\mu = +1$ we have

$$\left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=1} = \left[i\pi + \psi\left(\nu - \frac{1}{2}\right)\right] Q^{1}_{\nu-1/2}(z) - \left(z^2 - 1\right)^{-1/2} Q_{\nu-3/2}(z).$$

If we start with the expression for the associated Legendre function of the first kind in (2.76) and solve for $P^{-\mu}_{\nu-1/2}(z)$ we have

$$P^{-\mu}_{\nu-1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{\mu/2}}{\Gamma\left(\mu - \nu + \frac{1}{2}\right)\Gamma\left(\mu + \nu + \frac{1}{2}\right)} \int_0^\infty e^{-zt} K_\nu(t) t^{\mu-1/2} dt. \tag{2.81}$$

To justify differentiation under the integral sign in (2.81), with respect to ν, evaluated at $\nu = \pm n$, where $n \in \mathbf{N}_0$, we use as similar argument as in (2.77) only with modification $\mu \mapsto \nu$ and $m \mapsto n$. The same modified $\mathcal{L}^1$-majorant will work for the derivative of this integrand, since the integral (2.81) converges for $\operatorname{Re} z > -1$ and $\operatorname{Re}\nu > |\operatorname{Re}\mu| - \frac{1}{2}$.

The conditions for differentiating under the integral sign have been satisfied and we can re-write (2.81) as

$$\begin{aligned}\left[\frac{\partial}{\partial\nu} P^{-\mu}_{\nu-1/2}(p)\right]_{\nu=\pm n} &= \sqrt{\frac{2}{\pi}} \left(z^2-1\right)^{\mu/2} \left[\frac{\partial}{\partial\nu} \frac{1}{\Gamma\left(\mu - \nu + \frac{1}{2}\right)\Gamma\left(\mu + \nu + \frac{1}{2}\right)}\right]_{\nu=\pm n} \\ &\quad \times \int_0^\infty e^{-zt} K_{\pm n}(t) t^{\mu-1/2} dt \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{\mu/2}}{\Gamma\left(\mu \mp n + \frac{1}{2}\right)\Gamma\left(\mu \pm n + \frac{1}{2}\right)} \\ &\quad \times \int_0^\infty e^{-zt} t^{\mu-1/2} \left[\frac{\partial}{\partial\nu} K_\nu(t)\right]_{\nu=\pm n} dt.\end{aligned} \tag{2.82}$$

The derivative from the first term in (2.82) is given as

$$\left[\frac{\partial}{\partial\nu} \frac{1}{\Gamma\left(\mu - \nu + \frac{1}{2}\right)\Gamma\left(\mu + \nu + \frac{1}{2}\right)}\right]_{\nu=\pm n} = \frac{\psi\left(\mu \mp n + \frac{1}{2}\right) - \psi\left(\mu \pm n + \frac{1}{2}\right)}{\Gamma\left(\mu \pm n + \frac{1}{2}\right)\Gamma\left(\mu \mp n + \frac{1}{2}\right)}.$$

Substituting this expression for the derivative and that given in (2.78) yields the following general expression for the derivative of the associated Legendre function of the first kind with respect to its degree evaluated at odd-half-integer degrees as

$$\pm\left[\frac{\partial}{\partial\nu}P^{-\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n}=\left[\psi\left(\mu-n+\frac{1}{2}\right)-\psi\left(\mu+n+\frac{1}{2}\right)\right]P^{-\mu}_{n-1/2}(z)$$
$$+\frac{n!}{\Gamma\left(\mu+n+\frac{1}{2}\right)}\sum_{k=0}^{n-1}\frac{\Gamma\left(\mu-n+2k+\frac{1}{2}\right)\left(z^2-1\right)^{(n-k)/2}}{k!(n-k)2^{k-n+1}}P^{-\mu+n-k}_{k-1/2}(z).$$

If one makes a global replacement $-\mu\mapsto\mu$, using the properties of gamma and digamma functions, this result reduces to

$$\pm\left[\frac{\partial}{\partial\nu}P^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n}=\left[\psi\left(\mu+n+\frac{1}{2}\right)-\psi\left(\mu-n+\frac{1}{2}\right)\right]P^{\mu}_{n-1/2}(z)$$
$$+n!\,\Gamma\left(\mu-n+\frac{1}{2}\right)\sum_{k=0}^{n-1}\frac{\left(z^2-1\right)^{(n-k)/2}}{\Gamma\left(\mu+n-2k+\frac{1}{2}\right)k!(n-k)2^{k-n+1}}P^{\mu+n-k}_{k-1/2}(z).$$

Note that by using the recurrence relation for digamma functions (2.16) we establish

$$\psi\left(\mu+n+\frac{1}{2}\right)-\psi\left(\mu-n+\frac{1}{2}\right)=2\mu\sum_{l=1}^{n}\left[\mu^2-\left(l-\frac{1}{2}\right)^2\right]^{-1}.$$

We are now able to compute these derivatives for integer values of ν. For $\nu=0$ there is no contribution from the sum and we have

$$\left[\frac{\partial}{\partial\nu}P^{\mu}_{\nu-1/2}(z)\right]_{\nu=0}=0,$$

which agrees with that given in §4.4.3 of Magnus, Oberhettinger & Soni (1966) [67]. We also have for $\nu=\pm1$

$$\pm\left(\mu^2-\frac{1}{4}\right)\left[\frac{\partial}{\partial\nu}P^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm1}=2\mu P^{\mu}_{1/2}(z)+\left(z^2-1\right)^{1/2}P^{\mu+1}_{-1/2}(z).$$

Note that this method does not seem amenable to computing derivatives with respect to the degree of associated Legendre functions of the form P^{μ}_{ν} evaluated at integer-degrees, since shifting the degree by $+1/2$ in (2.81) converts the modified Bessel function of the second kind to a form like $K_{\nu+1/2}$, and the derivative with respect to order of this Bessel function (see Abramowitz & Stegun (1972) [1], Brychkov & Geddes (2005) [18], Magnus, Oberhettinger & Soni (1966) [67]) is not of a form which is easily integrated.

Parameter derivative formulas from $I_\nu(t)$

By starting with Gradshteyn & Ryzhik (2007) (6.624.5) [48] (see also Prudnikov et al. (1988) (2.15.3.2) [81]) and using the Whipple formulae (2.57), we have for $\operatorname{Re} z > 1$ and $\operatorname{Re}\mu > -\operatorname{Re}\nu - \frac{1}{2}$,

$$\begin{aligned}\int_0^\infty e^{-zt} I_\nu(t) t^{\mu-1/2} dt &= \sqrt{\frac{2}{\pi}} e^{-i\pi\mu} \left(z^2-1\right)^{-\mu/2} Q^\mu_{\nu-1/2}(z) \\ &= \Gamma\left(\mu+\nu+\frac{1}{2}\right) \left(z^2-1\right)^{-\mu/2-1/4} P^{-\nu}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right), \end{aligned} \tag{2.83}$$

where I_ν is a modified Bessel function of the first kind with order ν.

We will use this particular integral representation of associated Legendre functions to compute certain derivatives of the associated Legendre functions with respect to the degree and order. We start with the integral representation of the associated Legendre function of the second kind (2.83), namely

$$Q^\mu_{\nu-1/2}(z) = \sqrt{\frac{\pi}{2}} e^{i\pi\mu} \left(z^2-1\right)^{\mu/2} \int_0^\infty e^{-zt} t^{\mu-1/2} I_\nu(t) dt. \tag{2.84}$$

To justify differentiation under the integral sign in (2.84), with respect to ν, evaluated at $\nu_0 = \pm n$, where $n \in \mathbf{N}$, we use again Proposition 2.6.2. If we fix z and μ, the integrand of (2.84) can be given by the function $f : \mathbf{R} \times (0,\infty) \to \mathbf{C}$ defined by

$$f(\nu, t) := e^{-zt} t^{\mu-1/2} I_\nu(t).$$

We use the following integral representation for the derivative with respect to order of the modified Bessel function of the first kind (see (75) in Apelblat & Kravitsky (1985) [4])

$$\frac{\partial I_\nu(t)}{\partial \nu} = -\nu \int_0^t K_0(t-x) I_\nu(x) x^{-1} dx. \tag{2.85}$$

Let $\delta \in (0,1)$ and $M > 2$. Consider $g : (0,\infty) \to [0,\infty)$ defined by

$$g(t) := M e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu - 1/2} \int_0^t K_0(t-x) I_\delta(x) x^{-1} dx.$$

Using (2.85) we have for all $\nu \in (\delta, M)$

$$\begin{aligned}
\left|\frac{\partial f(\nu,t)}{\partial \nu}\right| &= e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu - 1/2} \left|\frac{\partial I_\nu(t)}{\partial \nu}\right| \\
&= \nu e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu - 1/2} \int_0^t K_0(t-x) I_\nu(x) x^{-1} dx \\
&\leq M e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu - 1/2} \int_0^t K_0(t-x) I_\delta(x) x^{-1} dx \\
&= g(t),
\end{aligned}$$

since for fixed t, $\nu \mapsto I_\nu(t)$ is strictly decreasing. Now we show that $g \in \mathcal{L}^1$. The integral of g over its domain is

$$\int_0^\infty g(t)dt = M \int_0^\infty e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu - 1/2} \int_0^t K_0(t-x) I_\delta(x) x^{-1} dx dt.$$

By making a change of variables in the integral, $(x,t) \mapsto (x,y)$ such that $y = t - x$, yields

$$\int_0^\infty g(t)dt = M \int_0^\infty e^{-y\operatorname{Re} z} K_0(y) \int_0^\infty e^{-x\operatorname{Re} z} (x+y)^{\operatorname{Re}\mu - 1/2} x^{-1} I_\delta(x) dx\ dy.$$

First we show that g is integrable in a neighbourhood of zero. Suppose $\operatorname{Re}\mu - 1/2 < 0$, $x, y \in (0,1]$ and $a \in (0,1)$ then

$$(x+y)^{\operatorname{Re}\mu - 1/2} = (x+y)^{-a}(x+y)^{\operatorname{Re}\mu - 1/2 + a} \leq y^{-a} \max\left(2^{\operatorname{Re}\mu - 1/2 + a}, x^{\operatorname{Re}\mu - 1/2 + a}\right)$$

then since $K_0(y) \sim -\log(y)$ ((9.6.8) in Abramowitz & Stegun (1972) [1]) it follows that

$$\int_0^1 K_0(y) y^{-a} dy < \infty.$$

Furthermore since $I_\delta(x) \sim (x/2)^\delta / \Gamma(\delta+1)$ ((9.6.7) in Abramowitz & Stegun (1972) [1]) it follows that

$$\int_0^1 I_\delta(x) x^{-1} dx < \infty.$$

Now we show that

$$\int_0^1 I_\delta(x) x^{\operatorname{Re}\mu - 1/2 + a - 1} dx < \infty, \tag{2.86}$$

which is convergent if $\mathrm{Re}\,\mu - 1/2 + a + \delta > 0$. If we define

$$\epsilon := \frac{\mathrm{Re}\,\mu + \nu_0 + \frac{1}{2}}{3} > 0,$$

then $\mathrm{Re}\,\mu = -\nu_0 - 1/2 + 3\epsilon$. Therefore if we take $a := 1 - \epsilon$ and $\delta := \nu_0 - \epsilon < \nu_0$ then

$$\mathrm{Re}\mu - \frac{1}{2} + a + \delta = \epsilon > 0,$$

and hence (2.86) is convergent and thus g is integrable near the origin. If $\mathrm{Re}\,\mu - \frac{1}{2} \geq 0$ then similarly g is integrable near the origin.

Now we show that g is integrable. Suppose $\mathrm{Re}\,\mu - 1/2 > 0$. Then

$$(x+y)^{\mathrm{Re}\,\mu - 1/2} \leq [2\max(x,y))]^{\mathrm{Re}\,\mu-1/2} = 2^{\mathrm{Re}\,\mu - 1/2}\max(x^{\mathrm{Re}\,\mu-1/2}, y^{\mathrm{Re}\,\mu-1/2})$$

for all $x, y \geq 0$. For $y \to \infty$ one has $K_\nu(y) \sim \sqrt{\pi/(2y)}e^{-y}$ ((8.0.4) in Olver (1997) [74]). Hence it follows that

$$\int_1^\infty K_0(y)e^{-y\mathrm{Re}\,z}y^{\mathrm{Re}\,\mu-1/2}dy < \infty,$$

and

$$\int_1^\infty K_0(y)e^{-y\mathrm{Re}\,z}dy < \infty.$$

Furthermore since for $x \to \infty$, $I_\delta(x) \sim e^x/\sqrt{2\pi x}$ (p. 83 in Olver (1997) [74]) it follows that

$$\int_1^\infty e^{-x\mathrm{Re}\,z}I_\delta(x)x^{\mathrm{Re}\,\mu-3/2}dx < \infty,$$

and

$$\int_1^\infty e^{-x\mathrm{Re}\,z}I_\delta(x)x^{-1}dx < \infty.$$

If $\mathrm{Re}\,\mu - \frac{1}{2} \leq 0$ then similarly g is integrable.

Therefore g is a $\mathcal{L}^1$-majorant for the derivative with respect to $\nu \neq 0$ of the integral (2.84). It is unclear whether differentiation under the integral sign is also possible for $\nu_0 = 0$. However, we show below that our derived results for derivatives with respect to the degree for associated Legendre functions match up with the to be derived results for degree $\nu = 0$. It is true that relatively little is known about the properties of Bessel functions in relation to operations (differentiation and integration) with respect to their order (cf. Apelblat & Kravitsky (1985) [4]).

Differentiating with respect to the degree ν and evaluating at $\nu = \pm n$, where $n \in \mathbf{N}$, one obtains

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = \sqrt{\frac{\pi}{2}}e^{i\pi\mu}\left(z^2-1\right)^{\mu/2}\int_0^\infty e^{-zt}t^{\mu-1/2}\left[\frac{\partial}{\partial\nu}I_\nu(t)\right]_{\nu=\pm n}dt. \tag{2.87}$$

The derivative of the modified Bessel function of the first kind (2.87) (see Abramowitz & Stegun (1972) [1], Brychkov (2010) [17], Brychkov & Geddes (2005) [18], Magnus, Oberhettinger & Soni (1966) [67]) is given by

$$\left[\frac{\partial}{\partial\nu}I_\nu(t)\right]_{\nu=\pm n} = (-1)^{n+1}K_n(t) \pm n!\sum_{k=0}^{n-1}\frac{(-1)^{k-n}}{k!(n-k)}\frac{t^{k-n}}{2^{k-n+1}}I_k(t) \tag{2.88}$$

(see for instance (1.13.2.1) in Brychkov (2008) [16]).

Inserting (2.88) into (2.87) and using (2.76) and (2.83), we obtain the following general expression for the derivative of the associated Legendre function of the second kind with respect to its degree evaluated at odd-half-integer degrees as

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu-n+\frac{1}{2}\right)\left(z^2-1\right)^{-1/4}Q^{n}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right)$$
$$\pm n!\sum_{k=0}^{n-1}\frac{\left(z^2-1\right)^{(n-k)/2}}{2^{k-n+1}k!(n-k)}Q^{\mu+k-n}_{k-1/2}(z). \tag{2.89}$$

We are now able to compute these derivatives for non-zero integer values of ν. For instance, we have for $\nu=\pm1$

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm 1} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu-\frac{1}{2}\right)\left(z^2-1\right)^{-1/4}Q^{1}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right)$$
$$\pm\left(z^2-1\right)^{1/2}\ Q^{\mu-1}_{-1/2}(z).$$

Note that

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=0} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu+\frac{1}{2}\right)\left(z^2-1\right)^{-1/4}Q_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

by Magnus, Oberhettinger & Soni (1966) [67]. Therefore (2.89) is also valid if $\nu=0$.

Similarly, we can see that this method will not be useful for computing derivatives with respect to the degree of associated Legendre functions of the form Q^{μ}_{ν} evaluated at integer-degrees. Shifting the degree by $+1/2$ in (2.87) converts the modified Bessel function of the first kind to $I_{\nu+1/2}(t)$, and the derivative with respect to order of this Bessel function (see

Abramowitz & Stegun (1972) [1], Brychkov & Geddes (2005) [18], Magnus, Oberhettinger & Soni (1966) [67]) is easily integrable.

Finally, we obtain a formula for the derivative with respect to the order for the associated Legendre function of the first kind evaluated at integer-orders. In order to do this we use the integral expression for the associated Legendre function of the first kind given by (2.83) and the map given in (2.56) to convert to the appropriate argument. Now use the negative-order condition for associated Legendre functions of the first kind (see for example (22) in Cohl et al. (2000) [27]) to convert to a positive order, namely

$$\begin{aligned} P^{\mu}_{\nu-1/2}(z) &= \frac{2}{\pi} e^{-i\mu\pi} \sin(\mu\pi) Q^{\mu}_{\nu-1/2}(z) \\ &\quad + \frac{(z^2-1)^{-\nu/2-1/4}}{\Gamma(\nu-\mu+\frac{1}{2})} \int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) I_\mu(t) t^{\nu-1/2} dt. \end{aligned} \tag{2.90}$$

To justify differentiation under the integral sign in (2.90), with respect to μ, evaluated at $\mu = \pm m$, where $m \in \mathbf{N}$, we use as similar argument as in (2.84) only with modification $\nu \mapsto \mu$ and $n \mapsto m$. The same modified $\mathcal{L}^1$-majorant will work for the derivative of this integrand, since the integral (2.90) converges for $\mathrm{Re}\,\dfrac{z}{\sqrt{z^2-1}} > 1$ and $\mathrm{Re}\,\mu > -\mathrm{Re}\,\nu - \frac{1}{2}$. Since we were unable to justify differentiation under the integral for $\nu = 0$ before, the case for differentiation under the integral (2.90) with respect to μ evaluated at $\mu = 0$ remains open. However, below we show that our derived results for derivatives with respect to the order for associated Legendre functions match up to previously established results for order $\mu = 0$.

Differentiating both sides of the resulting expression with respect to the order μ and evaluating at $\mu = \pm m$, where $m \in \mathbf{N}$ yields

$$\begin{aligned} \left[\frac{\partial}{\partial\mu} P^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} &= 2 Q^{\pm m}_{\nu-1/2}(z) \\ &\quad + \left(z^2-1\right)^{-\nu/2-1/4} \left\{\frac{\partial}{\partial\mu}\left[\Gamma\left(\nu-\mu+\frac{1}{2}\right)\right]^{-1}\right\}_{\mu=\pm m} \\ &\quad \times \int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) I_{\pm m}(t) t^{\nu-1/2} dt \\ &\quad + \frac{\left(z^2-1\right)^{-\nu/2-1/4}}{\Gamma\left(\nu \mp m + \frac{1}{2}\right)} \\ &\quad \times \int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} \left[\frac{\partial}{\partial\mu} I_\mu(t)\right]_{\mu=\pm m} dt. \end{aligned}$$

The derivative of the reciprocal of the gamma function reduces to

$$\left\{\frac{\partial}{\partial\mu}\left[\Gamma\left(\nu-\mu+\frac{1}{2}\right)\right]^{-1}\right\}_{\mu=\pm m}=\frac{\psi\left(\nu\mp m+\frac{1}{2}\right)}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)}.$$

The derivative with respect to order for the modified Bessel function of the first kind is given in (2.88). The integrals are easily obtained by applying the map given by (2.56) as necessary to (2.76) and (2.83). Hence by also using standard properties of associated Legendre, gamma, and digamma functions we obtain the following compact form

$$\frac{\Gamma(\nu\mp m+\frac{1}{2})}{\Gamma(\nu-m+\frac{1}{2})}\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m}=Q^{m}_{\nu-1/2}(z)+\psi\left(\nu\mp m+\frac{1}{2}\right)P^{m}_{\nu-1/2}(z)$$
$$\pm\, m!\sum_{k=0}^{m-1}\frac{(-1)^{k-m}\left(z^2-1\right)^{(k-m)/2}}{2^{k-m+1}k!(m-k)}P^{k}_{\nu+k-m-1/2}(z). \tag{2.91}$$

We are now able to obtain formulas for integer values of μ. For instance, for $\mu=-1$ we have

$$\left(\nu^2-\frac{1}{4}\right)\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=-1}=Q^{1}_{\nu-1/2}(z)+\psi\left(\nu+\frac{3}{2}\right)P^{1}_{\nu-1/2}(z)+\left(z^2-1\right)^{-1/2}P_{\nu-3/2}(z),$$

and for $\mu=+1$ we have

$$\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=1}=Q^{1}_{\nu-1/2}(z)+\psi\left(\nu-\frac{1}{2}\right)P^{1}_{\nu-1/2}(z)-\left(z^2-1\right)^{-1/2}P_{\nu-3/2}(z).$$

Note that

$$\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=0}=Q_{\nu-1/2}(z)+\psi\left(\nu+\frac{1}{2}\right)P_{\nu-1/2}(z),$$

by §4.4.3 of Magnus, Oberhettinger & Soni (1966) [67]. So (2.91) is also valid if $\mu=0$.

2.7 Orthogonal polynomials

Jacobi functions of both kinds are general with 3 complex parameters as well as a complex argument. Associated Legendre functions can be written in terms of Jacobi functions (see §7.4.3 in Vilenkin & Klimyk (1991) [101]), and these are both special cases of the Gauss hypergeometric function. We will restrict our attention mostly to associated Legendre functions. However, Jacobi polynomials, a restriction of Jacobi functions play an important role the study of hyperspherical harmonics in higher dimensions (see Chapter 6).

The Jacobi polynomials are a general class of orthogonal polynomials which are given in terms of a finite Gauss hypergeometric series

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right), \tag{2.92}$$

where $n \in \mathbf{N}_0$, $\alpha, \beta \in \mathbf{C}$ and $x \in [-1,1]$. Gegenbauer polynomials are specific cases of the Jacobi polynomials (see (6.4.9) in Andrews, Askey & Roy (1999) [3])

$$C_n^\lambda(x) = \frac{\Gamma(2\lambda+n)\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda)\Gamma\left(\lambda+n+\frac{1}{2}\right)} P_n^{(\lambda-1/2,\lambda-1/2)}(x), \tag{2.93}$$

where $\lambda \in \mathbf{C}$. One useful special value for the Gegenbauer polynomials is

$$C_n^\lambda(1) = \binom{2\lambda+n-1}{n} \tag{2.94}$$

(see (8.937.4) in Gradshteyn & Ryzhik (2007) [48]). The generating function for Gegenbauer polynomials is given by

$$\frac{1}{(1+z^2-2xz)^\lambda} = \sum_{n=0}^\infty C_n^\lambda(x) z^n, \tag{2.95}$$

where $|z| < 1$ (see for instance, p. 222 in Magnus, Oberhettinger & Soni (1966) [67]). Chebyshev polynomials of the first kind are also given as special cases of Jacobi and Gegenbauer polynomials (see (22.5.23) in Abramowitz & Stegun (1972) [1])

$$T_n(x) = \frac{4^n}{\binom{2n}{n}} P_n^{(-1/2,-1/2)}(x).$$

The Chebyshev polynomials of the first kind can also be written concisely as follows (see (22.3.15) in Abramowitz & Stegun (1972) [1])

$$T_n(\cos\psi) = \cos(n\psi).$$

The generating function for Chebyshev polynomials of the first kind (Magnus, Oberhettinger & Soni (1966) [67] and Fox & Parker (1968) [41], p. 51) is given as

$$\frac{1-z^2}{1+z^2-2xz} = \sum_{n=0}^\infty \epsilon_n T_n(x) z^n, \tag{2.96}$$

where $|z| < 1$ and $\epsilon_n := 2 - \delta_{n,0}$ is the Neumann factor (see p. 744 in Morse & Feshbach (1953) [71]), commonly-occurring in Fourier cosine series, with $\delta_{i,j}$ being the Kronecker delta symbol

$$\delta_{i,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for $i, j \in \mathbf{Z}$. Associated Legendre functions $P_l^m(x)$ for $l \in \mathbf{N}_0$ and $m \in \{-l, \ldots, l\}$ are expressible in terms of Gegenbauer polynomials and therefore also in terms of Jacobi polynomials

$$P_l^m(x) = (-1)^m (2m-1)!!(1-x^2)^{m/2} C_{l-m}^{m+1/2}(x) \tag{2.97}$$

(see (8.936.2) in Gradshteyn & Ryzhik (2007) [48]).

3

Normalized fundamental solutions for powers of the Laplacian in R^d

In this chapter we review some literature concerning normalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$. The classical results reviewed in this chapter will be necessary material for the rest of this thesis. One may gather further fundamental background material from Gilbarg & Trudinger (1983) [47], Folland (1976) [40], Friedman (1969) [42] and Boyling (1996) [14].

We begin this chapter by introducing the Laplace equation, Poisson's equation, and integral solutions of Poisson's equation through a normalized fundamental solution for the Laplacian. Then we discuss some known results relating to integral solutions of the inhomogeneous polyharmonic equation, and conclude this chapter by presenting a theorem which gives a closed-form expression for normalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$.

3.1 Normalized fundamental solution for the Laplacian

If Φ satisfies Laplace's equation, in Euclidean space $\mathbf{R}^d$, given by

$$-\Delta\Phi(\mathbf{x}) = 0,$$

where $d \in \mathbf{N}$, $\mathbf{x} \in \mathbf{R}^d$, and Δ is the Laplacian operator defined by

$$\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2},$$

then Φ is called a *harmonic function.* By Euclidean space $\mathbf{R}^d$, we mean the normed vector space given by the pair $(\mathbf{R}^d, \|\cdot\|)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^d$ defined by

$$\|\mathbf{x}\| := \sqrt{x_1^2 + \cdots + x_d^2}.$$

Next, Poisson's equation (the inhomogeneous Laplace equation) is given by

$$-\Delta\Phi(\mathbf{x}) = \rho(\mathbf{x}), \tag{3.1}$$

where $\rho : \mathbf{R}^d \to \mathbf{R}$ integrable, or even more generally $\rho \in (\mathcal{D}(\mathbf{R}^d))'$, where $\mathcal{D}(\mathbf{R}^d)$ is the space of smooth compactly supported functions on $\mathbf{R}^d$ with a suitable notion of convergence which makes it into a complete locally convex topological vector space (see §2.1 in Hörmander (2003) [52]).

A *fundamental solution* for the Laplacian in $\mathbf{R}^d$ is a function $\mathfrak{g}_1^d$ which satisfies the equation

$$-\Delta\mathfrak{g}_1^d(\mathbf{x}, \mathbf{x}') = c\delta(\mathbf{x} - \mathbf{x}'),$$

where $\mathbf{x}' \in \mathbf{R}^d$, δ is the d-dimensional Dirac delta function (see for instance p. 90 in John (1982) [61]), and $c \in \mathbf{R}, c \neq 0$. If $c \neq 1$ then we call a fundamental solution of the Laplacian in $\mathbf{R}^d$ *unnormalized.* A *normalized* fundamental solution $\mathcal{G}_1^d$ for the Laplacian satisfies the above equation, with $c = 1$, namely

$$-\Delta\mathcal{G}_1^d(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \tag{3.2}$$

In Euclidean space $\mathbf{R}^d$, a normalized Green's function for Laplace's equation (normalized fundamental solution for the Laplacian) is well-known and is given in the following theorem (see Folland (1976) [40], p. 94, Gilbarg & Trudinger (1983) [47], p. 17, Bers et al. (1964) [11], p. 211).

Theorem 3.1.1. *Let* $d \in \mathbf{N}$. *Define*

$$\mathcal{G}_1^d(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}\|\mathbf{x}-\mathbf{x}'\|^{2-d} & \text{if } d=1 \text{ or } d\geq 3, \\ \dfrac{1}{2\pi}\log\|\mathbf{x}-\mathbf{x}'\|^{-1} & \text{if } d=2, \end{cases}$$

then $\mathcal{G}_1^d$ *is a normalized fundamental solution for* $-\Delta$ *in Euclidean space* $\mathbf{R}^d$, *where* Δ *is the Laplace operator in* $\mathbf{R}^d$.

Note most authors only present the above theorem for the case $d \geq 2$ but it is easily-verified to also be valid for the case $d = 1$ as well. By expressing the gamma function in terms of the factorial and double factorial functions we can re-write $\mathcal{G}_1^d$ as follows

$$\mathcal{G}_1^d(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{(d-4)!!}{2^{(d+1)/2}\pi^{(d-1)/2}}\|\mathbf{x}-\mathbf{x}'\|^{2-d} & \text{if } d=1 \text{ or } d\geq 3 \quad \text{odd}, \\ \dfrac{1}{2\pi}\log\|\mathbf{x}-\mathbf{x}'\|^{-1} & \text{if } d=2, \\ \dfrac{(d/2-2)!}{4\pi^{d/2}}\|\mathbf{x}-\mathbf{x}'\|^{2-d} & \text{if } d\geq 4 \quad \text{even}. \end{cases}$$

It is clear that in general $\mathcal{G}_1^d$ is not unique since one can add any harmonic function $h : \mathbf{R}^d \to \mathbf{R}$ to $\mathcal{G}_1^d$ and still obtain a solution to (3.2) since h is in the kernel of $-\Delta$.

Proposition 3.1.2. *There exists precisely one* C^∞*-function* $G : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ *such that for all* $\mathbf{x}' \in \mathbf{R}^d$ *the function* $G_{\mathbf{x}'} : \mathbf{R}^d \setminus \{\mathbf{x}'\} \to \mathbf{R}$ *defined by* $G_{\mathbf{x}'}(\mathbf{x}) = G(\mathbf{x},\mathbf{x}')$ *is a distribution on* $\mathbf{R}^d$ *with*

$$-\Delta G_{\mathbf{x}'} = \delta(\cdot - \mathbf{x}')$$

and

$$\lim_{\|\mathbf{x}\|\to\infty} G_{\mathbf{x}'}(\mathbf{x}) = 0. \tag{3.3}$$

Proof. Existence: clear. Uniqueness. Suppose G and $\tilde{G}$ are two such functions. Let $\mathbf{x}' \in \mathbf{R}^d$. Define the C^∞-function $h : \mathbf{R}^d \setminus \{\mathbf{x}'\} \to \mathbf{R}$ by $h = G_{\mathbf{x}'} - \tilde{G}_{\mathbf{x}'}$. Then h is a distribution on $\mathbf{R}^d$ with $-\Delta h = 0$. By elliptic regularity h can be extended to a C^∞-function $\hat{h} : \mathbf{R}^d \to \mathbf{R}$. It follows from (3.3) for G and $\tilde{G}$ that $\lim_{\|\mathbf{x}\|\to\infty} \hat{h}(\mathbf{x}) = 0$. Hence $\hat{h}$ is bounded. But every bounded harmonic function on $\mathbf{R}^d$ is constant. So $\hat{h}$ is constant. Since $\lim_{\|\mathbf{x}\|\to\infty} \hat{h}(\mathbf{x}) = 0$, this constant vanishes. Therefore $h = 0$ and $G(\mathbf{x},\mathbf{x}') = \tilde{G}(\mathbf{x},\mathbf{x}')$ for all $\mathbf{x} \in \mathbf{R}^d \setminus \{\mathbf{x}'\}$.

We wish to consider solutions $\Phi : \mathbf{R}^d \to \mathbf{R}$ to Poisson's equation (3.1) given an integrable function $\rho : \mathbf{R}^d \to \mathbf{R}$. An integral solution of Poisson's equation is given by

$$\Phi(\mathbf{x}) = \int_{\mathbf{R}^d} \mathcal{G}_1^d(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}'. \tag{3.4}$$

This is called the Newtonian potential Φ with density ρ (see Gilbarg & Trudinger (1983) [47], p. 51).

By Proposition 3.1.2 the function $\mathcal{G}_1^d$ is the unique normalized fundamental solution which satisfies the decay (3.3), if $d \geq 3$.

3.2 Normalized fundamental solutions for powers of the Laplacian

If Φ satisfies the *polyharmonic equation* given by

$$(-\Delta)^k \Phi(\mathbf{x}) = 0, \tag{3.5}$$

where $k \in \mathbf{N}$ and $\Phi \in C^{2k}(\mathbf{R}^d)$ then Φ is called *polyharmonic*. If the power k of the Laplacian equals two, then (3.5) is called the *biharmonic equation* and Φ is called *biharmonic*. The *inhomogeneous polyharmonic equation* is given by

$$(-\Delta)^k \Phi(\mathbf{x}) = \rho(\mathbf{x}). \tag{3.6}$$

We would also like to take ρ to be an integrable function so that a solution to (3.6) exists. A *fundamental solution* for the polyharmonic equation in $\mathbf{R}^d$ is a function $\mathfrak{g}_k^d$ which satisfies the equation

$$(-\Delta)^k \mathfrak{g}_k^d(\mathbf{x}, \mathbf{x}') = c\delta(\mathbf{x} - \mathbf{x}'), \tag{3.7}$$

where $c \in \mathbf{R}, c \neq 0$. If $c \neq 1$ then we call a fundamental solution of the polyharmonic equation in $\mathbf{R}^d$ *unnormalized*. A *normalized* fundamental solution $\mathcal{G}_k^d$ for the Laplacian satisfies the above equation, with $c = 1$, namely

$$(-\Delta)^k \mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}').$$

Proposition 3.2.1. *Let $k \in \mathbf{N}$. There exists precisely one C^∞-function $G : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ such that for all $\mathbf{x}' \in \mathbf{R}^d$ the function $G_{\mathbf{x}'} : \mathbf{R}^d \setminus \{\mathbf{x}'\} \to \mathbf{R}$ defined by $G_{\mathbf{x}'}(\mathbf{x}) = G(\mathbf{x},\mathbf{x}')$ is a distribution on $\mathbf{R}^d$ with*

$$(-\Delta)^k G_{\mathbf{x}'} = \delta(\cdot - \mathbf{x}'),$$

and

$$\lim_{\|\mathbf{x}\|\to\infty} (-\Delta)^j G_{\mathbf{x}'}(\mathbf{x}) = 0, \tag{3.8}$$

for all $j \in \{0, \ldots, k-1\}$.

Proof. This follows by induction, similarly to the proof of Proposition 3.1.2.

We would like to construct a sequence of fundamental solutions such that

$$-\Delta \mathcal{G}^d_{k+1}(\mathbf{x},\mathbf{x}') = \mathcal{G}^d_k(\mathbf{x},\mathbf{x}'). \tag{3.9}$$

Standard references which refer to the form of a normalized fundamental solution for powers of the Laplacian include Aronszajn et al. (1983) ([6], p. 8), Boyling (1996) [14], Friedman (1969) ([42], p. 5), John (1955) ([60], p. 44), and Schwartz (1950) ([87], p. 45). A fundamental solution of the polyharmonic equation in $\mathbf{R}^d$ is given in terms of the Euclidean distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$. A straightforward computation verifies that the formula for the Laplacian acting on a power of the Euclidean distance between two points is given by

$$-\Delta \|\mathbf{x}-\mathbf{x}'\|^\nu = -\nu(d+\nu-2)\|\mathbf{x}-\mathbf{x}'\|^{\nu-2}, \tag{3.10}$$

and the formula for the Laplacian acting on a power of the Euclidean distance between two points multiplied by the logarithm of the Euclidean distance between two points is given by

$$-\Delta \|\mathbf{x}-\mathbf{x}'\|^\nu \log\|\mathbf{x}-\mathbf{x}'\| = -\|\mathbf{x}-\mathbf{x}'\|^{\nu-2}\left[\nu(d+\nu-2)\log\|\mathbf{x}-\mathbf{x}'\| + 2\nu + d - 2\right]. \tag{3.11}$$

These two expressions are very important for deriving fundamental solutions for the Laplacian in $\mathbf{R}^d$.

3.2.1 Integral solutions for arbitrary natural powers of the Laplacian

In what follows we will utilize Green's second identity, namely (see for instance p. 17 in Gilbarg & Trudinger (1983) [47])

$$\int_V \phi(-\Delta'\psi)d\mathbf{x}' = \int_V \psi(-\Delta'\phi)d\mathbf{x}' + \oint_{\partial V} \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS', \tag{3.12}$$

valid for all $\phi, \psi \in C^2(\bar{V})$, where we denote the Laplacian with respect to the primed coordinates by Δ', V is a bounded connected region in $\mathbf{R}^d$ with C^1 boundary ∂V, $\partial/\partial n$ is the normal derivative at the boundary, $d\mathbf{x}'$ is the infinitesimal 'volume' element, and dS' is the infinitesimal 'surface area' element. Note that when Green's second identity (3.12) is used, all derivatives with respect to variables under the integral sign are to be taken with respect to the primed variable.

Let $\rho \in C_c^2(\mathbf{R}^d)$. We use (3.12) with $V = B_s(\mathbf{x}) \setminus \bar{B}_r(\mathbf{x})$ with s large and r small, where $B_r(\mathbf{x}) = \{x \in \mathbf{R}^d : \|x\| < r\}$. In the limit $r \downarrow 0$ the surface terms originating from $\partial B_r(\mathbf{x})$ vanish (see for example p. 18 in Gilbarg & Trudinger (1983) [47]). Since $\rho \in C_c^2(\mathbf{R}^d)$ an integral solution of the inhomogeneous polyharmonic equation (3.6) is given by

$$\Phi(\mathbf{x}) = \int_{\mathbf{R}^d} \mathcal{G}_k^d(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}'. \tag{3.13}$$

We can repeatedly apply Green's second identity (3.12) to an integral solution of the inhomogeneous polyharmonic equation (3.13) to derive alternative forms of an integral solution for the inhomogeneous polyharmonic equation. For instance, for all $p \in \mathbf{N}$ and $\rho \in C_c^2(\mathbf{R}^d)$, one can verify that an integral solution of Poisson's equation, (3.6) with $k = 1$, is given by

$$\Phi_1(\mathbf{x}) = \int_{\mathbf{R}^d} ((-\Delta')^p \rho(\mathbf{x}'))\, \mathcal{G}_{1+p}^d(\mathbf{x}, \mathbf{x}')d\mathbf{x}'.$$

In exactly the same fashion, one can verify that an integral solution of the inhomogeneous biharmonic equation, (3.6) with $k = 2$, is given by

$$\Phi_2(\mathbf{x}) = \int_{\mathbf{R}^d} ((-\Delta')^p \rho(\mathbf{x}'))\, \mathcal{G}_{2+p}^d(\mathbf{x}, \mathbf{x}')d\mathbf{x}'.$$

And more generally, an integral solution of the inhomogeneous polyharmonic equation (3.6), is given by

$$\Phi_k(\mathbf{x}) = \int_{\mathbf{R}^d} ((-\Delta')^p \rho(\mathbf{x}'))\, \mathcal{G}_{k+p}^d(\mathbf{x}, \mathbf{x}')d\mathbf{x}'.$$

3.2.2 Normalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^3$ (and $\mathbf{R}$)

One method for computing a Green's function for higher-harmonic equations is to use appropriate powers of the Euclidean distance between two points and repeatedly integrating by parts (twice) by using Green's second identity (3.12) as applied to an integral solution of the inhomogeneous polyharmonic equation (3.13). In the following example we demonstrate how to compute a closed-form expression for a Green's function of the biharmonic equation and for higher-harmonic equations in $\mathbf{R}^3$. An analogous result is presented in $\mathbf{R}$. This example is given to illustrate how a fundamental solution of the polyharmonic equation may be generated in $\mathbf{R}^d$.

In $d = 3$ the Euclidean distance between two points is given by

$$\|\mathbf{x} - \mathbf{x}'\| = \left((x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\right)^{1/2}, \tag{3.14}$$

and using (3.10) we obtain

$$-\Delta\|\mathbf{x} - \mathbf{x}'\| = -2\|\mathbf{x} - \mathbf{x}'\|^{-1}. \tag{3.15}$$

A fundamental solution for the Laplacian in $\mathbf{R}^3$ is given by

$$\mathcal{G}_1^3(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}'\|},$$

then through (3.4), the Newtonian potential in $\mathbf{R}^3$ with integrable density ρ is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\rho(\mathbf{x}')}{\|\mathbf{x} - \mathbf{x}'\|} d\mathbf{x}'. \tag{3.16}$$

If $\rho \in C_c^2(\mathbf{R}^3)$ and we apply Green's second identity (3.12) to (3.16), and use (3.15) then

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi}\int_{\mathbf{R}^3} \rho(\mathbf{x}')\frac{1}{2}\Delta'\|\mathbf{x} - \mathbf{x}'\|\, d\mathbf{x}' \\
&= \frac{-1}{8\pi}\int_{\mathbf{R}^3} \rho(\mathbf{x}')(-\Delta'\|\mathbf{x} - \mathbf{x}'\|)\, d\mathbf{x}' \\
&= \frac{-1}{8\pi}\int_{\mathbf{R}^3} (-\Delta'\rho(\mathbf{x}'))\, \|\mathbf{x} - \mathbf{x}'\|\, d\mathbf{x}'.
\end{aligned}$$

This yields an alternative solution to Poisson's equation.

The above procedure produced an algebraic function such that when the Laplacian acts twice upon it, it produces a Dirac delta function. Through (3.15) we have

$$-\Delta\frac{1}{8\pi}\|\mathbf{x} - \mathbf{x}'\| = \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}'\|} = \mathcal{G}_1^3(\mathbf{x}, \mathbf{x}').$$

However, through (3.9) we know that

$$-\Delta\mathcal{G}_2^3(\mathbf{x},\mathbf{x}') = \mathcal{G}_1^3(\mathbf{x},\mathbf{x}').$$

Therefore a Green's function for the biharmonic equation is given by

$$\mathcal{G}_2^3(\mathbf{x},\mathbf{x}') = \frac{-1}{8\pi}\|\mathbf{x}-\mathbf{x}'\|.$$

Applying the above procedure repeatedly, yields for general natural powers of the Laplacian (3.10),

$$-\Delta\|\mathbf{x}-\mathbf{x}'\|^{2k-1} = -(2k-1)(2k)\|\mathbf{x}-\mathbf{x}'\|^{2k-3},$$

for all $k \in \mathbf{N}$. Repeated use of Green's second identity and requirements at least that $\rho \in C_c^{2k-2}(\mathbf{R}^3)$, for $k \geq 2$ yields alternative descriptions of a solution to Poisson's equation, namely

$$\Phi(\mathbf{x}) = \frac{(-1)^{k+1}}{4(2k-2)!\,\pi}\int_{\mathbf{R}^3}\left((-\Delta')^{k-1}\rho(\mathbf{x}')\right)\ \|\mathbf{x}-\mathbf{x}'\|^{2k-3}\,d\mathbf{x}'.$$

Our iterative procedure has produced an algebraic function such that when the Laplacian acts k times upon it, it produces a Dirac delta function. Therefore, we can now see that a normalized fundamental solution for the polyharmonic equation in $\mathbf{R}^3$ is given by

$$\mathcal{G}_k^3(\mathbf{x},\mathbf{x}') = \frac{(-1)^{k+1}\|\mathbf{x}-\mathbf{x}'\|^{2k-3}}{4\pi(2k-2)!}.$$

A similar procedure produces a normalized fundamental solution for natural powers of the Laplacian in $\mathbf{R}$. This is given by

$$\mathcal{G}_k^1(x,x') = \frac{(-1)^k|x-x'|^{2k-1}}{2(2k-1)!},$$

where the Euclidean norm $\|\cdot\|$ on $\mathbf{R}$ coincides with the absolute value function $|\cdot|$ on $\mathbf{R}$.

3.2.3 Logarithmic fundamental solutions for d even and $k \geq d/2$

In even dimensions d for powers of the Laplacian k greater than or equal to the dimension divided by two, a normalized fundamental solution seriously changes character. It is has logarithmic behaviour. The point is that there is no power-law function which when twice differentiated gives a reciprocal square. A fundamental solution of the Laplacian for all even dimensions greater than or equal to four is proportional to $\|\mathbf{x}-\mathbf{x}'\|^{2-d}$. For instance in $d=4$, a fundamental solution of the Laplacian is proportional to $\|\mathbf{x}-\mathbf{x}'\|^{-2}$. The only function which when twice differentiated is a reciprocal square is one which is logarithmic.

In order to treat this behaviour we look for a sequence of functions $\Theta_d^p(r)$, where $r := \|\mathbf{x} - \mathbf{x}'\|$, $\mathbf{x} \neq \mathbf{x}'$, whose first term in the sequence is given by

$$\Theta_d^0(r) := \log r.$$

The terms in the sequence satisfy the following recursion relation

$$-\Delta\Theta_d^{p+1}(r) = \Theta_d^p(r), \tag{3.17}$$

(cf. (3.9) and (3.11)). It is verified that the general form for such a sequence of functions is given by

$$\Theta_d^p(r) = \alpha_p(d) r^{2p} \left(\log r - \beta_p(d)\right), \tag{3.18}$$

where $\alpha_p, \beta_p : \{2, 4, \ldots\} \to \mathbf{R}$ and $p \in \mathbf{N}_0$. By inserting the general form for the solution (3.18) into (3.17), we obtain recursive formulae for α_p such that

$$\alpha_p(d) = \frac{-1}{2p(2p + d - 2)} \alpha_{p-1}(d),$$

and

$$\beta_p(d) = \beta_{p-1}(d) + \frac{4p + d - 2}{2p(2p + d - 2)}. \tag{3.19}$$

Now clearly since $\alpha_0 := 1$ and $\beta_0 := 0$ we obtain

$$\alpha_p(d) = \frac{(-1)^p (d - 2)!!}{2^p p! (d + 2p - 2)!!},$$

and

$$\beta_p(d) = \frac{1}{2} \sum_{i=1}^{p} \frac{d + 4i - 2}{i(d + 2i - 2)}. \tag{3.20}$$

Using partial fraction decomposition we can re-express β_p in terms of the H_j, the jth harmonic number

$$H_j := \sum_{i=1}^{j} \frac{1}{i},$$

as follows

$$\beta_p(d) = \frac{1}{2} \left[H_p + H_{d/2+p-1} - H_{d/2-1}\right]. \tag{3.21}$$

By getting a common denominator for (3.20) we see that one can extract a common term given by $(d-2)!!/(d+2p-2)!!$. Verification that $\beta_p(d)$ is divisible by $(d+2p)$ can be seen by noticing that it vanishes for $d = -2p$. Our sequence of functions is seen to behave as follows for $p \in \{1, 2, 3, 4\}$:

$$\Theta_d^1(r) = \frac{-r^2}{2d}\left(\log r - \frac{d+2}{2d}\right),$$

$$\Theta_d^2(r) = \frac{r^4}{8d(d+2)}\left(\log r - \frac{(d+4)(3d+2)}{4d(d+2)}\right),$$

$$\Theta_d^3(r) = \frac{-r^6}{48d(d+2)(d+4)}\left(\log r - \frac{(d+6)(11d^2+36d+16)}{12d(d+2)(d+4)}\right),$$

and

$$\Theta_d^4(r) = \frac{r^8}{384d(d+2)(d+4)(d+6)}\left(\log r - \frac{(d+8)(25d^3+196d^2+396d+144)}{24d(d+2)(d+4)(d+6)}\right).$$

One may compare the form of $\beta_p(d)$ used in the theorems which are presented in the next section to those given in Boyling (1996) [14]. Boyling has also computed a sequence of logarithmic functions given in this section subject to the recursion on the Laplacian given by (3.17). In order to make this recursion work, one has the freedom to replace an arbitrary function $f(d)$ with the first two terms in (3.21). We have chosen our function f in such a way that $\beta_0(d)$ vanishes for all even positive d, whereas Boyling has chosen his f such that $\beta_0(d)$ vanishes only for $d = 2$.

3.2.4 Normalized fundamental solution of the polyharmonic equation

We summarize the previous results.

Theorem 3.2.2. *Let $d, k \in \mathbf{N}$. Define*

$$\mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') = \begin{cases} \dfrac{(-1)^{k+d/2+1}\ \|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)!\ (k-d/2)!\ 2^{2k-1}\pi^{d/2}}\left(\log\|\mathbf{x}-\mathbf{x}'\| - \beta_{k-d/2}\right) & \text{if } d \text{ even},\ k \geq d/2, \\ \\ \dfrac{\Gamma(d/2-k)\|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)!\ 2^{2k}\pi^{d/2}} & \text{otherwise,} \end{cases}$$

then $\mathcal{G}_k^d$ is a normalized fundamental solution for $(-\Delta)^k$, where Δ is the Laplace operator in $\mathbf{R}^d$.

Theorem 3.2.3. *Let $d, k \in \mathbf{N}$. Define*

$$\mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') = \begin{cases} \dfrac{(-1)^{k+(d-1)/2} \ \|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)! \ (2k-d)!! \ 2^k (2\pi)^{(d-1)/2}} & \text{if } d \text{ odd} \geq 1, \quad k \geq \dfrac{d+1}{2}, \\ \dfrac{(-1)^{k+d/2+1} \ \|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)! \ (k-d/2)! \ 2^{2k-1}\pi^{d/2}} \left(\log \|\mathbf{x}-\mathbf{x}'\| - \beta_{k-d/2}\right) & \text{if } d \text{ even} \geq 2, \ k \geq \dfrac{d}{2}, \\ \dfrac{(d-2k-2)!! \ \|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)! \ 2^k (2\pi)^{(d-1)/2}} & \text{if } d \text{ odd} \geq 3, \quad k \leq \dfrac{d-1}{2}, \\ \dfrac{(d/2-k-1)! \ \|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)! \ 2^{2k}\pi^{d/2}} & \text{if } d \text{ even} \geq 4, \ k \leq \dfrac{d}{2}-1, \end{cases}$$

then $\mathcal{G}_k^d$ is a normalized fundamental solution for $(-\Delta)^k$, where Δ is the Laplace operator in $\mathbf{R}^d$.

By Proposition 3.2.1 the function $\mathcal{G}_k^d$ is the unique normalized fundamental solution which satisfies the decay (3.8), if $d \geq 3$ and $k \leq (d-1)/2$.

4

Separable subgroup type coordinates for Laplace's equation in R^d

In this chapter we review results in the literature concerning subgroup type coordinate systems, and their specific properties, in $\mathbf{R}^d$. The results reviewed in this chapter will be necessary material for the rest of this thesis. Related fundamental background material is in Vilenkin (1968) [100], Kalnins (1986) [62] and Izmest′ev et al. (2001) [59].

In particular, we review subgroup type coordinate systems in $\mathbf{R}^d$. Subgroup type coordinate systems in $\mathbf{R}^d$ are coordinate system which can be described by a chain of subgroups of the Euclidean group $E(d)$. We describe how certain subgroup type coordinate systems in $\mathbf{R}^d$ can be broken into two categories, pure hyperspherical coordinates and mixed Euclidean-hyperspherical coordinates.

Pure hyperspherical coordinates are a specific subgroup type coordinate system for $\mathbf{R}^d$ which is described by a radial coordinate $r \in [0, \infty)$ plus $d-1$ angles which together parametrize points on a $d-1$-dimensional hypersphere. A mixed Euclidean-hyperspherical coordinate system is a subgroup type coordinate system for the k-dimensional subspace $\mathbf{R}^k$ of $\mathbf{R}^d$. This type of coordinate system is given by k-dimensional pure hyperspherical coordinate system ($k \leq d$) with the remaining $d-k$ coordinates being Cartesian in nature.

Pure hyperspherical coordinate systems are those mixed-Euclidean hyperspherical coordinate systems in which $d = k$. We give a general procedure for constructing pure hyperspherical coordinate systems called a "method of trees." We conclude the chapter by describing several examples of these subgroup type coordinates, and in particular we describe what we refer to as standard hyperspherical coordinates, a pure hyperspherical coordinate system which is a generalization of spherical coordinates in $\mathbf{R}^3$ to arbitrary dimension.

The only coordinate systems we will describe in this thesis are those which can be obtained through the method of separation of variables. We refer to these coordinate systems as separable coordinates. We will ignore those coordinate systems which can be obtained in other ways. It is important to note that there clearly exist separable coordinate systems which are not of subgroup type, such as ellipsoidal coordinates (see for instance Kalnins (1986) [62]). *For the sake of simplicity in our discussion, we will not refer to non-subgroup type coordinate systems unless specifically mentioned.* When we refer to general coordinate systems we mean those particular subgroup type coordinate systems which satisfy a certain quality. For instance, if we refer to general hyperspherical (or general pure hyperspherical) coordinate systems, we mean all those coordinate systems which are of pure hyperspherical subgroup type in $\mathbf{R}^d$.

4.1 Subgroup type coordinates

Subgroup type coordinates are coordinates are obtained when we take a chain of subgroups

$$G_1 \supset G_2 \supset \cdots G_k.$$

A subgroup chain is defined such that for each chain link

$$G_k \supset G_{k+1},$$

G_{k+1} is a maximal subgroup of G_k, i.e. G_{k+1} is a maximal subgroup of G_k if $G_{k+1} \neq G_k$ and there does not exist a subgroup $K \subset G$ such that $G_k \supset K \supset G_{k+1}$. A chain of such links is referred to as a subgroup chain.

4.1.1 Subgroup type coordinates in $\mathbf{R}^d$

The class of coordinate systems which allow separation of variables for Laplace's equation in Euclidean space $\mathbf{R}^d$ (see §3.6 in Miller (1977) [69], Moon & Spencer (1988) [70]), can be broken up into two types, those which are subgroup type coordinates and those which are not. Take for instance the proper subgroups of the Euclidean $E(2)$

$$E(2) \supset O(2), \quad \text{and,} \tag{4.1}$$

$$E(2) \supset E(1) \otimes E(1), \tag{4.2}$$

or for instance the proper subgroups of $E(3)$

$$E(3) \supset O(3) \supset O(2), \tag{4.3}$$

$$E(3) \supset E(2) \otimes E(1) \supset O(2) \otimes E(1), \quad \text{and,} \tag{4.4}$$

$$E(3) \supset E(2) \otimes E(1) \supset E(1) \otimes E(1), \tag{4.5}$$

or for instance the proper subgroups of $E(4)$

$$E(4) \supset O(4) \supset O(3) \supset O(2), \tag{4.6}$$

$$E(4) \supset O(4) \supset O(2) \otimes O(2), \tag{4.7}$$

$$E(4) \supset E(3) \otimes E(1) \supset O(3) \supset O(2), \tag{4.8}$$

$$E(4) \supset E(2) \otimes E(2) \supset O(2) \otimes O(2), \tag{4.9}$$

$$E(4) \supset E(2) \otimes E(2) \supset O(2) \otimes E(1) \otimes E(1), \quad \text{and,} \tag{4.10}$$

$$E(4) \supset E(2) \otimes E(2) \supset E(1) \otimes E(1) \otimes E(1) \otimes E(1). \tag{4.11}$$

Possible subgroup chain links for the Euclidean group (often referred to as the Euclidean motion group) are as follows. Subgroups are either of Euclidean group type or Orthogonal group type. For subgroups of the Euclidean group, the subgroup links can be

$$E(p) \supset O(p),$$

$$E(p) \supset E(p_1) \otimes E(p_2),\ p_1 + p_2 = p,\ p_1 \geq p_2 \geq 1,$$

or for subgroups of the orthogonal group, the subgroup links can be

$$O(p) \supset O(p-1),$$

$$O(p) \supset O(p_1) \otimes O(p_2),\ p_1 + p_2 = p,\ p_1 \geq p_2 \geq 1.$$

The set of general hyperspherical coordinate systems are in correspondence with the different subgroup chains for $O(d)$ (see Izmest′ev et al. (1999) [58], Izmest′ev et al. (2001) [59], Wen & Avery (1985) [106]) of $O(d)$.

There are many examples of subgroup type coordinates in Euclidean space. One example is Cartesian coordinates, such as in the subgroup chains (4.2), (4.5) and (4.11). Also, there are pure hyperspherical coordinate systems in which the subgroup chain contains a maximal copy of $O(d)$, such as in the subgroup chains (4.1), (4.3), (4.6) and (4.7). Also there are mixed Euclidean-hyperspherical coordinates which contain a copy of $O(p)$ for $p \in \{2, \ldots, d-1\}$, such as in the subgroup chains (4.4), (4.8) and (4.10). Subgroup chains such as (4.9) will not be of interest to us, because when we Fourier expand about an angle in these coordinate systems, the results in these types of coordinates will be indistinguishable from a particular mixed Euclidean-hyperspherical coordinate system (as will be evident below). Pure Cartesian coordinate systems will also not be of interest to us either, since there is no natural angle in these coordinate systems in which to Fourier expand about. Non-subgroup type coordinates include those which are analogous to ellipsoidal coordinates, parabolic coordinates, toroidal coordinates, etc.

4.1.2 Method of trees

Subgroup type coordinates are coordinates in which the very useful "method of trees" (a graphical method for generating coordinate transformations for subgroup type coordinates) can be adopted. Corresponding to each of these subgroup chains are a set of tree diagrams. Each of these trees corresponds to a particular separable hyperspherical coordinate system in d-dimensions. See Izmest′ev et al. (1999) [58] (and below) for explicit parametrizations for the corresponding coordinate systems and trees. Describing the set of all general hyperspherical coordinate systems in terms of rooted trees was originally developed by Vilenkin (1968) [100] and has furthermore been used extensively by others in a variety of contexts (see Izmest′ev et al. (1999) [58], Izmest′ev et al. (2001) [59], Kil′dyushov (1972) [63], Vilenkin, Kuznetsov & Smorodinskiĭ (1965) [102]). In these rooted trees, there are two types of nodes, the leaf nodes and the branching nodes. In a particular tree, chosen from the set of all general hyperspherical coordinate systems, there are d leaf nodes, each corresponding to the particular Cartesian component of an arbitrary position vector $\mathbf{x} \in \mathbf{R}^d$. In our trees, the branching nodes always split into two separate branches, one up to the left and one up to the right. Each branch emanating from a branching node will end on either a leaf node or on another branching node. There are four possibilities for branching nodes:

- Type **a**, both branches of the branching node end on a leaf node. The angle corresponding to this type of branching node is $\phi_a \in [0, 2\pi)$.
- Type **b**, the left branch of a branching node ends on a leaf node and the right branch of the branching node ends on a branching node. The angle corresponding to this type

of branching node is $\theta_b \in [0, \pi]$.

- Type $\mathbf{b}'$, the left-branch of the branching node ends on a branching node and the right branch of the branching node ends on a leaf node. The angle corresponding to this type of branching node is $\theta_{b'} \in [-\pi/2, \pi/2]$.

- Type $\mathbf{c}$, both the left and right branches of the branching node ends on branching nodes (cells of type $\mathbf{c}$ are only possible for $d \geq 4$). The angle corresponding to this type of branching node is $\vartheta_c \in [0, \pi/2]$.

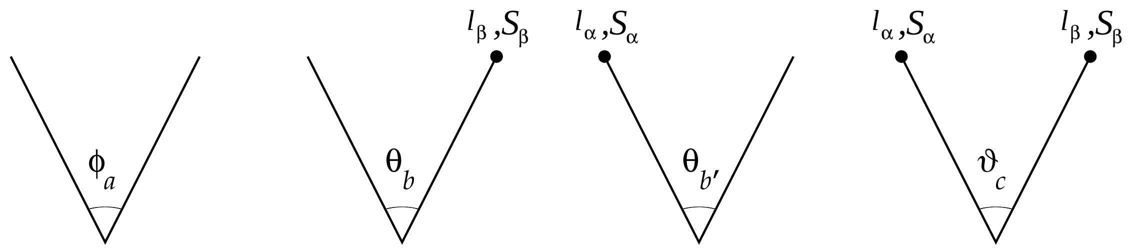

Figure 4.1: This figure show the possibilities from left to right for branching nodes of type $\mathbf{a}$, $\mathbf{b}$, $\mathbf{b}'$ and $\mathbf{c}$.

A linear partial differential equation on $\mathbf{R}^d$ may admit solutions via the separation of variables method. If the method works, then it produces d ordinary differential equations with $d-1$ separation constants which are determined by the conditions imposed on the problem being solved. The sets which the separation constants are contained in depend on the geometric structure of the problem being solved.

For instance, separation of variables in hyperspherical coordinates with $d-1$ angles, for Laplace's equation in $\mathbf{R}^d$ produces $d-1$ separation constants, each of which are called quantum numbers. The quantum numbers corresponding to these angles are all integers. This is due to the fact that a hypersphere is a compact manifold. As described above, each hyperspherical coordinate system is associated with a tree. Quantum numbers for a particular tree label the basis of separable solutions for Laplace's equation in that particular coordinate system. With each branching node of the tree, we associate a quantum number.

The quantum number corresponding to a 2π periodic (azimuthal) angle is called an azimuthal quantum number. Each azimuthal angle corresponds to a branching node of type $\mathbf{a}$, we associate with an azimuthal quantum number $m \in \mathbf{Z}$. A natural consequence of the "method of trees" is that there must exist at least one azimuthal angle for each tree, and therefore also for each pure hyperspherical coordinate system.

Each branching nodes of type $\mathbf{b}$, $\mathbf{b}'$ and $\mathbf{c}$, as described above, are associated with angles, which in turn are associated with quantum numbers which we call angular momentum quantum numbers $l \in \mathbf{N}_0$. The naming of these quantum numbers is due to the intimate connection between the properties of the rotation group $SO(d)$ and the quantum/classical description of angular momentum in physics. See for instance Chapter 10 in Fano & Rau (1996) [38].

There is always at least one branching node (the *root branching node*) and all branching nodes correspond to a particular angle and quantum number. Hence, let us conveniently associate each particular branching node with a particular angle and its corresponding quantum number. One particular general hyperspherical coordinate system is parametrized as follows. Starting at the root branching node, traverse the tree upward until you reach the leaf node corresponding to x_i. The parametrization for x_i is given by the hyperspherical radius r multiplied by cosine or sine of each angle encountered as you traversed the tree upward until you reached the leaf node corresponding to x_i. If you branched upwards to the left or upwards to the right at each branching node, multiply by the cosine or sine of the corresponding angle respectively. This procedure produces the appropriate transformation from hyperspherical coordinates to Cartesian coordinates. Note: The infinitesimal element of the solid angle is given by the absolute value of the Jacobian determinant of the transformation.

There are large numbers of equivalent trees and an even larger number of possible trees, each with their own specific hyperspherical coordinate system. The enumeration of these trees are characterized as follows. For all $d \in \mathbf{N}$, let b_d be the total number of possible trees dimension d. Then $b_1 = 1$ is the number of possible 1-branch trees. A 1-branch tree corresponding to a hyperspherical coordinate system does not exist in isolation. However, the concept of a 1-branch tree is useful. For instance, one can generate a new tree from a pre-existing tree by adding a single branch to either the left or to the right of a pre-existing tree and in order to perform the correct enumeration, one must incorporate 1-branch trees. The following recurrence relation gives the total number of possible hyperspherical coordinate systems for arbitrary dimension

$$b_d = \sum_{i=1}^{d-1} b_i b_{d-i}. \tag{4.12}$$

Using the recurrence relation given by (4.12), we can see that the first few elements of the sequence are given by

$$(b_d : d \in \{1,\ldots,13\}) = (1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ 208012).$$

These numbers, or the way to count these types of trees, are given in terms of the Catalan numbers (see for instance Sloane Integer Sequence A000108 [90] or p. 200 in Stanley (1999) [91]), i.e.

$$b_d = C_{d-1} = \frac{1}{d}\binom{2d-2}{d-1},$$

where C_n is the nth Catalan number.

If a_d is the total number of equivalent trees for dimension d, then the following recurrence relation gives their number for arbitrary d

$$a_d = \begin{cases} \displaystyle\sum_{i=1}^{\lfloor d/2 \rfloor} a_i a_{d-i}, & \text{if } d \text{ odd} \\ \displaystyle\sum_{i=1}^{d/2-1} a_i a_{d-i} + \frac{1}{2} a_{d/2}\left(a_{d/2}+1\right), & \text{if } d \text{ even.} \end{cases} \tag{4.13}$$

Using the recurrence relation given by (4.13), we can see that the first few elements of the sequence are given by

$$(a_d : d \in \{1, \ldots, 13\}) = (1,\ 1,\ 1,\ 2,\ 3,\ 6,\ 11,\ 23,\ 46,\ 98,\ 207,\ 451,\ 983).$$

These numbers, or the way to count these types of trees, are given in terms of the Wedderburn-Etherington numbers (see for instance Sloane Integer Sequence A001190 [90]).
As one can see from the recurrence relations, equivalence of the trees is established by a left-right symmetry in the topology of the trees. The types of spherical coordinates which one can construct, for a given dimension, are described in Izmest'ev et al. (1999, 2001) [58, 59]. In dimensions $d \geq 3$, there are several choices (some of them equivalent) on how one might parametrize the hyper-sphere.

We now describe a left-to-right recursive naming language for our trees, which is based on listing the types of branching nodes available in a particular tree. Each tree is described by a word composed of letters corresponding to the branching node types $\mathbf{a}, \mathbf{b}, \mathbf{b}', \mathbf{c}$. The naming of a tree is based on the naming of its subtrees. The word corresponding to a particular subtree is given as a list of branching node types. We order the branching nodes in a particular subtree by executing a *left-branch algorithm* (depth-first search, pp. 540-549 in Cormen et al. (2001) [31]) with input given by a particular branching node (subtree root) whose aim is to describe the subtree to the left of the subtree root. The left-branch algorithm proceeds as follows: The first letter of the word is the subtree root type. We then proceed upward along the left-branch of subtree root until we encounter another branching node, whose type is the next letter of the subtree word. After that, we continue to proceed left upward writing down the types of all the branching nodes we encounter. Once a leaf node is encountered, we then reverse direction and proceed downward along the left subtree branch until we encounter a branching node whose right branch proceeds upward to another branching node that we have not encountered before. This is the next letter of our word. Then we execute the left-branch algorithm on this branching node. We continue this process until we have described the entire subtree. The left-branch algorithm applied to the root branching node gives a naming convention for the tree.

4.2 Pure hyperspherical coordinates

A particular general hyperspherical coordinate system partitions $\mathbf{R}^d$ into a family of concentric $(d-1)$-dimensional hyper-spheres, each with a radius $r \in (0, \infty)$, on which all separable hyperspherical coordinate systems for $\mathbf{S}^{d-1}$ may be used. One must also consider the limiting case for $r = 0$ to fill out all of $\mathbf{R}^d$. The hyperspherical radius is computed using Cartesian coordinates through

$$r^2 := r_d^2 = \sum_{i=1}^{d} x_i^2.$$

A *general hyperspherical coordinate system*, one which is chosen to parametrize the $(d-1)$-dimensional hyper-sphere, yields solutions to Laplace's equation on $\mathbf{R}^d$ through separation of variables. We refer a particular choice from the the set of all separable pure hyperspherical coordinate systems as a general hyperspherical coordinate system. In a particular general hyperspherical coordinate system

$$dx_1 dx_2 \cdots dx_d = r^{d-1} dr d\Omega_d.$$

There will be a unique decomposition of $d\Omega_d$ in each general hyperspherical coordinate system, depending on the choice of angles used to parametrize the unit $(d-1)$-dimensional hyper-sphere. In the discussion below, we describe how to assemble each general hyperspherical coordinate system in $\mathbf{R}^d$.

In a hyperspherical coordinate system one may define the *separation angle* γ. The separation angle $\gamma \in [0, \pi]$, is the smallest angle measured between two arbitrary non-zero length position vectors $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$. It is defined through the relation

$$\cos\gamma := \frac{(\mathbf{x}, \mathbf{x}')}{\|\mathbf{x}\| \|\mathbf{x}'\|}, \tag{4.14}$$

where

$$(\mathbf{x}, \mathbf{x}') := x_1 x_1' + \ldots + x_d x_d', \tag{4.15}$$

is the Euclidean inner product of two arbitrary position vectors in $\mathbf{R}^d$. One can see from this definition that the method of trees can be used to construct the separation angle in a direct manner.

The cosine of the separation angle will be given by the sum of d terms, each corresponding a leaf node of the tree. There is a unique path starting from the root branching node to each leaf node. The cosine of the separation angle can be constructed using the following formula

$$\cos\gamma = \sum_{i=1}^{d} \prod_{j=1}^{N_i} A_{i,j}(\psi_{i,j}) A_{i,j}(\psi_{i,j}'),$$

where N_i is the number of branching nodes encountered from the root branching node to the leaf node, $A_{i,j}$ equals either the cos or sin function depending respectively on whether the left branch or right branch is chosen and $\psi_{i,j}$ is the angle corresponding to the jth branching node for each ith leaf node. The separation angle is unique for each tree.

4.2.1 Examples of pure hyperspherical coordinate systems

The simplest example of a hyperspherical coordinate system occurs in $d = 2$ where there is one branching node (the root branching node) and two leaf nodes. The left-branch ends on the leaf node corresponding to x_1 and the right branch ends on the leaf node corresponding to x_2. Therefore for $d = 2$ we have the following coordinate system whose transformation formulae to Cartesian coordinates is given by

$$\left.\begin{array}{rcl} x_1 & = & r\cos\phi \\ x_2 & = & r\sin\phi \end{array}\right\}, \tag{4.16}$$

which is unique. In $d = 2$ one parametrizes the corresponding hyper-sphere (circle) uniquely using polar coordinates with an angle, $\phi \in [0, 2\pi)$. Using the left-branch algorithm we see this is a tree of type **a**. These coordinates are adapted to the canonical subgroup chain

$$E(2) \supset O(2).$$

The infinitesimal volume element is given by

$$d\Omega_2 = d\phi,$$

the separation angle is given in terms of

$$\cos\gamma = \cos(\phi - \phi'),$$

and corresponding to the angle ϕ is the quantum number $m \in \mathbf{Z}$ (See Figure 4.2)

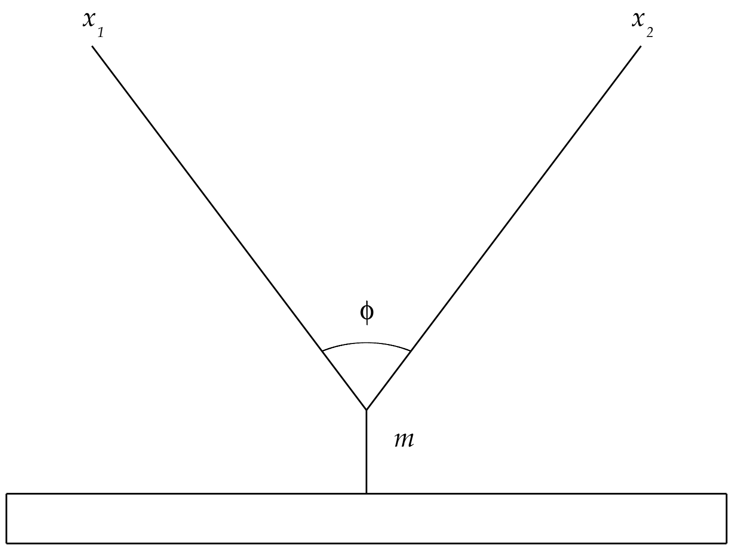

Figure 4.2: This figure is a tree diagram for type **a** pure hyperspherical coordinates in $\mathbf{R}^2$, where the root branching node is of type **a** corresponding to an angle ϕ and quantum number m. These coordinates correspond to Transformation (4.16).

In $d = 3$ there are 2 possible topological trees, each corresponding to one of two different trees. Both coordinate systems are adapted to the canonical subgroup chain

$$E(3) \supset O(3) \supset O(2).$$

The first tree corresponds to the following coordinate system whose transformation formulae to Cartesian coordinates is given by

$$\left.\begin{array}{rcl} x_1 & = & r\cos\theta \\ x_2 & = & r\sin\theta\cos\phi \\ x_3 & = & r\sin\theta\sin\phi \end{array}\right\}, \tag{4.17}$$

where $\theta \in [0, \pi]$. We call these coordinates standard spherical coordinates. Using the left-branch algorithm we see this is a tree of type **ba**. The infinitesimal volume element is given by

$$d\Omega_3 = \sin\theta d\theta d\phi,$$

the separation angle is given in terms of

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \tag{4.18}$$

and corresponding to the angles θ and ϕ are the quantum numbers $l \in \mathbf{N}_0$ and $m \in \mathbf{Z}$ respectively.

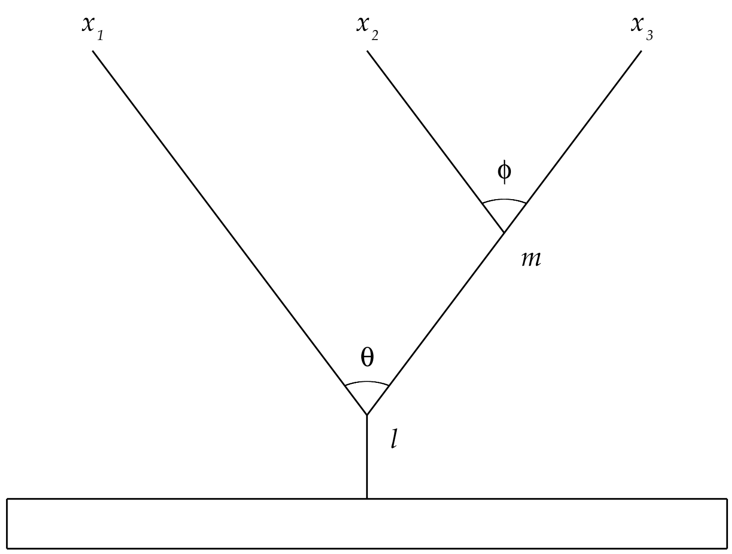

Figure 4.3: This figure is a tree diagram for type **ba** pure hyperspherical coordinates in $\mathbf{R}^3$. The root branching node is of type **b** corresponding to an angle θ and quantum number l. The other branching node is of type **a** corresponding to an angle ϕ and quantum number m. These coordinates correspond to Transformation (4.17).

The second tree corresponds to the following coordinate system whose transformation formulae to Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 &=& r\cos\theta\cos\phi \\ x_2 &=& r\cos\theta\sin\phi \\ x_3 &=& r\sin\theta \end{array}\right\}, \tag{4.19}$$

where $\theta \in [-\pi/2, \pi/2]$. Using the left-branch algorithm we see this is a tree of type $\mathbf{b'a}$. The infinitesimal volume element is given by

$$d\Omega_3 = \cos\theta d\theta d\phi,$$

the separation angle is given in terms of

$$\cos\gamma = \sin\theta\sin\theta' + \cos\theta\cos\theta'\cos(\phi - \phi'), \tag{4.20}$$

and corresponding to the angles θ and ϕ are the quantum numbers $l \in \mathbf{N}_0$ and $m \in \mathbf{Z}$ respectively. It is clear from examination of the corresponding trees, that the coordinate

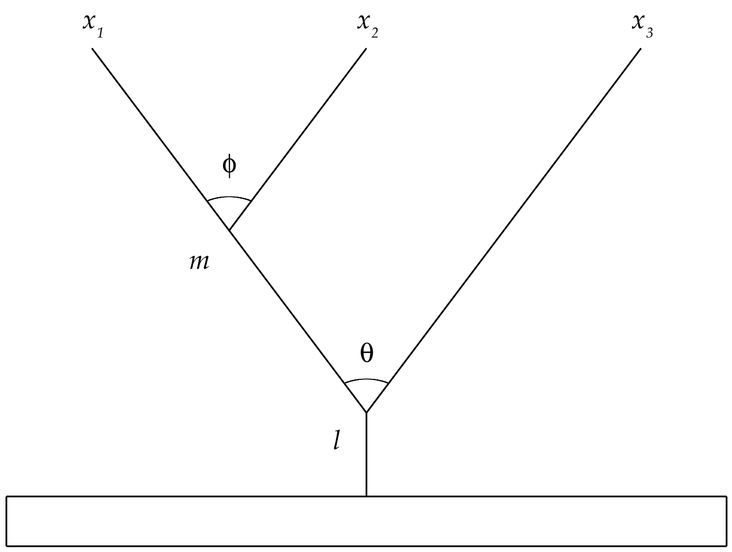

Figure 4.4: This figure is a tree diagram for type $\mathbf{b'a}$ pure hyperspherical coordinates in $\mathbf{R}^3$. The root branching node is of type $\mathbf{b'}$ corresponding to an angle θ and quantum number l. The other branching node is of type $\mathbf{a}$ corresponding to an angle ϕ and quantum number m. These coordinates correspond to Transformation (4.19).

systems of type **ba** and $\mathbf{b'a}$ are equivalent.

Circular cylindrical coordinates are also subgroup type coordinates in $d = 3$. However these coordinates are not of pure hyperspherical type, they are given in terms of a hyperspherical coordinate system, $d = 2$, which also includes one Cartesian dimension. These are referred to as mixed Euclidean-hyperspherical coordinates, and they will be covered in §4.3.

In $d = 4$ there are 5 possible topological trees for pure hyperspherical coordinates, each corresponding to a different tree. The first four coordinate systems are adapted to the canonical subgroup chain

$$E(4) \supset O(4) \supset O(3) \supset O(2).$$

The first tree corresponds to the following coordinate system, whose transformation formulae to Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 & = & r\cos\theta_1 \\ x_2 & = & r\sin\theta_1\cos\theta_2 \\ x_3 & = & r\sin\theta_1\sin\theta_2\cos\phi \\ x_4 & = & r\sin\theta_1\sin\theta_2\sin\phi \end{array}\right\}, \tag{4.21}$$

where $\theta_1, \theta_2 \in [0, \pi]$. Using the left-branch algorithm we see this is a tree of type $\mathbf{b^2a}$. The infinitesimal volume element is given by

$$d\Omega_4 = \sin^2\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi,$$

and

$$\cos\gamma = \cos\theta_1\cos\theta_1' + \sin\theta_1\sin\theta_1' \left(\cos\theta_2\cos\theta_2' + \sin\theta_2\sin\theta_2'\cos(\phi-\phi')\right) \tag{4.22}$$

(see Figure 4.5). The second tree corresponds to the following coordinate system whose transformation formulae to Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 & = & r\cos\theta_1 \\ x_2 & = & r\sin\theta_1\cos\theta_2\cos\phi \\ x_3 & = & r\sin\theta_1\sin\theta_2\sin\phi \\ x_4 & = & r\sin\theta_1\sin\theta_2 \end{array}\right\}, \tag{4.23}$$

where $\theta_1 \in [0, \pi]$ and $\theta_2 \in [-\pi/2, \pi/2]$. Using the left-branch algorithm we see this is a tree of type $\mathbf{bb'a}$. The infinitesimal volume element is given by

$$d\Omega_4 = \sin^2\theta_1 \cos\theta_2 d\theta_1 d\theta_2 d\phi,$$

and

$$\cos\gamma = \cos\theta_1\cos\theta_1' + \sin\theta_1\sin\theta_1' \left(\sin\theta_2\sin\theta_2' + \cos\theta_2\cos\theta_2'\cos(\phi-\phi')\right) \tag{4.24}$$

(see Figure 4.6). The third tree corresponds to the following coordinate system, whose

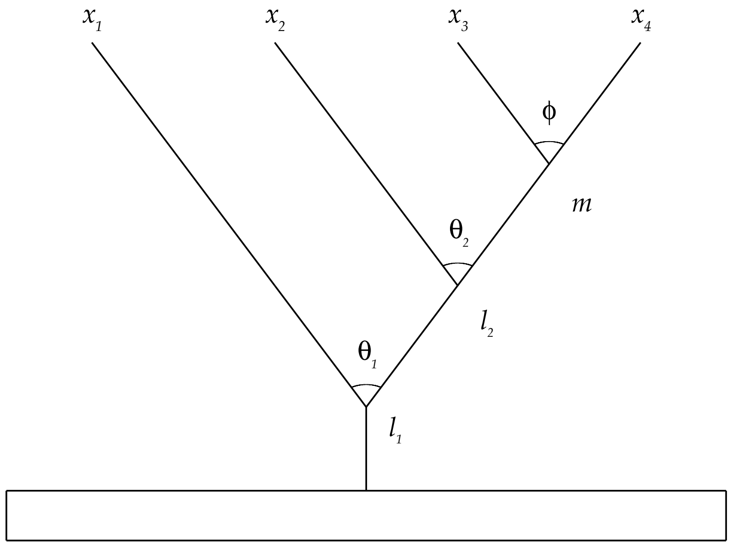

Figure 4.5: This figure is a tree diagram for type $\mathbf{b}^2\mathbf{a}$ pure hyperspherical coordinates in $\mathbf{R}^4$. The first two branching nodes are both of type $\mathbf{b}$. These correspond to the angles θ_1 and θ_2 with quantum numbers l_1 and l_2 respectively. The third branching node is of type $\mathbf{a}$ corresponding to an angle ϕ and quantum number m. These coordinates correspond to Transformation (4.21).

transformation formulae to Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 &=& r\cos\theta_1\cos\theta_2 \\ x_2 &=& r\cos\theta_1\sin\theta_2\cos\phi \\ x_3 &=& r\cos\theta_1\sin\theta_2\sin\phi \\ x_4 &=& r\sin\theta_1 \end{array}\right\}, \tag{4.25}$$

where $\theta_1 \in [-\pi/2, \pi/2]$ and $\theta_2 \in [0, \pi]$. Using the left-branch algorithm we see this is a tree of type $\mathbf{b'ba}$. The infinitesimal volume element is given by

$$d\Omega_4 = \cos^2\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi,$$

and

$$\cos\gamma = \sin\theta_1\sin\theta_1' + \cos\theta_1\cos\theta_1'\left(\cos\theta_2\cos\theta_2' + \sin\theta_2\sin\theta_2'\cos(\phi - \phi')\right) \tag{4.26}$$

(see Figure 4.7). The fourth tree corresponds to the following coordinate system, whose

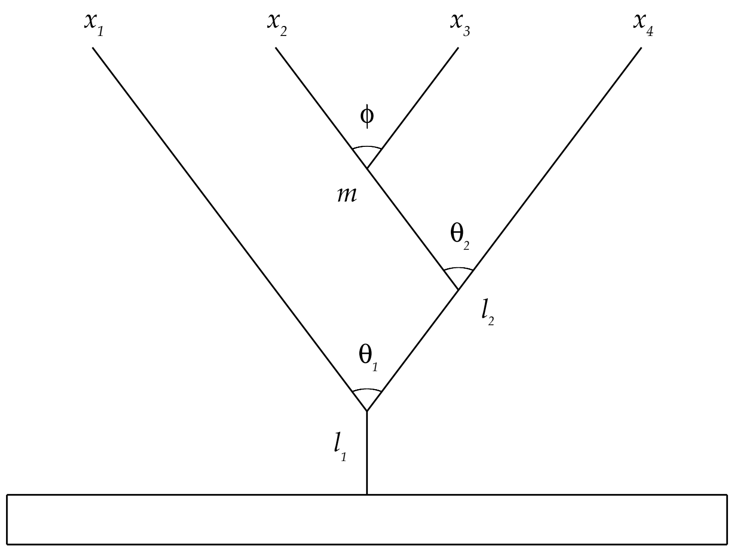

Figure 4.6: This figure is a tree diagram for type $\mathbf{bb'a}$ pure hyperspherical coordinates in $\mathbf{R}^4$. The root branching node is of type $\mathbf{b}$ which corresponds to the angle θ_1 and quantum number l_1. The second branching node is of type $\mathbf{b'}$ which corresponds to the angle θ_2 and quantum number l_2. The third branching node is of type $\mathbf{a}$ corresponding to the angle ϕ and quantum number m. These coordinates correspond to Transformation (4.23).

transformation formulae to Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 &=& r\cos\theta_1\cos\theta_2\cos\phi \\ x_2 &=& r\cos\theta_1\cos\theta_2\sin\phi \\ x_3 &=& r\cos\theta_1\sin\theta_2 \\ x_4 &=& r\sin\theta_2 \end{array}\right\}, \tag{4.27}$$

where $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$. Using the left-branch algorithm we see this is a tree of type $\mathbf{b'^2a}$. The infinitesimal volume element is given by

$$d\Omega_4 = \cos^2\theta_1\cos\theta_2 d\theta_1 d\theta_2 d\phi,$$

and

$$\cos\gamma = \sin\theta_1\sin\theta_1' + \cos\theta_1\cos\theta_1'\left(\sin\theta_2\sin\theta_2' + \cos\theta_2\cos\theta_2'\cos(\phi-\phi')\right) \tag{4.28}$$

(see Figure 4.8). The fifth tree corresponds to the following coordinate system (Hopf coordi-

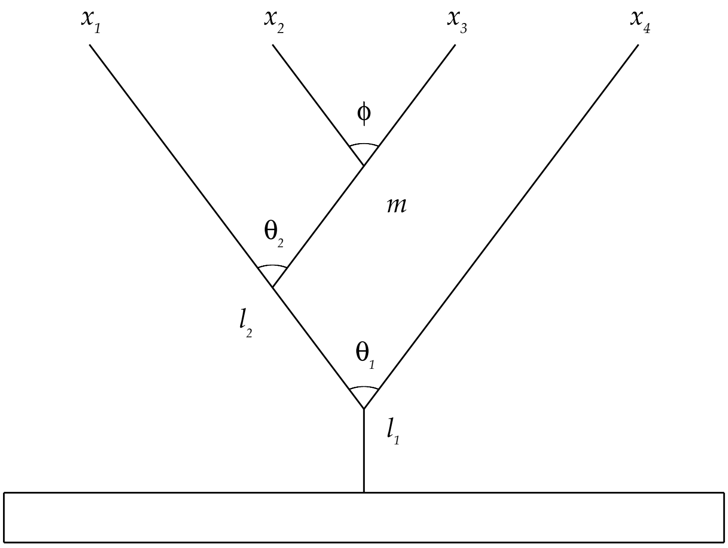

Figure 4.7: This figure is a tree diagram for type $\mathbf{b'ba}$ pure hyperspherical coordinates in $\mathbf{R}^4$. The root branching node is of type $\mathbf{b'}$ which corresponds to the angle θ_1 and quantum number l_1. The second branching node is of type $\mathbf{b}$ which corresponds to the angle θ_2 and quantum number l_2. The third branching node is of type $\mathbf{a}$ corresponding to the angle ϕ and quantum number m. These coordinates correspond to Transformation (4.25).

nates) whose transformation formulae to Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 &=& r\cos\vartheta\cos\phi_1 \\ x_2 &=& r\cos\vartheta\sin\phi_1 \\ x_3 &=& r\sin\vartheta\cos\phi_2 \\ x_4 &=& r\sin\vartheta\sin\phi_2 \end{array}\right\}, \tag{4.29}$$

where $\vartheta \in [0, \pi/2]$ and $\phi_1, \phi_2 \in [0, 2\pi)$. Using the left-branch algorithm we see this is a tree of type $\mathbf{ca^2}$. This coordinate system is adapted to the canonical subgroup chain

$$E(4) \supset O(4) \supset O(2) \otimes O(2).$$

The infinitesimal volume element is given by

$$d\Omega_4 = \cos\vartheta\sin\vartheta d\vartheta d\phi_1 d\phi_2,$$

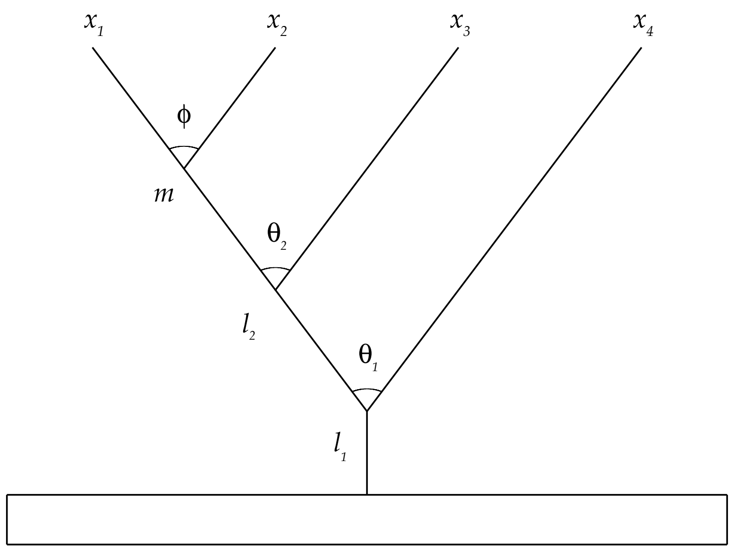

Figure 4.8: This figure is a tree diagram for type $\mathbf{b}'^2\mathbf{a}$ pure hyperspherical coordinates in $\mathbf{R}^4$. The first two branching nodes are of type $\mathbf{b}'$ which correspond to the angles θ_1 and θ_2 and quantum numbers l_1 and l_2 respectively. The third branching node is of type $\mathbf{a}$ corresponding to the angle ϕ and quantum number m. These coordinates correspond to Transformation (4.27).

and

$$\cos\gamma = \cos\vartheta\cos\vartheta'\cos(\phi_1 - \phi_1') + \sin\vartheta\sin\vartheta'\cos(\phi_2 - \phi_2'). \tag{4.30}$$

(see Figure 4.9). We can see that there are many choices for pure hyperspherical coordinate systems in $\mathbf{R}^d$. The simplest example is what we will call the *standard hyperspherical coordinate system*. Standard hyperspherical coordinates are a generalization of (4.21). This coordinate system (and all which are equivalent to it) is adapted to the canonical subgroup chain

$$E(d) \supset O(d) \supset O(d-1) \supset \cdots \supset O(2).$$

It is suitably defined for any number of dimensions, $d \geq 2$. The transformation formulae to

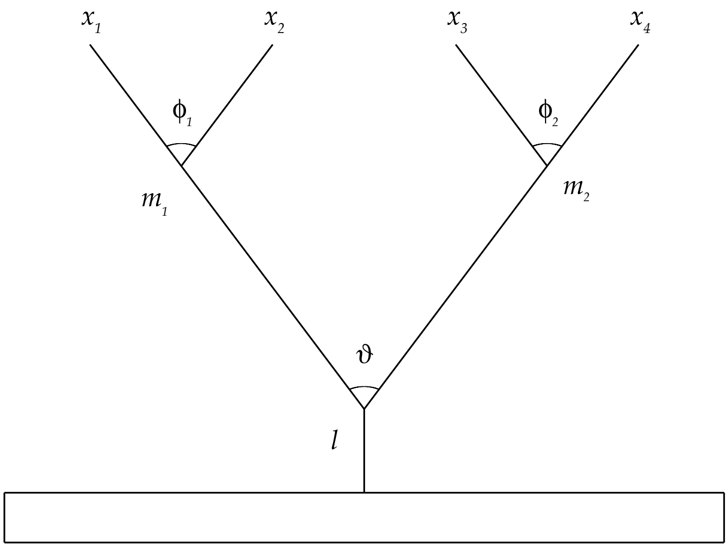

Figure 4.9: This figure is a tree diagram for type $\mathbf{ca}^2$ pure hyperspherical coordinates in $\mathbf{R}^4$. The first branching node is of type $\mathbf{c}$ which corresponds to the angle ϑ and quantum number l. The second and third branching nodes are of type $\mathbf{a}$ corresponding to angles ϕ_1 and ϕ_2 with quantum numbers m_1 and m_2 respectively. These coordinates correspond to Transformation (4.29).

Cartesian coordinates are given by

$$\left.\begin{array}{rcl} x_1 & = & r\cos\theta_1 \\ x_2 & = & r\sin\theta_1\cos\theta_2 \\ x_3 & = & r\sin\theta_1\sin\theta_2\cos\theta_3 \\ & \vdots & \\ x_{d-2} & = & r\sin\theta_1\cdots\sin\theta_{d-3}\cos\theta_{d-2} \\ x_{d-1} & = & r\sin\theta_1\cdots\sin\theta_{d-3}\sin\theta_{d-2}\cos\phi \\ x_d & = & r\sin\theta_1\cdots\sin\theta_{d-3}\sin\theta_{d-2}\sin\phi \end{array}\right\}, \tag{4.31}$$

where $\theta_i \in [0,\pi]$ for $i \in \{1,\ldots,d-2\}$ and $\phi \in [0,2\pi)$. Using the left-branch algorithm we see this is a tree of type $\mathbf{b}^{d-2}\mathbf{a}$. These are the standard hyperspherical coordinates that are usually adopted in multi-variable calculus and in physical applications (see for instance Fano & Rau (1996) [38]). In these coordinates

$$d\Omega_d = (\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\cdots(\sin\theta_{d-3})^2\sin\theta_{d-2}d\theta_1\cdots d\theta_{d-2}d\phi,$$

and

$$\cos\gamma = \sum_{i=1}^{d-2} \cos\theta_i \cos\theta_i{}' \prod_{j=1}^{i-1} \sin\theta_j \sin\theta_j{}' + \cos(\phi - \phi') \prod_{i=1}^{d-2} \sin\theta_i \sin\theta_i{}'$$

(see Figure 4.10).

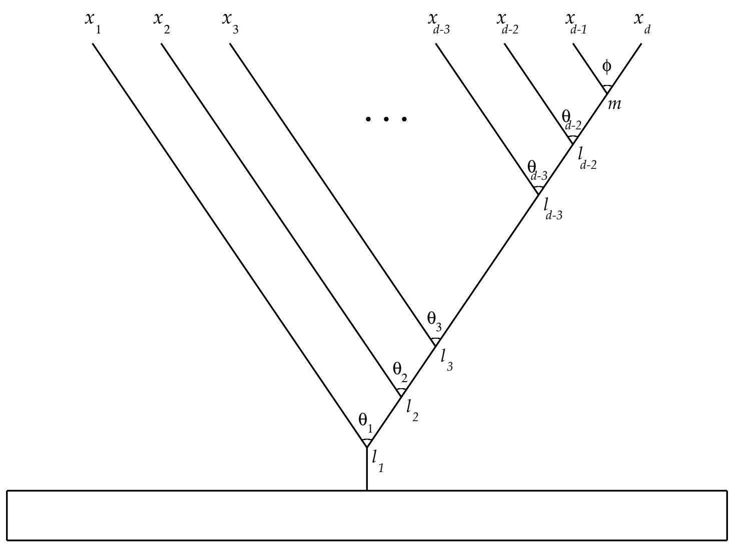

Figure 4.10: This figure is a tree diagram for type $\mathbf{b}^{d-2}\mathbf{a}$ pure hyperspherical coordinates in $\mathbf{R}^d$. The first $d-2$ branching nodes are of type **b** which correspond to the angles θ_i and quantum numbers l_i, where $i \in \{1, \ldots, d-2\}$. The final branching node is of type **a** corresponding to angle ϕ and quantum number m. These coordinates correspond to Transformation (4.31).

4.3 Mixed Euclidean-hyperspherical coordinate systems

In general we define mixed Euclidean-hyperspherical coordinates, (a generalization of pure hyperspherical coordinates) as follows. The (d,k) Euclidean-hyperspherical coordinate system, where $2 \leq k \leq d$, is one adapted from the following subgroup chain

$$E(d) \supset E(d-1) \supset \cdots E(k) \supset O(k) \supset \cdots .$$

Note that for the case for $k = d$, i.e. (d,d) Euclidean-hyperspherical coordinates, this corresponds to pure hyperspherical coordinates in $\mathbf{R}^d$. For mixed Euclidean-hyperspherical coordinates, all maximal subgroup chains for $O(k)$ are possible.

4.3.1 Examples of mixed Euclidean-hyperspherical coordinate systems

Now we present some examples of mixed Euclidean-hyperspherical coordinate systems. Take for instance, type **a** $(d, 2)$ Euclidean-hyperspherical coordinates. This coordinate system is given in terms of the following coordinate transformation

$$\left.\begin{array}{rcl} x_1 &=& r\cos\phi \\ x_2 &=& r\sin\phi \\ x_3 &=& x_3 \\ & \vdots & \\ x_{d-2} &=& x_{d-2} \\ x_{d-1} &=& x_{d-1} \\ x_d &=& x_d \end{array}\right\}, \tag{4.32}$$

and is adapted from the following subgroup chain

$$E(d) \supset E(d-1) \supset \cdots \supset E(2) \supset O(2).$$

The most common example of this type of coordinate system is circular cylindrical coordinates in $\mathbf{R}^3$, i.e. $(3, 2)$ Euclidean-hyperspherical coordinates. These coordinates are unique in that there is only one way to put coordinates on a 1-sphere. (see Figure 4.11).

Another example is type **ba** $(d, 3)$ Euclidean-hyperspherical coordinates (see Figure 4.12)

$$\left.\begin{array}{rcl} x_1 &=& r\cos\theta_1 \\ x_2 &=& r\sin\theta_1\cos\phi \\ x_3 &=& r\sin\theta_1\sin\phi \\ x_4 &=& x_4 \\ & \vdots & \\ x_{d-2} &=& x_{d-2} \\ x_{d-1} &=& x_{d-1} \\ x_d &=& x_d, \end{array}\right\}, \tag{4.33}$$

and is adapted from the following subgroup chain

$$E(d) \supset E(d-1) \supset \cdots \supset E(3) \supset O(3) \supset O(2).$$

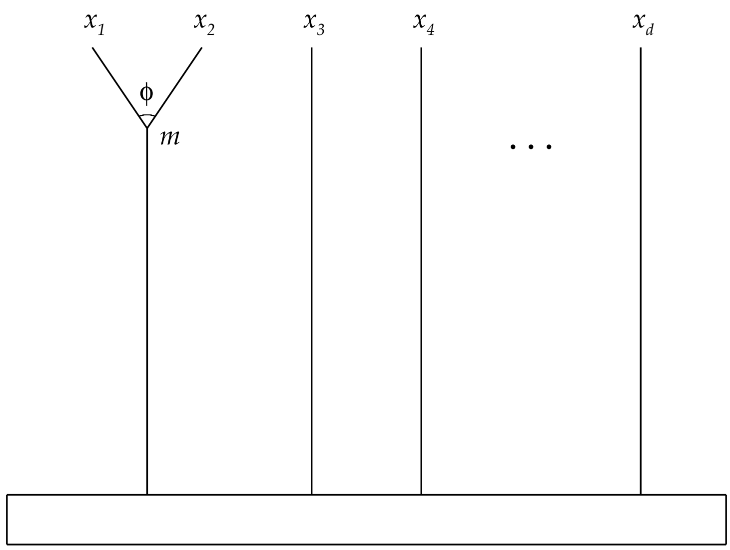

Figure 4.11: This figure is a tree diagram for type **a** $(d, 2)$ Euclidean-hyperspherical coordinates in $\mathbf{R}^d$. There is only the root branching node and it is of type **a** which correspond to the angle ϕ and quantum number m. There are $d-2$ Cartesian components. These coordinates correspond to Transformation (4.32).

The other coordinate system adapted from this particular subgroup chain is type $\mathbf{b'a}$ $(d, 3)$ Euclidean-hyperspherical coordinates.

Of course the number of choices of types for mixed coordinate systems increases as we increase k. For instance there are the types $\mathbf{b}^2\mathbf{a}$, $\mathbf{bb'a}$, $\mathbf{b'ba}$, $\mathbf{b'}^2\mathbf{a}$, and $\mathbf{ca}^2$ $(d, 4)$ Euclidean-hyperspherical coordinates. Standard type $\mathbf{b}^{k-2}\mathbf{a}$ (d, k) Euclidean-hyperspherical coordinates (mixed standard Euclidean-hyperspherical coordinates) is adapted from the following subgroup chain

$$E(d) \supset E(d-1) \supset \cdots \supset E(k) \supset O(k) \supset O(k-1) \supset \cdots \supset O(2),$$

and is given by the following coordinate transformation

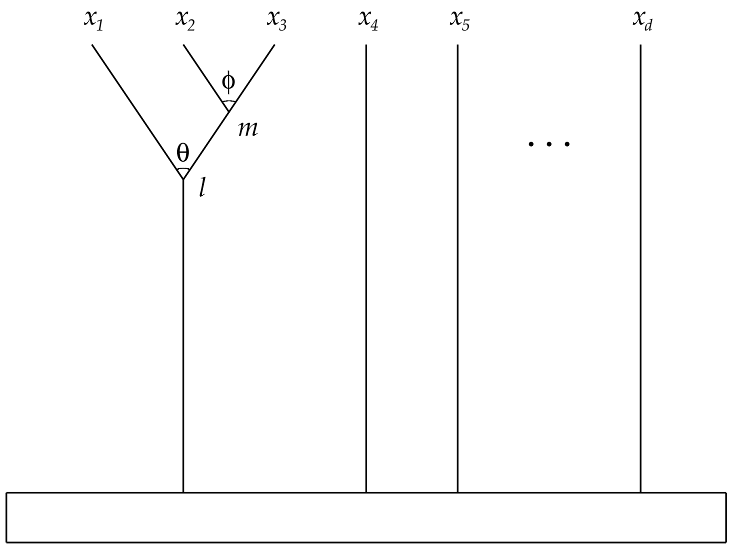

Figure 4.12: This figure is a tree diagram for type **ba** $(d,3)$ Euclidean-hyperspherical coordinates in $\mathbf{R}^d$. The root branching node is of type **b** which correspond to the angle θ and quantum number l. The second branching node is of type **a** which corresponds to the angle ϕ and quantum number m. There are $d-3$ Cartesian components. These coordinates correspond to Transformation (4.33).

$$\left.\begin{array}{rcl}
x_1 & = & r_k \cos\theta_1 \\
x_2 & = & r_k \sin\theta_1 \cos\theta_2 \\
x_3 & = & r_k \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
 & \vdots & \\
x_{k-2} & = & r_k \sin\theta_1 \cdots \sin\theta_{k-3} \cos\theta_{k-2} \\
x_{k-1} & = & r_k \sin\theta_1 \cdots \sin\theta_{k-3} \sin\theta_{k-2} \sin\phi \\
x_k & = & r_k \sin\theta_1 \cdots \sin\theta_{k-3} \sin\theta_{k-2} \cos\phi \\
x_{k+1} & = & x_{k+1} \\
 & \vdots & \\
x_d & = & x_d
\end{array}\right\}$$

(see Figure 4.13). For mixed standard Euclidean-hyperspherical coordinate systems we can

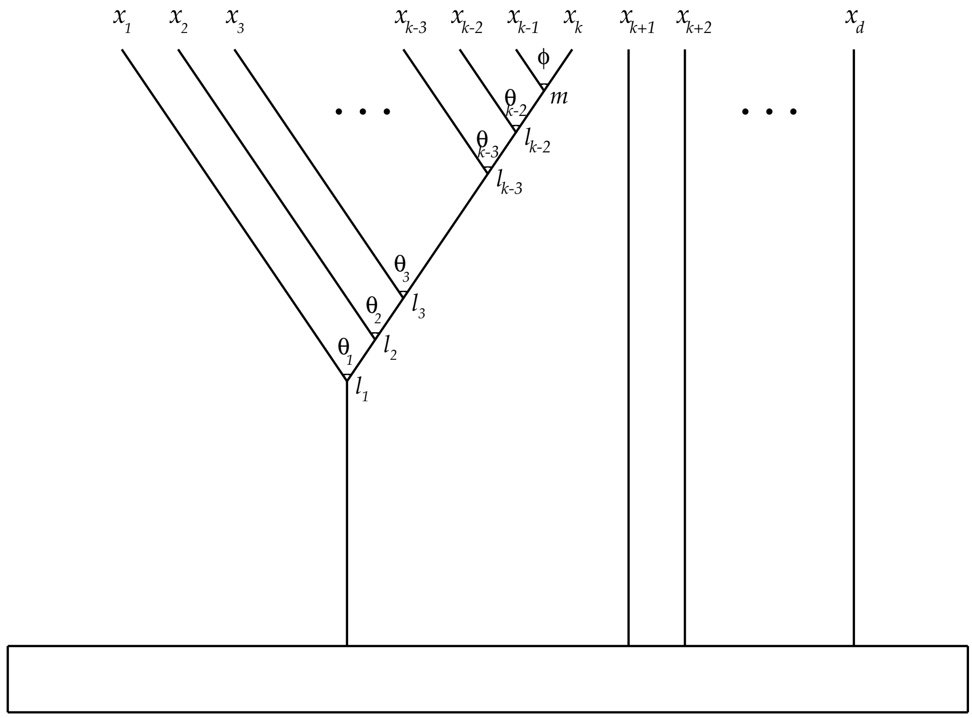

Figure 4.13: This figure is a tree diagram for type $\mathbf{b}^{k-2}\mathbf{a}$ (d,k) Euclidean-hyperspherical coordinates in $\mathbf{R}^d$ (mixed standard Euclidean-hyperspherical coordinates). The root branching nodes, angles, and quantum numbers are given by standard hyperspherical coordinates (4.31). There are $d-k$ Cartesian components. These coordinates correspond to Transformation (4.34).

naturally define the following quantities, the sub-radii

$$r_k^2 := \sum_{i=1}^{k} x_i^2,$$

and the separation angle

$$\cos\gamma_k = \cos(\phi-\phi')\prod_{i=1}^{k-2}\sin\theta_i\sin\theta_i{}' + \sum_{i=1}^{k-2}\cos\theta_i\cos\theta_i{}'\prod_{j=1}^{i-1}\sin\theta_j\sin\theta_j{}'.$$

The Euclidean distance between two points in these coordinates is given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2r_k r_k' \prod_{i=1}^{k-2} \sin\theta_i \sin{\theta_i}'} \left[\chi_k^d - \cos(\phi-\phi')\right]^{1/2},$$

where the toroidal parameter χ_k^d (see Cohl & Tohline (1999) [26]), is given by

$$\chi_k^d := \frac{r^2+r'^2-2rr'\sum_{i=1}^{k-2}\cos\theta_i\cos{\theta_i}'\prod_{j=1}^{i-1}\sin\theta_j\sin{\theta_j}' + \sum_{i=k+1}^{d}(x_i-x_i')^2}{2rr'\prod_{i=1}^{k-2}\sin\theta_i\sin{\theta_i}'}, \tag{4.34}$$

where $\chi_k^d > 1$. Note that in pure standard hyperspherical coordinates

$$\chi_d^d = \frac{r^2+r'^2-2rr'\sum_{i=1}^{d-2}\cos\theta_i\cos{\theta_i}'\prod_{j=1}^{i-1}\sin\theta_j\sin{\theta_j}'}{2rr'\prod_{i=1}^{d-2}\sin\theta_i\sin{\theta_i}'}, \tag{4.35}$$

and if $d=2$ or if $\theta_i=\theta_i'=\dfrac{\pi}{2}$ for all $i\in\{1,\ldots,k-2\}$ then

$$\chi = \frac{r^2+r'^2}{2rr'}.$$

Now consider a general type (d,k) Euclidean-hyperspherical coordinate system, namely one with a k-dimensional Euclidean subspace whose points are parametrized by the method of trees. The Euclidean distance between two points in any of these coordinate systems are given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2r_k r_k'}\left[\chi_k^d - \cos\gamma_k\right]^{1/2}, \tag{4.36}$$

where the toroidal parameter χ_k^d, is given by

$$\chi_k^d := \frac{r_k^2+{r_k'}^2+\sum_{i=k+1}^{d}(x_i-x_i')^2}{2r_k r_k'}, \tag{4.37}$$

and the separation angle is given by

$$\cos\gamma_k = \frac{(\mathbf{y},\mathbf{y}')}{r_k r_k'}, \tag{4.38}$$

where $\mathbf{y} = (x_1, \ldots, x_k)$ and $\mathbf{y}' = (x_1', \ldots, x_k')$. Note that the sub-radii are given as $r_k = \|\mathbf{y}\|$ and $r_k' = \|\mathbf{y}'\|$.

In every curvilinear coordinate system there will be a unique representation of χ. However the hypersurfaces given by χ equals constant are independent of coordinate system and represent hyper-tori of revolution. We use these quantities in Chapters 5 and 6 to perform Fourier expansions of normalized fundamental solutions for powers of the Laplacian in these coordinate systems and to generate multi-summation addition theorems.

5

Fourier expansions for fundamental solutions in R^d

In this chapter we present joint research with Diego Dominici (see Cohl & Dominici (2010) [24]) and Tom ter Elst.

We compute Fourier cosine series of unnormalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$ which was presented in Chapter 3. As far as the author is aware, the main results of this chapter are new, namely the Fourier cosine series of a fundamental solution for powers of the Laplacian with $k \geq (d+1)/2$. The Fourier cosine series for the functional form of a fundamental solution for powers of the Laplacian in $\mathbf{R}^d$ has been previously given for $k \leq (d-1)/2$ (see p. 182 in Magnus, Oberhettinger & Soni (1966) [67]). The new results which apply in this range for unnormalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$ are those functions appearing in §5.2 on the generalized Heine identity, and the particularly challenging logarithmic functions in §5.3. Also new are the results in §5.2.3 on finite-summation expressions for associated Legendre functions of the second kind and §5.4.1 on Fourier expansions in mixed Euclidean-hyperspherical coordinate systems.

5.1 The functions for Fourier expansions

Consider the following functions. For d odd and for d even with $k \le d/2-1$, define

$$\mathfrak{g}_k^d(\mathbf{x},\mathbf{x}') := \|\mathbf{x}-\mathbf{x}'\|^{2k-d},$$

and for d even, $k \ge d/2$, define

$$\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}') := \|\mathbf{x}-\mathbf{x}'\|^{2p}\left(\log\|\mathbf{x}-\mathbf{x}'\| - \beta_{k-\frac{d}{2}}\right),$$

where $p = k - d/2$. By Theorem 3.2.2 we see that the functions $\mathfrak{g}_k^d$ and $\mathfrak{l}_k^d$ equal real non-zero constant multiples of $\mathcal{G}_k^d$ for the same parameters. Therefore by (3.7), $\mathfrak{g}_k^d$ and $\mathfrak{l}_k^d$ are unnormalized fundamental solutions of the polyharmonic equation with the same parameters. In Chapter 4, for mixed Euclidean-hyperspherical coordinates, we used the letter k to enumerate the dimension of the subspace of $\mathbf{R}^d$ which was pure hyperspherical coordinates. This now gives a clash with the k used for the power of the Laplacian operator. Therefore in the next paragraph we temporarily write κ for the dimension of this subspace.

Using (4.36) we can write $\mathfrak{g}_k^d$ in terms of mixed Euclidean-hyperspherical coordinates as follows. For d odd, and for d even with $k \le d/2-1$ we have

$$\mathfrak{g}_k^d(\mathbf{x},\mathbf{x}') = (2r_\kappa r'_\kappa)^{k-d/2}\left[\chi_\kappa^d - \cos\gamma_\kappa\right]^{k-d/2}, \tag{5.1}$$

where χ_κ^d is given in (4.37), $\cos\gamma_\kappa$ is given in (4.38). For d even with $k \ge d/2$ we have

$$\begin{aligned}\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}') = (2r_\kappa r'_\kappa)^p &\left[\frac{1}{2}\log(2r_\kappa r'_\kappa) - \beta_{k-d/2}\right]\left[\chi_\kappa^d - \cos\gamma_\kappa\right]^p \\ &+ \frac{1}{2}(2r_\kappa r'_\kappa)^p\left[\chi_\kappa^d - \cos\gamma_\kappa\right]^p \log\left(\chi_\kappa^d - \cos\gamma_\kappa\right).\end{aligned} \tag{5.2}$$

By examining (5.1) and (5.2), we see that for computation of Fourier expansions about the separation angle γ_κ of $\mathfrak{g}_k^d$ and $\mathfrak{l}_k^d$, all that is required is to compute the Fourier cosine series for the following four functions, defined for $\chi > 1$,

$$\frac{1}{[\chi - \cos\psi]^\mu} \quad \text{if } d \text{ odd}, \quad k \in \mathbf{N}, \qquad \mu \text{ being an odd-half-integer},$$

$$\frac{1}{[\chi - \cos\psi]^\mu} \quad \text{if } d \text{ even}, \quad k \leq d/2 - 1, \quad \mu \in \mathbf{N},$$

$$[\chi - \cos\psi]^p \quad \text{if } d \text{ even}, \quad k \geq d/2, \qquad p \in \mathbf{N}_0, \quad \text{and}$$

$$[\chi - \cos\psi]^p \log(\chi - \cos\psi) \quad \text{if } d \text{ even}, \quad k \geq d/2, \qquad p \in \mathbf{N}_0.$$

Construction of a hyperspherical coordinate system using the method of trees ensures the existence of an azimuthal angle ϕ in that coordinate system. One may also compute Fourier expansions for $\mathfrak{g}_k^d$ and $\mathfrak{l}_k^d$ about an angle $\phi - \phi'$ in $\mathbf{R}^d$. With an appropriately modified $\chi = \chi_{\kappa-1}^d$, azimuthal Fourier expansions may be computed using these same four functions. In the mixed Euclidean-hyperspherical coordinate system used in (5.1) and (5.2), consider the angle

$$\gamma_{\kappa-1} := \gamma_\kappa \Big|_{\phi=\phi'}.$$

For azimuthal Fourier expansions, the property $\chi_{\kappa-1}^d > 1$ is logically equivalent to

$$r^2 + r'^2 - 2rr' \cos\gamma_{\kappa-1} > 0,$$

where $r \neq r'$.

In the following section we derive an identity which will actually give us the Fourier cosine series for the first three cases. The fourth Fourier cosine series will be established in Section 5.3.

5.2 Generalized Heine identity

Gauss (1812) ([46], p. 128 in Werke III) was able to write down closed-form expressions for the Fourier series for the related function $[r_1^2 + r_2^2 - 2r_1r_2\cos\psi]^{-\mu}$. Gauss recognised that the coefficients of the expansion are given in terms of the ${}_2F_1$ hypergeometric function, and he was able to write down a closed-form solution (where we have used modern notations) given by

$$\frac{1}{[r_1^2 + r_2^2 - 2r_1r_2\cos\psi]^\mu} = \sum_{n=0}^\infty \epsilon_n \cos(n\psi) \frac{(\mu)_n}{n!} \frac{r_2^n}{r_1^{2\mu+n}} {}_2F_1\left(n+\mu, \mu; n+1; \frac{r_2^2}{r_1^2}\right), \tag{5.3}$$

assuming $r_1, r_2 \in \mathbf{R}$, $0 \le r_2 < r_1$, $(\mu)_n$ is the Pochhammer symbol for rising factorials (see (2.10) and (2.11)), and the Gauss hypergeometric function is defined through (2.19). Neumann (1864) ([72], pp. 33–34) was one of the first to study separable solutions to Laplace's equation in toroidal coordinates. He examined and wrote down the Fourier expansion in term of Gauss hypergeometric functions for the instance $\mu = \frac{1}{2}$.

But it was not until Heine (1881) ([50], p. 286) in his *Handbuch der Kugelfunctionen* that it was recognised that this particular Gauss hypergeometric function represented a certain special class of higher transcendental functions, namely associated Legendre functions of the second kind with odd-half-integer degree. These associated Legendre functions, and in particular, those with integer-order, are toroidal harmonics, the functions which separate Laplace's equation in toroidal coordinates. The fact that the algebraic function of present study relates to behaviour on toroids of revolution is an important consequence of a non-axisymmetric Fourier description of potential theory in rotationally-invariant coordinate systems. We now proceed to build upon Heine's original identity (5.11) in order to derive a complex generalization of that identity (5.10).

5.2.1 Derivation of the identity

We are interested in computing the following Fourier expansion

$$\frac{1}{[z-\cos\psi]^\mu} = \sum_{n=0}^{\infty} \cos(n\psi) A_{\mu,n}(z), \tag{5.4}$$

where $\mu \in \mathbf{C}$, $z \in \mathbf{C} \setminus (-\infty, 1]$, and $|z| > 1$. The expression for these Fourier coefficients is given in the standard manner by

$$A_{\mu,n}(z) = \frac{\epsilon_n}{\pi} \int_0^\pi \frac{\cos(n\psi)}{[z-\cos\psi]^\mu} d\psi. \tag{5.5}$$

We can take advantage of the fact that the summand is an even function of the Fourier quantum number $n \in \mathbf{Z}$ in order to sum over $n \in \mathbf{N}_0$, in that case

$$\sum_{n=-\infty}^{\infty} A_{\mu,n}(z) e^{in\psi} = \sum_{n=0}^{\infty} \epsilon_n A_{\mu,n}(z) \cos(n\psi).$$

In order to use the complex binomial theorem, we first show that

$$\arg(z-\cos\psi) = \arg(z) + \arg\left(1 - \frac{\cos\psi}{z}\right), \tag{5.6}$$

which is verified as follows. We define the function $f : [-1,1] \to \mathbf{R}$ by

$$f(x) = \arg(z+x) - \arg(z) - \arg\left(1 + \frac{x}{z}\right).$$

f is clearly continuous and since $[-1,1]$ is connected, $f([-1,1])$ must be connected and $f([-1,1]) \subset 2\pi\mathbf{Z}$. Hence $f([-1,1])$ is a one-point set and since $f(0) = 0$, f is a constant equal to zero. Therefore we have shown that (5.6) is true and therefore we can re-write the left-hand side of (5.4) without loss of generality as

$$\frac{1}{[z-\cos\psi]^\mu} = \frac{1}{z^\mu\left[1 - \dfrac{\cos\psi}{z}\right]^\mu}.$$

Since $|z| > 1$ and $\cos\psi \in [-1,1]$ this implies that $|(\cos\psi)/z| < 1$, and we are in a position to employ Newton's binomial series (2.20), where the generalized binomial coefficient is defined in (2.13). Combining (2.13), (2.12), and (2.20), we obtain

$$\frac{1}{[z-\cos\psi]^\mu} = \sum_{k=0}^\infty \frac{(\mu)_k}{k!} z^{-\mu-k}\cos^k\psi. \tag{5.7}$$

We can expand the powers of cosine using the following trigonometric identity

$$\cos^k\psi = \frac{1}{2^k}\sum_{n=0}^k \binom{k}{n}\cos[(2n-k)\psi],$$

which is the standard expansion for powers using Chebyshev polynomials (see for instance p. 52 in Fox & Parker (1968) [41]). Inserting this expression in (5.7), we obtain the following double-summation expression

$$\frac{1}{[z-\cos\psi]^\mu} = \sum_{k=0}^\infty\sum_{n=0}^k \frac{(\mu)_k}{k!}\frac{1}{2^k z^{\mu+k}}\binom{k}{n}\cos[(2n-k)\psi]. \tag{5.8}$$

Now we perform a double-index replacement in (5.8). There are two separate cases $k \le 2n$ and $k \ge 2n$. There is an overlap if both have an equality, and in that case we must multiply by $1/2$ after we sum over both cases. If $k \le 2n$ make the substitution $k' = k - n$ and $n' = 2n - k$. It follows that $k = 2k' + n'$ and $n = n' + k'$, therefore

$$\binom{k}{n} = \binom{2k'+n'}{n'+k'} = \binom{2k'+n'}{k'}.$$

If $k \ge 2n$ make the substitution $k' = n$ and $n' = k - 2n$. Then $k = 2k' + n'$ and $n = k'$

therefore

$$\binom{k}{n} = \binom{2k'+n'}{k'} = \binom{2k'+n'}{k'+n'},$$

where the equalities of the binomial coefficients are confirmed using the binomial identity (2.14). To take into account the double-counting which occurs when $k = 2n$ (which occurs when $n' = 0$), we introduce a factor of $\epsilon_{n'}/2$ into the expression (and relabel $k' \mapsto k$ and $n' \mapsto n$). We are left with

$$\frac{1}{[z-\cos\psi]^\mu} = \frac{1}{2z^\mu}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)\sum_{k=0}^{\infty}\frac{(\mu)_{2k+n}}{(2k+n)!}\frac{1}{(2z)^{2k+n}}\left[\binom{2k+n}{k}+\binom{2k+n}{k+n}\right],$$

which is straightforwardly simplified by using the definition of the binomial coefficients and (2.14)

$$\frac{1}{[z-\cos\psi]^\mu} = \frac{1}{z^\mu}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)\frac{1}{(2z)^n}\sum_{k=0}^{\infty}\frac{(\mu)_{2k+n}}{k!(k+n)!}\frac{1}{4^k}\left(\frac{1}{z^2}\right)^k.$$

The second sum is given in terms of a Gauss hypergeometric series

$$\frac{1}{[z-\cos\psi]^\mu} = \sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)\frac{(\mu)_n}{n!2^n z^{\mu+n}}\,{}_2F_1\left(\frac{\mu+n}{2},\frac{\mu+n+1}{2};n+1;\frac{1}{z^2}\right). \tag{5.9}$$

This expression (5.9) with

$$z = \frac{r_1^2+r_2^2}{2r_1r_2},$$

is equivalent to Gauss' original formula (5.3) by utilization of (2.34) with

$$z = \left(\frac{r_<}{r_>}\right)^2.$$

Only now we have shown that $\mu \in \mathbf{C}$, and $z \in \{z : z \in \mathbf{C}, |z| > 1\}$. This Gauss hypergeometric function is expressible in terms of the associated Legendre function of the second kind by using (2.41) and taking $\nu \mapsto \nu - \frac{1}{2}$ and $\mu \mapsto \mu - \frac{1}{2}$.

If we substitute $\nu = n \in \mathbf{Z}$ in the hypergeometric function and take advantage of the following property of associated Legendre functions of the second kind for $z \in \mathbf{C} \setminus (-\infty, 1]$ (Cohl et al. (2000) [27])

$$Q^\mu_{-n-1/2}(z) = Q^\mu_{n-1/2}(z),$$

in (5.9), for all $n \in \mathbf{Z}$ and $\mu \in \mathbf{C}$, we obtain a complex generalization of Heine's reciprocal square root identity given as follows

$$\frac{1}{[z-\cos\psi]^\mu} = \sqrt{\frac{2}{\pi}}\frac{e^{-i\pi(\mu-1/2)}(z^2-1)^{-\mu/2+1/4}}{\Gamma(\mu)} \sum_{n=-\infty}^{\infty} e^{in\psi} Q_{n-1/2}^{\mu-1/2}(z), \tag{5.10}$$

where Heine's original identity Heine (1881) ([50], p. 286) is the case for $\mu = \frac{1}{2}$ given by

$$\frac{1}{\sqrt{z-\cos\psi}} = \frac{\sqrt{2}}{\pi} \sum_{m=-\infty}^{\infty} Q_{m-\frac{1}{2}}(z)\, e^{im\psi}. \tag{5.11}$$

(see Cohl & Tohline (1999) [26] for exact forms of these toroidal harmonics and for their recurrence relations.)

The expansion given by (5.10) is actually given in Magnus, Oberhettinger & Soni (1966) ([67], p. 182) and more recently in Conway (2007) [28]. Both of the results contained in these references contain the restriction that $\mathrm{Re}\,\mu > 0$. Recently Selvaggi et al. (2008) ([89], p. 033913-6) generalized (5.11) for μ given by odd-half-integers, even those μ less than or equal to $-\frac{1}{2}$ (this was also suggested in Cohl (2003) [22]). The main result of §5.2 is that our proof confirms that which is suggested by Selvaggi et al. (2008), i.e. that (5.10) is not only valid for $\mathrm{Re}\,\mu > 0$, but over the entire complex μ-plane.

The generalization given by (5.10) is also expressible in terms of associated Legendre functions of the first kind, through the Whipple formulae for the associated Legendre functions (Abramowitz & Stegun (1972) [1], Cohl et al. (2000) [27]) as

$$\frac{1}{[z-\cos\psi]^\mu} = \frac{(z^2-1)^{-\mu/2}}{\Gamma(\mu)} \sum_{n=-\infty}^{\infty} e^{in\psi}\Gamma(\mu-n)P_{\mu-1}^{n}\left(\frac{z}{\sqrt{z^2-1}}\right).$$

In the following section we describe some specific examples of the generalized identity, and present some interesting implications.

5.2.2 Examples and implications

We now have from (5.5), the value of the following definite integral

$$\int_{-\pi}^{\pi} \frac{\cos(nt)dt}{[z-\cos t]^\mu} = 2^{3/2}\sqrt{\pi}\frac{e^{-i\pi(\mu-1/2)}(z^2-1)^{-\mu/2+1/4}}{\Gamma(\mu)} Q_{n-1/2}^{\mu-1/2}(z). \tag{5.12}$$

Note that this equation makes sense even for μ being a negative integer, $\mu=-q$, $q\in\mathbf{N}_0$, since

$$\frac{1}{\Gamma(-q)}Q_{n-1/2}^{-q-1/2}(z) = \frac{-(-q)_n}{\Gamma(n+q+1)}Q_{n-1/2}^{q+1/2}(z), \tag{5.13}$$

where we have used the definition of the Pochhammer symbol for $(-q)_n$, (2.11), combined with negative-order condition for the associated Legendre functions of the second kind (Cohl et al. (2000) [27], p. 367).

In the previous section, we computed the integral (5.12), valid for for $z \in \mathbf{C}\backslash(-\infty, 1]$, $\mu \in \mathbf{C}$. However, we may compare this definite integral with the integral representation of the associated Legendre function of the second kind, $Q_\nu^\mu(z)$ given in Gradshteyn & Ryzhik (2007) [48], (8.713.1). In that integral representation, if we make the replacement $\mu \mapsto \mu - \frac{1}{2}$ and set $\nu = n - \frac{1}{2}$, where $n \in \mathbf{Z}$, we can obtain the same form as (5.12). However, in that reference, the restriction is given that $\operatorname{Re}\mu > 0$. There is no such restriction for μ and n for our derivation of the integral representation, (5.12); μ is arbitrary complex and $n \in \mathbf{Z}$.

The principal example for the generalized Heine identity which was first proven in Selvaggi et al. (2008) [89] (see also Magnus, Oberhettinger & Soni (1966) [67], p. 182 and Cohl (2003) [22]) for $\mu = q + \frac{1}{2}$ where $q \in \mathbf{Z}$ is given by

$$\frac{1}{[z - \cos\psi]^{q+1/2}} = \frac{2^{q+1/2}(-1)^q}{\pi(2q-1)!!(z^2-1)^{q/2}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) Q_{n-1/2}^q(z),$$

where $(2q-1)!!$ is the double factorial. For instance, for $q = -1$ (2.7) we have

$$\sqrt{z - \cos\psi} = \frac{\sqrt{z^2-1}}{\sqrt{2}\pi} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\psi)}{n^2 - \frac{1}{4}} Q_{n-1/2}^1(z),$$

and for $q = +1$

$$[z - \cos\psi]^{-3/2} = \frac{-2^{3/2}}{\pi\sqrt{z^2-1}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) Q_{n-1/2}^1(z). \tag{5.14}$$

Note that the minus sign in (5.14) expansion might seem initially troublesome, except that it is important to notice that the unit-order associated Legendre functions of the second kind are all negative in sign, as can be easily seen from the Gauss hypergeometric function representation (2.41). Using (2.65) and (2.66), the rest of the unit-order, odd-half-integer degree associated Legendre functions of the second kind can be computed, using the following recurrence relation (cf. (2.50))

$$Q_{m+1/2}^1(z) = \frac{4mz}{2m-1} Q_{m-1/2}^1(z) - \frac{2m+1}{2m-1} Q_{m-3/2}^1(z).$$

Notice that all odd-half-integer degree, integer-order associated Legendre functions can be written in terms of elliptic integrals of the first and second kind (see §2.6.3). The analogous formulas for $q = 0$ are given in Cohl & Tohline (1999) [26], (22)–(26).

Let us look at the behaviour of the generalized Heine identity (5.10) for μ being a negative-integer such that the binomial expansion should reduce to a polynomial in z. Using (5.13) and setting $\mu = -q$ in (5.10), we obtain

$$(z - \cos\psi)^q = -i\sqrt{\frac{2}{\pi}}(-1)^q(z^2-1)^{q/2+1/4}\sum_{n=0}^{q}\epsilon_n\cos(n\psi)\frac{(-q)_n}{(q+n)!}Q_{n-1/2}^{q+1/2}(z), \tag{5.15}$$

where $q \in \mathbf{N}_0$.

Let us explicitly verify that

$$(z - \cos\psi)^0 = 1.$$

The right-hand side of (5.15) is

$$\sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{1/4}}{i}\sum_{n=0}^{0}\frac{\epsilon_n\cos(n\psi)}{n!}(0)_nQ_{n-1/2}^{1/2}(z).$$

Hence for $q = 0$ in (5.15), the sum consists of merely one term. Using the associated Legendre function (2.59), the right-hand side of (5.15) reduces to 1, as expected. Similarly (5.15) can be verified to be true for all $q \in \mathbf{N}_0$.

Now we look at the behaviour of (5.10) for μ being a natural number. Substituting $\mu = q \in \mathbf{N}$ in (5.10) yields

$$\frac{1}{[z-\cos\psi]^q} = i\sqrt{\frac{2}{\pi}}\frac{(-1)^q(z^2-1)^{-q/2+1/4}}{(q-1)!}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)Q_{n-1/2}^{q-1/2}(z), \tag{5.16}$$

where the right-hand side is verified to be positive real upon examination of the definition of the associated Legendre function of the second kind in term of the Gauss hypergeometric function, (2.41). For $q = 1$ the coefficients are associated Legendre functions with odd-half-integer degree and $\frac{1}{2}$ order. These can be evaluated using (2.60) and (2.58).

If we take $z = \cosh\eta$ and $\nu = n - \frac{1}{2}$, where $n \in \mathbf{Z}$ and insert the resulting expression in (5.16) we obtain

$$\frac{1}{\cosh\eta - \cos\psi} = \frac{1}{\sinh\eta}\sum_{n=0}^{\infty}\epsilon_n e^{-n\eta}\cos(n\psi). \tag{5.17}$$

This formula is elementary. It can be obtained directly by summing the geometric series introduced by the definition of the cosine function (2.1) into the right-hand side of (5.17). Similarly if we use (2.60) and the order recurrence relation for associated Legendre functions (2.48), we are able to compute all required odd-half-integer-order associated Legendre functions appearing in (5.16).

By taking $z = \cosh\eta$ we have

$$\frac{1}{(\cosh\eta - \cos\psi)^2} = \frac{1}{\sinh^3\eta}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)e^{-n\eta}(\cosh\eta + n\sinh\eta), \tag{5.18}$$

$$\frac{1}{(\cosh\eta - \cos\psi)^3} = \frac{1}{2\sinh^5\eta} \times\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)e^{-n\eta}\left((n^2-1)\sinh^2\eta + 3n\sinh\eta\cosh\eta + 3\cosh^2\eta\right), \tag{5.19}$$

and

$$\frac{1}{(\cosh\eta - \cos\psi)^4} = \frac{1}{6\sinh^7\eta}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)e^{-n\eta} \times\Big((n^3-4n)\sinh^3\eta + (6n^2-9)\sinh^2\eta\cosh\eta + 15n\sinh\eta\cosh^2\eta + 15\cosh^3\eta\Big). \tag{5.20}$$

One way to verify these formulae is to start with the generating function for Chebyshev polynomials of the first kind (2.96) and substitute $z = \cosh\eta$, yielding (5.17). The rest of the examples can be verified by direct repeated differentiation with respect to η.

5.2.3 Closed-form expressions for certain associated Legendre functions

From (5.4), (5.9) and (5.10) we have

$$\begin{aligned} A_{\mu,n}(z) &= \frac{(\mu)_n}{2^n n! z^{\mu+n}}\,{}_2F_1\left(\frac{\mu+n}{2}, \frac{\mu+n+1}{2}; n+1; z^{-2}\right) \\ &= \sqrt{\frac{2}{\pi}}\frac{e^{-i\pi(\mu-1/2)}(z^2-1)^{-\mu/2+1/4}}{\Gamma(\mu)}Q_{n-1/2}^{\mu-1/2}(z). \end{aligned} \tag{5.21}$$

Letting

$$z = \cosh\eta = \frac{1}{2}\left(x + x^{-1}\right),$$

with $x = e^{-\eta}$, and since $\eta > 0$, and $0 < x < 1$, we have have

$$A_{\mu,n}(\cosh\eta) = \frac{(\mu)_n}{n!}2^\mu x^{\mu+n}\left(1+x^2\right)^{-(\mu+n)}\,{}_2F_1\left(\frac{\mu+n}{2}, \frac{\mu+n+1}{2}; n+1; \frac{4x^2}{(1+x^2)^2}\right).$$

Using (2.34) with $a = \mu + n$ and $b = \mu$, we get

$$A_{\mu,n}\left(\cosh\eta\right) = \frac{(\mu)_n}{n!} 2^\mu x^{\mu+n} {}_2F_1\left(\mu+n, \mu; n+1; x^2\right).$$

Using Pfaff's transformation (2.25) we obtain

$$A_{\mu,n}\left(\cosh\eta\right) = \frac{(\mu)_n}{n!} 2^\mu x^{\mu+n} \left(1-x^2\right)^{-\mu} {}_2F_1\left(\mu, 1-\mu; n+1; \frac{x^2}{x^2-1}\right).$$

Taking $\mu = q \in \mathbf{N}$, we can write

$$A_{q,n}\left(\cosh\eta\right) = \frac{(q)_n}{n!} 2^q \frac{x^{q+n}}{(1-x^2)^q} \sum_{k=0}^{q-1} \frac{(q)_k\,(1-q)_k}{(n+1)_k} \frac{1}{k!} \left(\frac{x^2}{x^2-1}\right)^k$$

or

$$\begin{aligned} A_{q,n}\left(\cosh\eta\right) &= 2^q \frac{x^{q+n}}{(1-x^2)^q} \sum_{k=0}^{q-1} \binom{q+n-1}{n+k} \binom{q+k-1}{k} (-1)^k \left(\frac{x^2}{x^2-1}\right)^k \\ &= \frac{x^n}{\left(\frac{1-x^2}{2x}\right)^q} \sum_{k=0}^{q-1} \binom{q+n-1}{n+k} \binom{q+k-1}{k} 2^{-k} \left(\frac{x}{\frac{1-x^2}{2x}}\right)^k. \end{aligned}$$

Since $x = e^{-\eta}$, we get

$$A_{q,n}\left(\cosh\eta\right) = \frac{e^{-n\eta}}{\sinh^q\left(\eta\right)} \sum_{k=0}^{q-1} \binom{q+n-1}{n+k} \binom{q+k-1}{k} \frac{e^{-k\eta}}{2^k \sinh^k \eta},$$

or for instance by using (2.14)

$$A_{q,n}(\cosh\eta) = \frac{e^{-n\eta}}{\sinh^q(\eta)} \sum_{k=0}^{q-1} \binom{n+q-1}{q-k-1} \binom{q+k-1}{q-1} \frac{e^{-k\eta}}{2^k \sinh^k \eta}.$$

Using this formula and (5.21), we are able to write down a concise formula for odd-half-integer degree, odd-half-integer-order associated Legendre functions

$$Q_{n-1/2}^{q-1/2}(z) = \sqrt{\frac{\pi}{2}} \frac{i(-1)^{q+1}(z-\sqrt{z^2-1})^n}{(q-1)!(z^2-1)^{1/4}} \sum_{k=0}^{q-1} \binom{n+q-1}{q-k-1} \binom{q+k-1}{q-1} \left[\frac{z-\sqrt{z^2-1}}{2(z^2-1)^{1/2}}\right]^k,$$

or in terms of Pochhammer symbols

$$Q_{n-1/2}^{q-1/2}(z) = i(-1)^{q+1}\sqrt{\frac{\pi}{2}}\frac{\Gamma(q+n)}{n!}\frac{(z-\sqrt{z^2-1})^n}{(z^2-1)^{1/4}}$$
$$\times\sum_{k=0}^{q-1}\frac{(q)_k(1-q)_k}{(n+1)_k k!}\left[\frac{-z+\sqrt{z^2-1}}{2(z^2-1)^{1/2}}\right]^k, \tag{5.22}$$

since

$$(-1)^k\frac{(q)_n(q)_k(1-q)_k}{(n+1)_k k!n!} = \binom{q+k-1}{k}\binom{q+n-1}{n+k}.$$

We have seen that (5.22) can be generalized using entry (31) on p. 162 of Magnus, Oberhettinger & Soni (1966) [67], $q\in\mathbf{Z}$ to obtain

$$Q_{\nu}^{q-1/2}(z) = i(-1)^{q+1}\sqrt{\frac{\pi}{2}}\frac{\Gamma\left(q+\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{(z+\sqrt{z^2-1})^{-\nu-1/2}}{(z^2-1)^{1/4}}$$
$$\times\sum_{k=0}^{|q-\frac{1}{2}|-\frac{1}{2}}\frac{(q)_k(1-q)_k}{\left(\nu+\frac{3}{2}\right)_k k!}\left[\frac{-z+\sqrt{z^2-1}}{2(z^2-1)^{1/2}}\right]^k. \tag{5.23}$$

This is a generalization of the very important formulae given by (8.6.10) and (8.6.11) in Abramowitz & Stegun (1972) [1] p. 334. Taking $z=\cosh\eta$ we have

$$Q_{\nu}^{q-1/2}(\cosh\eta) = i(-1)^{q+1}\sqrt{\frac{\pi}{2}}\frac{\Gamma\left(q+\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{e^{-\eta(\nu+1/2)}}{\sqrt{\sinh\eta}}\sum_{k=0}^{|q-\frac{1}{2}|-\frac{1}{2}}\frac{(q)_k(1-q)_k}{\left(\nu+\frac{3}{2}\right)_k k!}\frac{(-1)^k e^{-k\eta}}{2^k\sinh^k\eta}. \tag{5.24}$$

An alternative procedure for computing these associated Legendre functions is to start with (2.60) and (2.58) and use the associated Legendre function order recurrence relation. On the other hand, the expressions given by (5.23) and (5.24), directly give closed-form expressions for the associated Legendre functions, simply by evaluating a finite sum.

Not only is the generalized Heine identity useful for studying Poisson's equation in three-dimensions, but it is equally valid with fundamental solutions for k powers of the Laplacian in $\mathbf{R}^d$. These are given in terms of a functions which match exactly the left-hand side of the generalized Heine identity, particularly those in odd dimensions and in the even dimensions for $k\le\frac{n}{2}-1$. As will be seen in follow-up publications, the generalized Heine identity can be used as a powerful tool for expressing geometric properties (multi-summation addition theorems) for these unnormalized fundamental solutions in rotationally-invariant coordinate systems which yield solutions to such equations through separation of variables.

5.3 Algebraic approach to logarithmic Fourier series

The algebraic form of an unnormalized fundamental solution for d even and $k \geq d/2$ is given in (5.2). It is given in the form of $(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi)$, where $p \in \mathbf{N}_0$. For $p = 0$ the result is well-known (see for instance Magnus, Oberhettinger & Soni (1966) [67] p. 259)

$$\log(\cosh\eta - \cos\psi) = \eta - \log 2 - 2\sum_{n=1}^{\infty} \frac{e^{-n\eta}}{n}\cos(n\psi). \tag{5.25}$$

Now let us examine the $p = 1$ case

$$\begin{aligned}(\cosh\eta - \cos\psi)\log(\cosh\eta - \cos\psi) &= (\eta - \log 2)(\cosh\eta - \cos\psi) \\ &\quad - 2\sum_{n=1}^{\infty} \frac{e^{-n\eta}}{n}\cos(n\psi)(\cosh\eta - \cos\psi).\end{aligned} \tag{5.26}$$

Taking advantage of the formula

$$\cos(n\psi)\cos\psi = \frac{1}{2}\left\{\cos[(n+1)\psi] + \cos[(n-1)\psi]\right\}, \tag{5.27}$$

which is a direct consequence of (2.3), we have

$$\begin{aligned}(\cosh\eta - \cos\psi)\log(\cosh\eta - \cos\psi) &= (\eta - \log 2)\cosh\eta - (\eta - \log 2)\cos\psi \\ - 2\cosh\eta\sum_{n=1}^{\infty} \frac{e^{-n\eta}}{n}\cos(n\psi) &+ \sum_{n=1}^{\infty} \frac{e^{-n\eta}}{n}\cos[(n+1)\psi] + \sum_{n=1}^{\infty} \frac{e^{-n\eta}}{n}\cos[(n-1)\psi].\end{aligned} \tag{5.28}$$

Collecting the contributions to the Fourier cosine series, we obtain (compare with (5.18))

$$\begin{aligned}(\cosh\eta - \cos\psi)\log(\cosh\eta - \cos\psi) &= (1 + \eta - \log 2)\cosh\eta - \sinh\eta \\ + \cos\psi\left(\log 2 - 1 - \eta - \frac{1}{2}e^{-2\eta}\right) &+ 2\sum_{n=2}^{\infty} \frac{e^{-n\eta}\cos n\psi}{n(n^2-1)}(\cosh\eta + n\sinh\eta).\end{aligned} \tag{5.29}$$

For $p = 2$ and using (5.27), we similarly have

$$(\cosh\eta - \cos\psi)^2 \log(\cosh\eta - \cos\psi) = (\eta - \log 2)\cosh^2\eta - 2(\eta - \log 2)\cosh\eta\cos\psi$$
$$+ (\eta - \log 2)\cos^2\psi - (2\cosh^2\eta + 1)\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos n\psi + 2\cosh\eta\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+1)\psi]$$
$$+ 2\cosh\eta\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n-1)\psi] - \frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+2)\psi] - \frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n-2)\psi].$$

Collecting the contributions to the Fourier cosine series, we obtain (compare with (5.19))

$$(\cosh\eta - \cos\psi)^2 \log(\cosh\eta - \cos\psi) = (\eta - \log 2)\left(\cosh^2\eta + \frac{1}{2}\right) + 2(\cosh\eta)e^{-\eta} - \frac{1}{4}e^{-2\eta}$$
$$+ \left(-2(\eta - \log 2)\cosh\eta - \left(2\cosh^2\eta + \frac{3}{2}\right)e^{-\eta} + (\cosh\eta)\,e^{-2\eta} - \frac{1}{6}e^{-3\eta}\right)\cos\psi$$
$$+ \left(\frac{1}{2}(\eta - \log 2) + 2(\cosh\eta)e^{-\eta} - \frac{1}{2}(2\cosh^2\eta + 1)e^{-2\eta} + \frac{2}{3}(\cosh\eta)e^{-3\eta} - \frac{1}{8}e^{-4\eta}\right)\cos 2\psi$$
$$- 4\sum_{n=3}^{\infty}\frac{e^{-n\eta}\cos(n\psi)}{n(n^2-1)(n^2-4)}\left[(n^2-1)\sinh^2\eta + 3n\sinh\eta\cosh\eta + 3\cosh^2\eta\right]. \tag{5.30}$$

We now see how to generalize this process. Starting with (5.25) and repeatedly multiplying by factors of $\cosh\eta - \cos\psi$, we see that the general Fourier expansion is by

$$(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) = (\eta - \log 2)(\cosh\eta - \cos\psi)^p$$
$$+ 2\sum_{k=-p}^{p}(-1)^{k+1}R_p^k(\cosh\eta)\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+k)\psi], \tag{5.31}$$

where $R_p^k(x)$ are what we refer to as logarithmic polynomials with $x = \cosh\eta$ (in our notation p and k are both indices) which are nonvanishing only for $k \in \{-p, \ldots, 0, \ldots, p\}$, and are satisfied by the following recurrence relation

$$R_p^k(x) = \frac{1}{2}R_{p-1}^{k-1}(x) + xR_{p-1}^k(x) + \frac{1}{2}R_{p-1}^{k+1}(x).$$

We see that $R_0^0(x) = 1$ from (5.25), which gives us the starting point for the recursion. These polynomials are even in the index k, i.e.

$$R_p^k(x) = R_p^{-k}(x).$$

The derivative of the polynomials are given by

$$\frac{d}{dx}R_p^{\pm k}(x) = pR_{p-1}^{\pm k}(x).$$

Some of the first few logarithmic polynomials are given by

$$R_0^0(x) = 1$$

$$R_1^0(x) = x,\ R_1^{\pm 1}(x) = \frac{1}{2}$$

$$R_2^0(x) = \frac{1}{2} + x^2,\ R_2^{\pm 1}(x) = x,\ R_2^{\pm 2}(x) = \frac{1}{4}$$

$$R_3^0(x) = \frac{3}{2}x + x^3,\ R_3^{\pm 1}(x) = \frac{3}{8} + \frac{3}{2}x^2,\ R_3^{\pm 2}(x) = \frac{3}{4}x,\ R_3^{\pm 3}(x) = \frac{1}{8}$$

$$R_4^0(x) = \frac{3}{8} + 3x^2 + x^4,\ R_4^{\pm 1}(x) = \frac{3}{2}x + 2x^3,\ R_4^{\pm 2}(x) = \frac{1}{4} + \frac{3}{2}x^2,\ R_4^{\pm 3}(x) = \frac{1}{2}x,\ R_4^{\pm 4}(x) = \frac{1}{16}.$$

We can find the generating function for the polynomials $R_p^k(x)$ as follows. Let

$$F(x, y, z) = \sum_{p=0}^{\infty} \sum_{k=-\infty}^{\infty} R_p^k(x) y^k z^p$$

be the generating function for the polynomials $R_p^k(x)$. If we define the function

$$S_p(x, y) = \sum_{k=-\infty}^{\infty} R_p^k(x) y^k,$$

then using the recurrence relation for $R_p^k(x)$ we can show

$$S_p(x, y) = \left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right) S_{p-1}(x, y).$$

Combining this result along with the fact that $R_0^0(x) = 1$ we have

$$S_p(x, y) = \left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right)^p,$$

so therefore the generating function for the polynomials $R_p(x)$ is given by

$$F(x,y,z) = \frac{1}{1 - z\left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right)}.$$

We can re-write (5.31) by rearranging the order to the k and n summations

$$\begin{aligned}(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) &= (\eta - \log 2)(\cosh\eta - \cos\psi)^p \\ + \sum_{n=1}^{p-1} \cos(n\psi) e^{n\eta}\, \mathfrak{r}_{-n,p}^{-n-1,-p}(\cosh\eta) &+ \sum_{n=0}^{p} \cos(n\psi) e^{-n\eta}\, \mathfrak{r}_{n,p}^{-p,-1}(\cosh\eta) \\ + \sum_{n=1}^{p} \cos(n\psi) e^{-n\eta}\, \mathfrak{r}_{n,p}^{0,n-1}(\cosh\eta) &+ \sum_{n=p+1}^{\infty} \frac{\cos(n\psi) e^{-n\eta}}{(n^2-p^2)\cdots(n^2-1)n} \mathfrak{R}_{n,p}(\cosh\eta), \end{aligned} \tag{5.32}$$

where

$$\mathfrak{r}_{n,p}^{k_1,k_2}(\cosh\eta) := \sum_{k=k_1}^{k_2} \frac{\rho_p^k(\cosh\eta)}{n-k},$$

$$\rho_p^k(\cosh\eta) := 2(-1)^{k+1} e^{k\eta} R_p^k(\cosh\eta),$$

and

$$\mathfrak{R}_{n,p}(\cosh\eta) := \sum_{k=-p}^{p} \frac{(n+p)!\rho_p^k(\cosh\eta)}{(n-p-1)!(n-k)}.$$

Note that the above expression is well-behaved because $\mathfrak{R}_{n,p}$ is only defined for $n \ge p+1$. It is also interesting to notice that for $\eta > 0$ we may write e^η and therefore η as a function of $\cosh\eta$ since

$$\sinh\eta = \sqrt{\cosh^2\eta - 1},$$

$$e^\eta = \cosh\eta + \sqrt{\cosh^2\eta - 1},$$

and therefore

$$\eta = \log\left(\cosh\eta + \sqrt{\cosh^2\eta - 1}\right).$$

If we make the following definition

$$\mathfrak{P}_{n,p}(\cosh\eta) = \begin{cases} \mathfrak{r}_{0,p}^{-p,-1}(\cosh\eta) & \text{if } n = 0, \\ e^{-n\eta}\mathfrak{r}_{n,p}^{-p,-1}(\cosh\eta) + D_{n,p}(\cosh\eta) + E_{n,p}(\cosh\eta) & \text{if } 1 \leq n \leq p-1, \\ e^{-p\eta}\mathfrak{r}_{p,p}^{-p,p-1}(\cosh\eta) & \text{if } n = p, \\ \dfrac{e^{-n\eta}}{(n^2-p^2)\cdots(n^2-1)n}\,\mathfrak{R}_{n,p}(\cosh\eta) & \text{if } n \geq p+1, \end{cases}$$

where

$$D_{n,p}(\cosh\eta) = \begin{cases} e^{n\eta}\mathfrak{r}_{-n,p}^{-n-1,-p}(\cosh\eta) & \text{if } p \geq 2, \\ 0 & \text{if } p \in \{0,1\}, \end{cases}$$

and

$$E_{n,p}(\cosh\eta) = \begin{cases} e^{-n\eta}\mathfrak{r}_{n,p}^{0,n-1}(\cosh\eta) & \text{if } p \geq 1, \\ 0 & \text{if } p = 0, \end{cases}$$

then we can write (5.32) as

$$(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) = (\eta - \log 2)(\cosh\eta - \cos\psi)^p + \sum_{n=0}^{\infty} \cos(n\psi)\mathfrak{P}_{n,p}(\cosh\eta). \tag{5.33}$$

In fact, if we use (5.15) then we can write

$$(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) = i(\eta - \log 2)\sqrt{\frac{2}{\pi}}(-1)^{p+1}(\sinh\eta)^{p+1/2}\sum_{n=0}^{p}\epsilon_n\cos(n\psi)\frac{(-p)_n}{(p+n)!}Q_{n-1/2}^{p+1/2}(\cosh\eta) + \sum_{n=0}^{\infty}\cos(n\psi)\mathfrak{P}_{n,p}(\cosh\eta). \tag{5.34}$$

If we define

$$\mathfrak{Q}_{n,p}(\cosh\eta) = \mathfrak{R}_{n,p}(\cosh\eta) + i\epsilon_n(\eta - \log 2)\sqrt{\frac{2}{\pi}}(-1)^{p+1}(\sinh\eta)^{p+1/2}\frac{(-p)_n}{(p+n)!}Q_{n-1/2}^{p+1/2}(\cosh\eta),$$

then we have

$$(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) = \sum_{n=0}^{\infty}\cos(n\psi)\mathfrak{Q}_{n,p}(\cosh\eta). \tag{5.35}$$

We shall prove in (6.46) in §6.3 that the "ending" function $\mathfrak{R}_{n,p}$ (cf. (5.32), (5.29), (5.30), (5.18), and (5.19)) is directly related to associated Legendre functions of the second kind. Explicitly,

$$\mathfrak{R}_{n,q}(\cosh\eta) = 2\sqrt{\frac{2}{\pi}} iq! e^{n\eta} (\sinh\eta)^{q+1/2} Q_{n-1/2}^{q+1/2}(\cosh\eta). \tag{5.36}$$

Therefore we have

$$\frac{1}{(\cosh\eta - \cos\psi)^q} = \frac{(-1)^q}{2[(q-1)!]^2 \sinh^{2q-1}\eta} \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) e^{-n\eta} \mathfrak{R}_{n,q-1}(\cosh\eta), \tag{5.37}$$

and also that

$$\begin{aligned}\Big(\cosh\eta - \cos\psi\Big)^p \log(\cosh\eta - \cos\psi) &= (\eta - \log 2)\Big(\cosh\eta - \cos\psi\Big)^p \\ &+ \sum_{n=1-p}^{-1} \cos(n\psi) e^{-n\eta}\, \mathfrak{r}_{n,p}^{n-1,-p}(\cosh\eta) + \sum_{n=0}^{p} \cos(n\psi) e^{-n\eta}\, \mathfrak{r}_{n,p}^{-p,-1}(\cosh\eta) \\ &+ \sum_{n=1}^{p} \cos(n\psi) e^{-n\eta}\, \mathfrak{r}_{n,p}^{0,n-1}(\cosh\eta) \\ &+ 2\sqrt{\frac{2}{\pi}} ip!\, (\sinh\eta)^{p+1/2} \sum_{n=p+1}^{\infty} \frac{\cos(n\psi)}{(n^2-p^2)\cdots(n^2-1)n} Q_{n-1/2}^{p+1/2}(\cosh\eta).\end{aligned} \tag{5.38}$$

This establishes a firm connection between the results in this section and the potential theoretic results in §5.2.

5.4 Fourier expansions

In this section we take advantage of the above derived Fourier cosine series to compute Fourier expansions for an unnormalized fundamental solution for powers of the Laplacian in $\mathbf{R}^d$.

5.4.1 Azimuthal Fourier expansions in pure and mixed coordinates

Fourier expansions of fundamental solutions for powers of the Laplacian in pure and mixed Euclidean-hyperspherical coordinate systems can now be obtained. We write down the algebraic expressions for unnormalized fundamental solutions in §5.1 and in these coordinate systems use the Fourier cosine series obtained this chapter to write down the Fourier expansions in these coordinate systems.

The quintessential example of mixed Euclidean-hyperspherical coordinates is given by circular cylindrical coordinates in $\mathbf{R}^3$ i.e. type **a** $(3,2)$ Euclidean-hyperspherical coordinates (see Figure 4.11). In this case we have

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|}=\frac{1}{\sqrt{2rr'}}\frac{1}{[\chi-\cos(\phi-\phi')]^{1/2}},$$

where

$$\chi=\frac{r^2+r'^2+(x_3-x_3')^2}{2rr'}.$$

By applying (5.11) we have

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|}=\frac{1}{\pi\sqrt{rr'}}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}Q_{m-1/2}(\chi).$$

More generally, in type **a** $(d,2)$ Euclidean-hyperspherical coordinates (see Figure 4.11), for arbitrary powers of ν using (5.10) we have

$$\|\mathbf{x}-\mathbf{x}'\|^\nu=\sqrt{\frac{\pi}{2}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}(2rr')^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi),$$

where

$$\chi=\frac{r^2+r'^2+(x_3-x_3')^2+\ldots+(x_d-x_d')^2}{2rr'}.$$

Another example of Fourier expansions for mixed Euclidean-hyperspherical coordinates is type **ba** $(d,3)$ Euclidean-hyperspherical coordinates (see Figure 4.12). The Fourier expansion for arbitrary powers of ν is

$$\|\mathbf{x}-\mathbf{x}'\|^\nu=\sqrt{\frac{\pi}{2}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}(2rr'\sin\theta\sin\theta')^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi),$$

where

$$\chi=\frac{r^2+r'^2-2rr'\cos\theta\cos\theta'+(x_4-x_4')^2+\ldots+(x_d-x_d')^2}{2rr'\sin\theta\sin\theta'}.$$

Many more examples of Fourier expansions are now possible in general mixed Euclidean-hyperspherical coordinate systems. Take for instance standard type $\mathbf{b}^{k-2}\mathbf{a}$ (d,k) Euclidean-hyperspherical coordinates (see Figure 4.13). In these coordinates we have

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \sqrt{\frac{\pi}{2}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}\left(2rr'\prod_{i=1}^{k-2}\sin\theta_i \sin\theta_i{}'\right)^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}$$
$$\times\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi_k^d), \tag{5.39}$$

where χ_k^d is given in (4.34).

The fact that these Fourier expansions are valid in every rotationally invariant coordinate system which contains these embedded k-dimensional hyperspherical coordinate systems (of which there are many), and yields solutions via separation of variables presents a powerful method for constructing multi-integration addition theorems/definite integrals (for an example of this utility in $\mathbf{R}^3$ see Cohl et al. (2000) [27]). Your ability to construct these addition theorems rests on your capacity to generate eigenfunction expansions (or integrals) for fundamental solutions in certain coordinate systems. It happens that for pure hyperspherical coordinate systems, this capacity is available, and this is the subject of the following chapter.

In the pure standard hyperspherical coordinate system (4.31) we have

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \sqrt{\frac{\pi}{2}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}\left(2rr'\prod_{i=1}^{k-2}\sin\theta_i \sin\theta_i{}'\right)^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}$$
$$\times\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi_d^d),$$

where χ_d^d is given in (4.35), and many more examples of such expressions may be written down in any other pure hyperspherical coordinate system using the appropriate expansion of $\cos\gamma$ in the distance function to generate the toroidal parameter.

5.4.2 Fourier expansions for the separation angle

It should be noted that an unnormalized fundamental solution for powers of the Laplacian in $\mathbf{R}^d$ can also be Fourier expanded in terms of a pure hyperspherical separation angle, γ. They are computed using the specific point parametrization for $\cos\gamma$ on the unit-hyper-sphere $\mathbf{S}^{d-1}$. In pure hyperspherical coordinates we have fundamental solutions for powers of the

Laplacian in $\mathbf{R}^d$ through (5.1) and (5.2) with $\kappa = d$, namely for d odd, and for d even with $k \le d/2 - 1$ we have

$$\mathfrak{g}_k^d(\mathbf{x}, \mathbf{x}') = (2r_d r_d')^{k-d/2} \left[\chi_d^d - \cos\gamma_d\right]^{k-d/2},$$

where

$$\chi_d^d = \frac{r_d^2 + {r_d'}^2}{2r_d r_d'}, \tag{5.40}$$

$\cos\gamma_d$ is given in (4.38), and for d even with $k \ge d/2$ we have

$$\mathfrak{g}_k^d(\mathbf{x}, \mathbf{x}') = (2r_d r_d')^p \left[\frac{1}{2}\log(2r_d r_d') - \beta_{k-d/2}\right] \left[\chi_d^d - \cos\gamma_d\right]^p$$
$$+ \frac{1}{2}(2r_d r_d')^p \left[\chi_d^d - \cos\gamma_d\right]^p \log\left(\chi_d^d - \cos\gamma_d\right).$$

Now we can use the previous Fourier cosine series results from this chapter to compute separation angle Fourier expansions in pure hyperspherical coordinate systems for fundamental solutions of the Laplacian in $\mathbf{R}^d$.

For instance in $\mathbf{R}^3$ we can use Heine's reciprocal square root identity (5.11) to reproduce the result in Cohl et al. (2001) [25], namely

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{1}{\pi\sqrt{r_3 r_3'}} \sum_{n=-\infty}^{\infty} e^{in\gamma} Q_{n-1/2}(\chi_3^3), \tag{5.41}$$

where $\cos\gamma$ is given in (4.18) for type **ba** coordinates. However (5.41) is also valid in type $\mathbf{b'a}$ coordinates. In that case the only modification is to use (4.20) to define our γ.

Take also for instance, type $\mathbf{b}^2\mathbf{a}$, $\mathbf{b'}^2\mathbf{a}$, $\mathbf{bb'a}$, $\mathbf{b'ba}$ and $\mathbf{ca}^2$ coordinates in $\mathbf{R}^4$. Then if we use (5.17), we have

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|^2} = \frac{1}{r_>^2 - r_<^2} \sum_{n=0}^{\infty} \epsilon_n \cos(n\gamma) \left(\frac{r_<}{r_>}\right)^n,$$

where $r_\lessgtr$ is defined in the Glossary and $\cos\gamma$ is given by (4.22), (4.28), (4.24), (4.26), and (4.30).

These types of expansions are now possible in all pure hyperspherical coordinate systems because of our derivation of the Fourier series of unnormalized fundamental solutions for powers of the Laplacian. Using (5.10) for any pure hyperspherical coordinate system in $\mathbf{R}^d$, we now have the separation angle Fourier expansion for arbitrary powers of the distance, namely

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|^s} = \frac{e^{i\pi(1-s)/2}(r_>^2 - r_<^2)^{(1-s)/2}}{\sqrt{\pi r r'}\Gamma(s/2)} \sum_{n=-\infty}^{\infty} e^{in\gamma} Q_{n-1/2}^{(1-s)/2}(\chi),$$

for $s \in \mathbf{C}$ except if $s \in \{0, -2, -4, \ldots\}$, i.e. $s = -2p$ for $p \in \mathbf{N}_0$, then we have

$$\|\mathbf{x}-\mathbf{x}'\|^{2p} = \frac{i(-1)^{p+1}}{\sqrt{rr'}}(r_>^2 - r_<^2)^{p+1/2} \sum_{n=0}^{\infty} \cos(n\gamma) \frac{(-p)_n}{(p+n)!} Q_{n-1/2}^{p+1/2}(\chi). \tag{5.42}$$

Through simple algebra we have the separation angle Fourier expansion for the logarithmic fundamental solutions, namely through (5.35)

$$\|\mathbf{x}-\mathbf{x}'\|^{2p} \log \|\mathbf{x}-\mathbf{x}'\| = \log(2rr')\|\mathbf{x}-\mathbf{x}'\|^{2p} + (2rr')^p \sum_{n=0}^{\infty} \cos(n\gamma) \mathfrak{Q}_{n,p}(\chi),$$

where the expansion of $\|\mathbf{x}-\mathbf{x}'\|^{2p}$ is given by (5.42).

6

Addition theorems for pure hyperspherical coordinates in R^d

In §6.1 we review the addition theorem for hyperspherical harmonics and in §6.2 we review the computation of normalized hyperspherical harmonics in $\mathbf{R}^d$. We shall use these results to derive, in §6.3, new Gegenbauer polynomial expansions in $\mathbf{R}^d$ of a fundamental solution for powers of the Laplacian in $\mathbf{R}^d$. In §6.4 we present new multi-summation addition theorems for hyperspherical harmonics in $\mathbf{R}^d$. Fundamental background material for this chapter is in Vilenkin (1968) [100], Izmest′ev et al. (2001) [59], and Wen & Avery (1985) [106]. For directly relevant discussions, see §10.2.1 in Fano & Rau (1996), Chapter 9 in Andrews, Askey & Roy (1999) [3] and especially Chapter XI in Erdélyi et al. Vol. II (1981) [36].

The aim of this chapter is to compute, for a general hyperspherical coordinate system, new addition theorems using basis function expansions for unnormalized fundamental solutions of Laplace's equation in $\mathbf{R}^d$. This is a generalization of the $\mathbf{R}^3$ results in Cohl et al. (2001) [25]. In that reference, we used the well-known addition theorem for spherical harmonics in order to obtain a new addition theorem for spherical harmonics, one that involves a toroidal harmonic. In the present case we have Fourier expansions of unnormalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$ and we generalize our previous result for expansions in terms of hyperspherical harmonics, the normalized angular solutions to Laplace's equation in a general hyperspherical coordinate system in $\mathbf{R}^d$.

6.1 Addition theorem for hyperspherical harmonics

The *addition theorem for spherical harmonics* (in $\mathbf{R}^3$) is given by

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{l,m}(\widehat{\mathbf{x}})Y^*_{l,m}(\widehat{\mathbf{x}}'), \tag{6.1}$$

where P_l is a Legendre polynomial of degree $l \in \mathbf{N}_0$, γ is the separation angle defined by (4.14), $m \in \{-l,\ldots,0,\ldots,l\}$, $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{x}}'$ are the unit vectors in the direction of $\mathbf{x},\mathbf{x}' \in \mathbf{R}^3$ respectively, and $Y_{l,m}(\widehat{\mathbf{x}})$ is the normalized angular solution to Laplace's equation in spherical coordinates, the *spherical harmonics*.

The *addition theorem for hyperspherical harmonics*, which reduces to (6.1) for $d=3$, is given in terms of Gegenbauer polynomials. For the proof of the addition theorem, see Wen & Avery (1985) [106] and for a relevant discussion, see §10.2.1 in Fano & Rau (1996) [38]. We use the addition theorem for hyperspherical harmonics to generate basis function expansions for unnormalized fundamental solutions of Laplace's equation in $\mathbf{R}^d$, in order to generate new addition theorems in terms of the coefficients of the Fourier expansions computed in the previous chapter. The addition theorem for hyperspherical harmonics is

$$C_\lambda^{d/2-1}(\cos\gamma) = \frac{N_d}{(2\lambda+d-2)(d-4)!!}\sum_{\mu\in\boldsymbol{\mu}} Y_{\lambda,\mu}(\widehat{\mathbf{x}})Y^*_{\lambda,\mu}(\widehat{\mathbf{x}}'), \tag{6.2}$$

where $Y_{\lambda,\mu}(\widehat{\mathbf{x}})$ for $\lambda \in \mathbf{N}_0$ and $\mu \in \boldsymbol{\mu}$ are the normalized separable hyperspherical harmonics, $\boldsymbol{\mu}$ stands for a set of $d-2$ quantum numbers identifying degenerate harmonics for each λ, $\boldsymbol{\lambda} := \{\lambda\} \cup \boldsymbol{\mu}$ represents the set of all angular momentum quantum numbers in a general hyperspherical coordinate system, $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{x}}'$ are the unit vectors in the direction of $\mathbf{x},\mathbf{x}' \in \mathbf{R}^d$ respectively, γ is the separation angle (4.14), and N_d is defined below. The proper normalization of the hyperspherical harmonics is given by the following integral

$$\int d\Omega_d Y_{\lambda,\mu}(\widehat{\mathbf{x}})Y^*_{\lambda',\mu'}(\widehat{\mathbf{x}}) = \delta_{\lambda,\lambda'}\ \delta_{\mu,\mu'}.$$

The degeneracy of the space of hyperspherical harmonics as a function of d and λ is given by

$$\dim(d) = \frac{(d-2+2\lambda)(d-3+\lambda)!}{\lambda!(d-2)!} \tag{6.3}$$

(see (9.2.11) in Vilenkin (1968) [100]). The degeneracy (6.3) as a function of the dimension d tells you how many linearly independent solutions exist for a particular λ value. The total number of linearly independent solutions are described uniquely by the sum over $\boldsymbol{\mu}$ in (6.2) for each λ. Note that this formula reduces to the standard result in $d = 3$ with a degeneracy given by $2\lambda + 1$ and in $d = 4$ with a degeneracy given by $(\lambda + 1)^2$. The "surface area" S_d (the $(d-1)$-dimensional volume of the $(d-1)$-dimensional hyper-sphere at the boundary of the d-dimensional hyper-sphere) of the d-dimensional unit hyper-sphere is given in terms of

$$\int d\Omega_d = S_d := \frac{N_d}{(d-2)!!},$$

where $d\Omega_d$ is the infinitesimal element of the solid angle,

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \begin{cases} \dfrac{(2\pi)^{d/2}}{(d-2)!!} & \text{if } d \text{ even,} \\ \dfrac{2(2\pi)^{(d-1)/2}}{(d-2)!!} & \text{if } d \text{ odd,} \end{cases}$$

and therefore

$$N_d := \frac{2\pi^{d/2}(d-2)!!}{\Gamma(d/2)} = \begin{cases} (2\pi)^{d/2} & \text{if } d \text{ even,} \\ 2(2\pi)^{(d-1)/2} & \text{if } d \text{ odd.} \end{cases} \tag{6.4}$$

6.2 Hyperspherical harmonics

The general basis functions that one gets by putting coordinates on the d-dimensional hyper-sphere $\mathbf{S}^d$, can be specified as solutions to the angular part of Laplace's equation on $\mathbf{R}^{d+1}$ (these correspond to separated solutions of Laplace's equation, using the Laplace-Beltrami operator, on the hyper-sphere $\mathbf{S}^d$) as

$$\Psi = \prod_{k=1}^{d} \Psi_k(\theta_k).$$

The following numbers are associated with each cell m, l, l_α, l_β (see Figure 4.1 in §4.2). The number of vertices above each branching node l_α and l_β are represented by S_α and S_β respectively. The numbers $m \in \mathbf{Z},\ l, l_\alpha, l_\beta \in \mathbf{N}_0$ label representations for the corresponding rotation subgroup in the subgroup chain, i.e. angular momentum type quantum numbers (see §4.1 and especially Izmest′ev et al. (2001) [59]).

The following eigenfunctions are generated at each branching node for normalized hyperspherical harmonics using the method of trees:

- Type **a**:

$$\Psi_m(\phi_a) = \frac{1}{\sqrt{2\pi}} e^{im\phi_a}; \quad m \in \mathbf{Z}. \tag{6.5}$$

- Type **b**:

$$\left.\begin{aligned} \Psi^{\alpha}_{n,l_\beta}(\theta_b) &= N_n^{\alpha,\alpha}(\sin\theta_b)^{l_\beta} P_n^{(\alpha,\alpha)}(\cos\theta_b); \\ n = l - l_\beta, \quad \alpha &= l_\beta + \frac{S_\beta}{2}, \quad n \in \mathbf{N}_0 \end{aligned}\right\} \tag{6.6}$$

  where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.

- Type **b′**:

$$\left.\begin{aligned} \Psi^{\beta}_{n,l_\alpha}(\theta_{b'}) &= N_n^{\beta,\beta}(\cos\theta_{b'})^{l_\alpha} P_n^{(\beta,\beta)}(\sin\theta_{b'}); \\ n = l - l_\alpha, \quad \beta &= l_\alpha + \frac{S_\alpha}{2}, \quad n \in \mathbf{N}_0. \end{aligned}\right\} \tag{6.7}$$

- Type **c**:

$$\left.\begin{aligned} &\Psi^{\alpha,\beta}_{n,l_\beta,l_\alpha}(\vartheta_c) = 2^{(\alpha+\beta)/2+1} N_n^{\alpha,\beta}(\sin\vartheta_c)^{l_\beta}(\cos\vartheta_c)^{l_\alpha} P_n^{(\alpha,\beta)}(\cos 2\vartheta_c); \\ &n = \frac{1}{2}\left(l - l_\alpha - l_\beta\right), \quad \alpha = l_\beta + \frac{S_\beta}{2}, \quad \beta = l_\alpha + \frac{S_\alpha}{2}, \quad n \in \mathbf{N}_0. \end{aligned}\right\} \tag{6.8}$$

Please see Figure 4.1. The normalization constants are given by

$$N_n^{\alpha,\beta} = \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}},$$

such that

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{\delta_{m,n}}{\left(N_n^{\alpha,\beta}\right)^2},$$

where $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > -1$ (see (7.391.1) in Gradshteyn & Ryzhik (2007) [48]). Notice that the eigenfunctions for cells of type **b** and **b′** can be expressed in terms of Gegenbauer polynomials using (2.93). We can re-write (6.6) as

$$\Psi^{\alpha}_{n,l_\beta}(\theta_b) = \frac{(2\alpha)!}{\Gamma(\alpha+1)}\sqrt{\frac{(2\alpha+2n+1)n!}{2^{2\alpha+1}(2\alpha+n)!}}(\sin\theta_b)^{l_\beta} C_n^{\alpha+1/2}(\cos\theta_b),$$

and (6.7) as

$$\Psi^{\beta}_{n,l_\alpha}(\theta_{b'}) = \frac{(2\beta)!}{\Gamma(\beta+1)}\sqrt{\frac{(2\beta+2n+1)n!}{2^{2\beta+1}(2\beta+n)!}}(\sin\theta_{b'})^{l_\alpha}C_n^{\beta+1/2}(\cos\theta_{b'}).$$

6.2.1 Examples of hyperspherical harmonics

The simplest example occurs in $d = 2$, where the only hyperspherical coordinate system generated is of type **a** through (4.16, see Figure 4.2). In this case there is one angle $\phi \in [0, 2\pi)$ in correspondence with an azimuthal quantum number $m \in \mathbf{Z}$, the only branching node is of type **a** and the normalized harmonics are

$$Y_m(\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}}.$$

In $d = 3$ there are two possible coordinate systems, that of type **ba** (4.17), and **b′a** (4.19). In both cases there are two branching nodes. First we treat the type **ba** coordinate system (see Figure 4.3) At the root branching node, there is an angle $\theta \in [0, \pi]$ which we correspond to an angular momentum quantum number $l \in \mathbf{N}_0$ (6.6) and at the type **a** cell we have $m \in \mathbf{Z}$ (6.5). Using (6.6) we see that $\alpha = m$, $n = l - m$, $l_\beta = m$, and $S_\beta = 0$ since there are no vertices above the branching node m. Through reduction and multiplication by the ϕ eigenfunction, the normalized spherical harmonics are

$$Y_{l,m}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}. \tag{6.9}$$

These are the standard spherical harmonics.

Now we treat the type **b′a** coordinate system (see Figure 4.4). This tree is corresponding to this coordinate system is equivalent to the previous tree. At the root branching node, there is an angle $\theta \in [-\pi/2, \pi/2]$ which we correspond to an angular momentum quantum number $l \in \mathbf{N}_0$ (6.7) and at the type **a** cell we have $m \in \mathbf{Z}$ (6.5). Using (6.7) we see that $\alpha = m$, $n = l - m$, $l_\alpha = m$, and $S_\alpha = 0$ since there are no vertices above the branching node m. Through reduction and multiplication by the ϕ eigenfunction, the normalized spherical harmonics are

$$Y_{l,m}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\sin\theta)e^{im\phi}. \tag{6.10}$$

For certain combinations of indices, Gegenbauer polynomials reduce to Legendre polynomials (2.97). We have relied on (2.97) in our use of (6.6) and (6.7), for these particular trees, to derive the spherical harmonics (6.9) and (6.10). The natural domain of the function P_l^m is

$[-1,1]$ (see for instance §12.5 of Arfken & Weber (1995) [5]). It will be understood that the range of θ is chosen to be $\theta \in [0,\pi]$ or $\theta \in [-\pi/2,\pi/2]$ for nodes of type $\mathbf{b}$ and $\mathbf{b}'$ respectively, so that the appropriate domain for the associated Legendre functions is taken.

In $d = 4$ there are five possible pure hyperspherical coordinate systems, those of type $\mathbf{b}^2\mathbf{a}$ (4.21), $\mathbf{bb}'\mathbf{a}$ (4.23), $\mathbf{b}'\mathbf{ba}$ (4.25), $\mathbf{b}'^2\mathbf{a}$ (4.27), and $\mathbf{ca}^2$ (4.29). Spherical coordinates in $d = 4$ can be represented in one of five ways.

The normalized hyperspherical harmonic in type $\mathbf{b}^2\mathbf{a}$ coordinates (see Figure 4.5) is

$$\begin{aligned} Y_{l_1,l_2,m}(\theta_1,\theta_2,\phi) = {} & \frac{(-1)^m(2l_2)!!}{\pi}\sqrt{\frac{(2l_2+1)(l_1+1)(l_1-l_2)!(l_2-m)!}{2(l_1+l_2+1)!(l_2+m)!}} \\ & \times (\sin\theta_1)^{l_2} C_{l_1-l_2}^{l_2+1}(\cos\theta_1) P_{l_2}^m(\cos\theta_2) e^{im\phi}. \end{aligned} \tag{6.11}$$

The normalized hyperspherical harmonic in type $\mathbf{b}'^2\mathbf{a}$ coordinates (see Figure 4.8) is

$$\begin{aligned} Y_{l_1,l_2,m}(\theta_1,\theta_2,\phi) = {} & \frac{(-1)^m(2l_2)!!}{\pi}\sqrt{\frac{(2l_2+1)(l_1+1)(l_1-l_2)!(l_2-m)!}{2(l_1+l_2+1)!(l_2+m)!}} \\ & \times (\cos\theta_1)^{l_2} C_{l_1-l_2}^{l_2+1}(\sin\theta_1) P_{l_2}^m(\sin\theta_2) e^{im\phi}. \end{aligned}$$

The normalized hyperspherical harmonic in type $\mathbf{bb}'\mathbf{a}$ coordinates (see Figure 4.6) is

$$\begin{aligned} Y_{l_1,l_2,m}(\theta_1,\theta_2,\phi) = {} & \frac{(-1)^m(2l_2)!!}{\pi}\sqrt{\frac{(2l_2+1)(l_1+1)(l_1-l_2)!(l_2-m)!}{2(l_1+l_2+1)!(l_2+m)!}} \\ & \times (\sin\theta_1)^{l_2} C_{l_1-l_2}^{l_2+1}(\cos\theta_1) P_{l_2}^m(\sin\theta_2) e^{im\phi}. \end{aligned}$$

The normalized hyperspherical harmonic in type $\mathbf{b}'\mathbf{ba}$ coordinates (see Figure 4.7) is

$$\begin{aligned} Y_{l_1,l_2,m}(\theta_1,\theta_2,\phi) = {} & \frac{(-1)^m(2l_2)!!}{\pi}\sqrt{\frac{(2l_2+1)(l_1+1)(l_1-l_2)!(l_2-m)!}{2(l_1+l_2+1)!(l_2+m)!}} \\ & \times (\cos\theta_1)^{l_2} C_{l_1-l_2}^{l_2+1}(\sin\theta_1) P_{l_2}^m(\cos\theta_2) e^{im\phi}. \end{aligned}$$

The normalized hyperspherical harmonics in type $\mathbf{ca}^2$ hyperspherical coordinates (see Figure

4.9) is

$$Y_{l,m_1,m_2}(\vartheta,\phi_1,\phi_2) = \frac{e^{i(m_1\phi_1+m_2\phi_2)}}{\pi}\sqrt{\frac{l+1}{2}\frac{\left[\frac{1}{2}(l+|m_1|+|m_2|]!\left[\frac{1}{2}(l-|m_1|-|m_2|)\right]!}{\left[\frac{1}{2}(l-|m_1|+|m_2|)\right]!\left[\frac{1}{2}(l+|m_1|-|m_2|)\right]!}}$$
$$\times (\sin\vartheta)^{|m_2|}(\cos\vartheta)^{|m_1|}P^{(|m_2|,|m_1|)}_{(l-|m_1|-|m_2|)/2}(\cos 2\vartheta),$$

with the restriction to the parameter space given by $\frac{1}{2}(l-|m_1|-|m_2|)\in\mathbf{N}_0$.

For arbitrary dimensions we can use standard hyperspherical coordinates. The hyperspherical harmonics corresponding to this coordinate system are basis functions for the irreducible representations of $O(d)$. In terms of these coordinates, the properly normalized hyperspherical harmonics, i.e. the functions which yield separation of variables for the d-dimensional Euclidean Laplace equation in standard hyperspherical coordinates are given as

$$Y_{\lambda,\boldsymbol{\mu}}(\widehat{\mathbf{x}}) = \left[\prod_{j=1}^{d-3}\Theta(l_j,l_{j+1};\theta_j)\right]Y_{\ell,m}(\theta_{d-2},\phi), \tag{6.12}$$

where $\boldsymbol{\mu}=\{l_2,l_3,\ldots,l_{d-1}\}$,

$$l_1=\lambda\geq l_2\geq l_3\geq\cdots\geq l_{d-3}\geq l_{d-2}=\ell\geq|l_{d-1}=m|\geq 0, \tag{6.13}$$

and

$$\Theta(l_j,l_{j+1};\theta_j) = \frac{\Gamma\left(l_{j+1}+\frac{d-j+1}{2}\right)}{2l_{j+1}+d-j-1}\sqrt{\frac{2^{2l_{j+1}+d-j-1}(2l_j+d-j-1)(l_j-l_{j+1})!}{\pi(l_j+l_{j+1}+d-j-2)!}}$$
$$\times(\sin\theta_j)^{l_{j+1}}C^{l_{j+1}+(d-j-1)/2}_{l_j-l_{j+1}}(\cos\theta_j). \tag{6.14}$$

The computation of (6.14) is a straightforward consequence of (6.6), and doing the proper node counting for S_β, in the tree depicted in Figure 4.10, for type $\mathbf{b}^{d-2}\mathbf{a}$ coordinates.

6.3 Fundamental solution expansions in Gegenbauer polynomials

One may compute eigenfunction expansions for unnormalized fundamental solutions for powers of the Laplacian in $\mathbf{R}^d$ in terms hyperspherical harmonics by using the generating function for Gegenbauer polynomials (2.95). The generating function for Gegenbauer polynomials only matches up to the addition theorem for hyperspherical harmonics with superscript $d/2-1$, see (6.2). This corresponds to an unnormalized fundamental solution for a single power of the Laplacian for $d \geq 3$ in pure hyperspherical coordinates ($r = r_d$, $\gamma = \gamma_d$) where

$$\|\mathbf{x}-\mathbf{x}'\| = \left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{1/2}.$$

If in the generating function for Gegenbauer polynomials (2.95) we substitute $z = r_</r_>$, $\lambda = d/2-1$, and $x = \cos\gamma$, then in the context of (6.2) we have

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|^{d-2}} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+d-2}} C_n^{d/2-1}(\cos\gamma). \tag{6.15}$$

This is a generalization of the generating function for Legendre polynomials (Laplace expansion), and reduces to such for $d = 3$. We may use (6.15) to compute single summation fundamental solution expansions ($d \geq 3$) for a single power of the Laplacian. But due to the discrepancy of the superscript of the Gegenbauer polynomial in the generating function for general powers of the Laplacian vs. that in (6.2), we are unable to straightforwardly use this formula to compute eigenfunction expansions for an unnormalized fundamental solution for the Laplacian in $\mathbf{R}^d$ in pure hyperspherical coordinates for general powers of the Laplacian k.

We would like to be able to use the addition theorem for hyperspherical harmonics (6.2) to compute eigenfunction expansions of unnormalized fundamental solutions for general powers of the Laplacian k. In order to do this we must compute the angular projection for unnormalized fundamental solutions for powers of the Laplacian onto this particular Gegenbauer polynomial with superscript $d/2-1$. We now see that not only are we able to expand an unnormalized fundamental solutions for general powers of the Laplacian in a Fourier series, but we are also able to expand unnormalized fundamental solutions in hyperspherical harmonics. By comparison we are able to generate an addition theorem, by extracting out the Fourier contribution from both sides of the equality.

One may compute the appropriate expansion as follows. For the case described above, namely using the standard hyperspherical coordinates in (4.31) which are adapted to a specific canonical subgroup chain, in that case see (6.13) and we have

$$\sum_{\mu\in\boldsymbol{\mu}}=\sum_{l_1=0}^{\lambda}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{d-4}=0}^{l_{d-5}}\sum_{\ell=0}^{l_{d-4}}\sum_{m=-\ell}^{\ell}. \tag{6.16}$$

For these coordinates, it is this multi-index sum which is used in the addition theorem for hyperspherical harmonics. In order to be able to use this addition theorem, we need to be able to expand unnormalized fundamental solutions in terms of Gegenbauer polynomials (see p. 15 of Faraut & Harzallah (1987) [39]), namely

$$\|\mathbf{x}-\mathbf{x}'\|^{\nu}=\sum_{\lambda=0}^{\infty}R_{\nu,\lambda}(r,r')C_{\lambda}^{d/2-1}(\cos\gamma). \tag{6.17}$$

Through the use of the addition theorem for hyperspherical harmonics we see that the Gegenbauer polynomials $C_{\lambda}^{d/2-1}(\cos\gamma)$ are hyperspherical harmonics when regarded as a function of $\mathbf{x}$ only. We would also like to consider Gegenbauer polynomial expansion formulas, in even dimensions, of fundamental solutions for powers of the Laplacian greater than or equal to $d/2$, namely

$$\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\|=\sum_{\lambda=0}^{\infty}S_{2p,\lambda}(r,r')C_{\lambda}^{d/2-1}(\cos\gamma), \tag{6.18}$$

for $p\in\mathbf{N}_0$. These will be discussed below.

Sack (1964) [84] computed expansions in terms of Legendre polynomials for functions which are a power of the distance in $\mathbf{R}^3$ (see also Hausner (1997) [49]). This is the limiting case for $d=3$ in (6.17). The result given in Sack (1964) [84] for $R_{\nu,\lambda}(r,r')$ in $\mathbf{R}^3$ is given as a specific Gauss hypergeometric function. Other relevant works give expressions for expanding arbitrary functions of $f(\|\mathbf{x}-\mathbf{x}'\|)$ in terms of the correct Gegenbauer polynomials (appropriate for its dimension) in $\mathbf{R}^3$ (see Avery (1979) [7]) and in $\mathbf{R}^d$ (see Wen & Avery (1985) [106]). However, those formulae do not seem to be normalized properly and because of their construction they will not work for fundamental solutions of Laplace's equation. The result given in Sack (1964) [84] is a general solution for our problem in $\mathbf{R}^3$.

We generalize Sack's proof for $\mathbf{R}^d$. Starting with (6.17), we apply the Laplacian in $\mathbf{R}^d$ to both sides. We denote the Laplacian with respect to the unprimed coordinates in general hyperspherical coordinates by Δ, and the Laplacian with respect to the primed coordinates by Δ'. The Laplacian with respect to the unprimed and primed coordinates, in a general hyperspherical coordinate system, satisfies the following partial differential equations

$$\Delta\|\mathbf{x}-\mathbf{x}'\|^{\nu}=\Delta'\|\mathbf{x}-\mathbf{x}'\|^{\nu}=\nu(\nu+d-2)\|\mathbf{x}-\mathbf{x}'\|^{\nu-2}. \tag{6.19}$$

Orthogonality for Gegenbauer polynomials (cf. (8.939.8) in Gradshteyn & Ryzhik (2007)) gives

$$\int_0^\pi C_\lambda^{d/2-1}(\cos\gamma)C_{\lambda'}^{d/2-1}(\cos\gamma)(\sin\gamma)^{d-2}d\gamma = \frac{\pi 2^{4-d}\Gamma(\lambda+d-2)\delta_{\lambda,\lambda'}}{\lambda!(2\lambda+d-2)[\Gamma(d/2-1)]^2},$$

therefore in conjunction with (6.17) we have

$$R_{\nu,\lambda}(r,r') = \frac{\lambda!(2\lambda+d-2)[\Gamma(d/2-1)]^2}{\pi 2^{4-d}\Gamma(\lambda+d-2)} \times \int_0^\pi C_\lambda^{d/2-1}(\cos\gamma)\left(r^2+r'^2-2rr'\cos\gamma\right)^{\nu/2}(\sin\gamma)^{d-2}d\gamma. \tag{6.20}$$

Since the integrand in (6.20) is regular, we may differentiate under the integral sign with respect to the radial coordinates. For instance if we define

$$a_{\lambda,d} := \frac{\lambda!(d-2+2\lambda)\left[\Gamma(d/2-1)\right]^2}{\pi 2^{4-d}\Gamma(\lambda+d-2)},$$

then

$$\frac{\partial^2 R_{\nu,\lambda}}{\partial r^2}(r,r') = a_{\lambda,d}\int_0^\pi (\sin\gamma)^{d-2}C_\lambda^{d/2-1}(\cos\gamma)\left(r^2+r'^2-2rr'\cos\gamma\right)^{\nu/2-2} \times \left[\nu\left(r^2+r'^2-2rr'\cos\gamma\right)+\nu(\nu-2)(r-r'\cos\gamma)^2\right]d\gamma,$$

and

$$\frac{d-1}{r}\frac{\partial R_{\nu,\lambda}}{\partial r}(r,r') = a_{\lambda,d}\int_0^\pi (\sin\gamma)^{d-2}C_\lambda^{d/2-1}(\cos\gamma)\left(r^2+r'^2-2rr'\cos\gamma\right)^{\nu/2-2} \times \nu(d-1)\left(1-\frac{r'}{r}\cos\gamma\right)\left(r^2+r'^2-2rr'\cos\gamma\right)d\gamma.$$

Since Gegenbauer polynomials are hyperspherical harmonics in a single variable, namely

$$C_\lambda^{d/2-1}(\cos\gamma) = \frac{-1}{\lambda(\lambda+d-2)}\Delta_{\mathbf{S}^{d-1}}C_\lambda^{d/2-1}(\cos\gamma), \tag{6.21}$$

and with respect to the integration variable, the entire integrand of (6.20) is a function of $\cos\gamma$, we can use (6.20) to re-write $-\lambda(\lambda+d-2)R_{\nu,\lambda}(r,r')/r^2$. In normal coordinates, the hyperspherical Laplacian in (6.21), acting on a function $f\in C^2([-1,1])$, is

$$\Delta_{\mathbf{S}^{d-1}}f = \frac{1}{(\sin\gamma)^{d-2}}\frac{\partial}{\partial\gamma}(\sin\gamma)^{d-2}\frac{\partial}{\partial\gamma}f \tag{6.22}$$

(see for example p. 494 in Vilenkin (1968) [100]). Using (6.22) and (6.21) in (6.20), and integrating by parts twice, demonstrates that the spherical Laplacian is symmetric. The surface terms which appear when integrating by parts vanish due to the appearance of factors proportional to $\sin\gamma$. Note that the Gegenbauer polynomials are regular at the end points, namely

$$C_\lambda^{d/2-1}(1)=\frac{(d-2)_\lambda}{\lambda!}=\frac{(d-3+\lambda)!}{\lambda!(d-3)!},$$

i.e. (2.94), and by the parity of Gegenbauer polynomials

$$C_\lambda^{d/2-1}(-1)=(-1)^\lambda\frac{(d-2)_\lambda}{\lambda!}.$$

Evaluation of the double integration by parts results in

$$-\frac{\lambda(\lambda+d-2)}{r^2}R_{\nu,\lambda}(r,r')=a_{\lambda,d}\int_0^\pi(\sin\gamma)^{d-2}C_\lambda^{d/2-1}(\cos\gamma)\left(r^2+r'^2-2rr'\cos\gamma\right)^{\nu/2-2}$$
$$\times\,\nu\frac{r'}{r}\left[(d-1)\cos\gamma\left(r^2+r'^2-2rr'\cos\gamma\right)+(\nu-2)rr'(1-\cos^2\gamma)\right]d\gamma,$$

and therefore by symmetry

$$\frac{\partial^2R_{\nu,\lambda}}{\partial r^2}+\frac{d-1}{r}\frac{\partial R_{\nu,\lambda}}{\partial r}-\frac{\lambda(\lambda+d-2)R_{\nu,\lambda}}{r^2}$$
$$=\frac{\partial^2R_{\nu,\lambda}}{\partial r'^2}+\frac{d-1}{r'}\frac{\partial R_{\nu,\lambda}}{\partial r'}-\frac{\lambda(\lambda+d-2)R_{\nu,\lambda}}{r'^2}=\nu(\nu+d-2)R_{\nu-2,\lambda},\qquad(6.23)$$

which is satisfied for each $\lambda\in\mathbf{N}_0$.

Furthermore $R_{\nu,\lambda}$ is a homogeneous function of degree ν in the variables r and r', and since $\|\mathbf{x}-\mathbf{x}'\|^\nu$ is a continuous function if $r_<=0$, it must contain the factor $r_<^\lambda$ so that

$$R_{\nu,\lambda}(r,r')=r_<^\lambda r_>^{\nu-\lambda}G_{\nu,\lambda}\left(\frac{r_<}{r_>}\right),$$

where $G_{\nu,\lambda}(x)$ is an analytic function for $x\in[0,1)$. Expressing $G_{\nu,\lambda}$ as a power series

$$G_{\nu,\lambda}\left(\frac{r_<}{r_>}\right)=\sum_{s=0}^\infty c_{\nu,\lambda,s}\left(\frac{r_<}{r_>}\right)^s,$$

we have

$$R_{\nu,\lambda}(r,r')=r_<^\lambda r_>^{\nu-\lambda}\sum_{s=0}^\infty c_{\nu,\lambda,s}\left(\frac{r_<}{r_>}\right)^s.\qquad(6.24)$$

Substituting (6.24) into (6.23), we obtain the recurrence relations

$$(s+2)(2\lambda+s+d)c_{\nu,\lambda,s+2} = (\nu-2\lambda-s)(\nu-s+d-2)c_{\nu,\lambda,s}.$$

The sequence of coefficients thus begins with $s=0$, as the other possibility $s=-2\lambda-1$ would violate the continuity condition and $c_{\nu,\lambda,s}=0$ for all $s\in\{1,3,5,\ldots\}$. Hence for even $s=2p$, where $p\in\mathbf{N}_0$,

$$c_{\nu,\lambda,2p} = \frac{\left(\lambda-\frac{\nu}{2}\right)_p\left(1-\frac{\nu+d}{2}\right)_p}{p!\left(\lambda+\frac{d}{2}\right)_p}c_{\nu,\lambda,0},$$

where $(z)_p$ is a Pochhammer symbol for rising factorial. Hence with the definition of the Gauss hypergeometric function given in (2.19), we have

$$R_{\nu,\lambda}(r,r') = K(\nu,\lambda)r_<^{\lambda}r_>^{\nu-\lambda}{}_2F_1\left(\lambda-\frac{\nu}{2},1-\frac{\nu+d}{2};\lambda+\frac{d}{2};\left(\frac{r_<}{r_>}\right)^2\right). \tag{6.25}$$

The coefficients $K(\nu,\lambda)$ are most easily determined by considering the case $\gamma=0$. In this case, using the binomial theorem on the left-hand side of (6.17) we have

$$(r_>-r_<)^{\nu} = \sum_{k=0}^{\infty}\frac{(-\nu)_k}{k!}r_<^k r_>^{\nu-k}, \tag{6.26}$$

which also equals

$$\|\mathbf{x}-\mathbf{x}'\| = \sum_{\lambda=0}^{\infty}R_{\nu,\lambda}(r,r')\frac{(d-2)_\lambda}{\lambda!}. \tag{6.27}$$

If we insert (6.25) into (6.27) and compare with the $r_<^k r_>^{\nu-k}$ coefficients of (6.26) we obtain the following recurrence relation for $K(\nu,\lambda)$

$$\frac{(-\nu)_k}{k!} = \sum_{n=0}^{\lfloor k/2\rfloor}K(\nu,k-2n)\frac{(d-2)_{k-2n}}{(k-2n)!}\frac{\left(k-2n-\frac{\nu}{2}\right)_n}{\left(k-2n+\frac{d}{2}\right)_n}\frac{\left(1-\frac{\nu+d}{2}\right)_n}{n!}. \tag{6.28}$$

Clearly (6.28) has a unique solution. We propose that

$$K(\nu,\lambda) = \frac{(-\nu/2)_\lambda}{(d/2-1)_\lambda}. \tag{6.29}$$

We emphasize that this choice of coefficients is seen to be consistent with the generating function for Gegenbauer polynomials (6.15) with $\nu=2-d$ since $K(2-d,\lambda)=1$ (see also (6.34) below). This form of the coefficient clearly matches that given in Sack (1964) [84] for $\mathbf{R}^3$. However Sack has failed to give a fully-rigorous proof of the form of his coefficient. We are able to prove that (6.29) is correct as follows. By inserting (6.29) into the right-hand-side of (6.28) we obtain

$$\sum_{n=0}^{\lfloor k/2\rfloor}\frac{\left(-\frac{\nu}{2}\right)_{k-2n}}{\left(\frac{d}{2}-1\right)_{k-2n}}\frac{(d-2)_{k-2n}}{(k-2n)!}\frac{\left(k-2n-\frac{\nu}{2}\right)_n}{\left(k-2n+\frac{d}{2}\right)_n}\frac{\left(1-\frac{\nu+d}{2}\right)_n}{n!}$$

$$=\sum_{n=0}^{\lfloor k/2\rfloor}\frac{\sqrt{\pi}2^{2-d}(d-2+2k-4n)(d-3+k-2n)!\Gamma(k-n-\frac{\nu}{2})\Gamma(n-\frac{d+\nu}{2}+1)}{\Gamma(\frac{d-1}{2})\Gamma(-\frac{\nu}{2})\Gamma(1-\frac{d+\nu}{2})(k-2n)!\Gamma(\frac{d}{2}+k-n)n!}. \tag{6.30}$$

This sum can be calculated using Zeilberger's algorithm (see Petkovšek, Wilf & Zeilberger (1996) [77]). Let $F(k,n)$ denote the individual terms in the finite sum. Then Zeilberger's algorithm produces the rational function

$$R(k,n)=\frac{2n(2n-k-d+1)(2n-k-d+2)(2n-2k+\nu)}{(2n-2k-d)(2n-k-1)(4n-2k-d+2)}$$

(see Paule & Schorn (1995) [76]). Now define $G(k,n)=R(k,n)F(k,n)$. We will next show that the telescoping recurrence relation

$$(k+1)F(k+1,n)+(\nu-k)F(k,n)=G(k,n)-G(k,n+1), \tag{6.31}$$

is valid. Expressing $F(k,n)$ and $G(k,n)$ in terms of gamma functions yields

$$(k+1)F(k+1,n)$$
$$=\frac{2^{2-d}\sqrt{\pi}(1+k)(d+2k-4n)\Gamma(-1+d+k-2n)\Gamma(1+k-n-\frac{\nu}{2})\Gamma(1+n-\frac{d+\nu}{2})}{n!\Gamma(\frac{d-1}{2})\Gamma(2+k-2n)\Gamma(1+\frac{d}{2}+k-n)\Gamma(1-\frac{d+\nu}{2})\Gamma(-\frac{\nu}{2})},$$

$$(\nu-k)F(k,n)$$
$$=-\frac{2^{2-d}\sqrt{\pi}(-2+d+2k-4n)(k-\nu)\Gamma(2+d+k-2n)\Gamma(k-n-\frac{\nu}{2})\Gamma(1+n-\frac{d+\nu}{2})}{n!\Gamma(\frac{d-1}{2})\Gamma(1+k-2n)\Gamma(\frac{d}{2}+k-n)\Gamma(1-\frac{d+\nu}{2})\Gamma(-\frac{\nu}{2})},$$

$$G(k,n)=\frac{2^{3-d}\sqrt{\pi}\Gamma(d+k-2n)\Gamma(1+k-n-\frac{\nu}{2})\Gamma(1-\frac{d+\nu}{2}+n)}{(n-1)!\Gamma(\frac{d-1}{2})\Gamma(2+k-2n)\Gamma(1+\frac{d}{2}+k-n)\Gamma(1-\frac{d+\nu}{2})\Gamma(-\frac{\nu}{2})},$$

and

$$G(k,n+1)=\frac{2^{3-d}\sqrt{\pi}\Gamma(2+d+k-2n)\Gamma(k-n-\frac{\nu}{2})\Gamma(2+n-\frac{d+\nu}{2})}{n!\Gamma(\frac{d-1}{2})\Gamma(k-2n)\Gamma(\frac{d}{2}+k-n)\Gamma(1-\frac{d+\nu}{2})\Gamma(-\frac{\nu}{2})}.$$

Then

$$(k+1)F(k+1,n)+(\nu-k)F(k,n)-G(k,n)+G(k,n+1)=$$
$$\frac{1}{(k-2n)(1+k-2n)(d-1+k-2n)(d+2k-2n)(d+k-2(n+1))}\times$$
$$\times\frac{2^{2-d}\sqrt{\pi}\Gamma(d+k-2n)\Gamma(k-n-\frac{\nu}{2})\Gamma(1+n-\frac{d+\nu}{2})}{n!\Gamma(\frac{d-1}{2})\Gamma(k-2n)\Gamma(\frac{d}{2}+k-n)\Gamma(1-\frac{d+\nu}{2})\Gamma(-\frac{\nu}{2})}\times$$
$$\Big((2k-2n-\nu)(d+k-2(1+n))[(d+2k-4n)(1+k)-2n(d-1+k-2n)]$$
$$+(d+2k-2n)(1+k-2n)[-(k-\nu)(d-2+2k-4n)-(k-2n)(d-2-2n+\nu)]\Big).$$

Since

$$(d+2k-4n)(1+k)-2n(d-1+k-2n)=(1+k-2n)(d+2k-2n),$$

and

$$(k-\nu)(d-2+2k-4n)-(k-2n)(d-2-2n+\nu)=(2k-2n-\nu)(d+k-2(1+n)),$$

it follows that

$$(k+1)F(k+1,n)+(\nu-k)F(k,n)-G(k,n)+G(k,n+1)=0.$$

Hence (6.31) is satisfied.

Set

$$f(k)=\sum_{n\in\mathbf{Z}}F(k,n)=\sum_{n=0}^{\lfloor k/2\rfloor}F(k,n).$$

It follows from (6.31) that

$$f(k+1)=\frac{-\nu+k}{k+1}f(k),$$

for all $k\in\mathbf{N}_0$. Clearly $f(0)=1$. Therefore

$$f(k)=\frac{(-\nu)_k}{k!},$$

for all $k\in\mathbf{N}_0$. Hence (6.29) is a solution of the recurrence relation (6.28). By the uniqueness one deduces that (6.29) is valid.

It is interesting to point out that an alternate proof, that (6.29) is the correct choice for $K(\nu,\lambda)$, can be given by expressing the sum in (6.30) as an instance of a *very-well poised* terminating generalized hypergeometric series ${}_5F_4$ (see §3.4 in Andrews, Askey & Roy (1999)

[3] and also [30]). The generalized hypergeometric series ${}_5F_4$ can be defined for $|z|<1$ in terms of the following sum

$$ {}_5F_4\left(\begin{array}{c} a_1,a_2,a_3,a_4,a_5 \\ b_1,b_2,b_3,b_4 \end{array};z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n(a_5)_n}{(b_1)_n(b_2)_n(b_3)_n(b_4)_n n!} z^n. $$

In analogy with (2.24), it can be shown that this series converges absolutely for $|z|=1$ if

$$ \mathrm{Re}\,(b_1+b_2+b_3+b_4-a_1-a_2-a_3-a_4-a_5)>0. $$

By using properties of Pochhammer symbols, we can re-write (6.30) as

$$ \sum_{n=0}^{\lfloor k/2\rfloor} \frac{\left(-\frac{\nu}{2}\right)_{k-2n}}{\left(\frac{d}{2}-1\right)_{k-2n}} \frac{\left(d-2\right)_{k-2n}}{(k-2n)!} \frac{\left(k-2n-\frac{\nu}{2}\right)_n}{\left(k-2n+\frac{d}{2}\right)_n} \frac{\left(1-\frac{\nu+d}{2}\right)_n}{n!} $$
$$ = \frac{(-\frac{\nu}{2})_k(d-2)_k(\frac12-\frac{d}{4}-\frac{k}{2})}{(\frac{d}{2})_k k!(\frac12-\frac{d}{4})} {}_5F_4\left(\begin{array}{c} 1-\frac{d}{2}-k,\frac32-\frac{d}{4}-\frac{k}{2},1-\frac{\nu+d}{2},-\frac{k}{2},\frac{1-k}{2} \\ \frac12-\frac{d}{4}-\frac{k}{2},1+\frac{\nu}{2}-k,2-\frac{d+k}{2},\frac32-\frac{d+k}{2} \end{array};1\right). \qquad (6.32) $$

This series terminates because of the numerator parameters $-k/2$ (for k even) and $(1-k)/2$ (for k odd). Upon examination, it is seen that the parameters in this ${}_5F_4$ have quite a lot of structure. A series of this form is called a very-well poised terminating ${}_5F_4$. Its sum has a simple closed form which can be found in Theorem 3 of Cooper (2002) [29], namely

$$ {}_5F_4\left(\begin{array}{c} A,\frac{A}{2}+1,B,C,-n \\ \frac{A}{2},A-B+1,A-C+1,A+n+1 \end{array};1\right) = \frac{(A-B-C+1)_n(A+1)_n}{(A-B+1)_n(A-C+1)_n}, $$

which when applied to (6.32) yields the desired result.

The final result for $R_{\nu,\lambda}(r,r')$ matches Sack's $d=3$ case exactly and it is given by

$$ R_{\nu,\lambda}(r,r') = \frac{(-\nu/2)_\lambda}{(d/2-1)_\lambda} r_<^\lambda r_>^{\nu-\lambda} {}_2F_1\left(\lambda-\frac{\nu}{2},1-\frac{\nu+d}{2};\lambda+\frac{d}{2};\left(\frac{r_<}{r_>}\right)^2\right). \qquad (6.33) $$

Sack noticed that his Gauss hypergeometric function satisfied a quadratic transformation for hypergeometric functions. The same is true for our generalization. Hypergeometric functions which satisfy quadratic transformations are related to associated Legendre functions (see

§2.4.3 of Magnus, Oberhettinger & Soni (1966) [67]). By using (2.34) we can show that

$$
{}_2F_1\left(\lambda-\frac{\nu}{2},1-\frac{\nu+d}{2};\lambda+\frac{d}{2};\left(\frac{r_<}{r_>}\right)^2\right)=
$$
$$
\left(\frac{r_>^2+r_<^2}{r_>^2}\right)^{\nu/2-\lambda}{}_2F_1\left(\frac{\lambda}{2}-\frac{\nu}{4},\frac{\lambda}{2}-\frac{\nu}{4}+\frac{1}{2};\lambda+\frac{d}{2};\left(\frac{r^2+r'^2}{2rr'}\right)^{-2}\right).
$$

Now we apply (2.41), this obtains

$$
{}_2F_1\left(\frac{\lambda}{2}-\frac{\nu}{4},\frac{\lambda}{2}-\frac{\nu}{4}+\frac{1}{2};\lambda+\frac{d}{2};\left(\frac{r^2+r'^2}{2rr'}\right)^{-2}\right)=
$$
$$
\frac{e^{i\pi(\nu+d-1)/2}2^{\lambda+(d-1)/2}}{\sqrt{\pi}}\frac{\Gamma\left(\lambda+\frac{d}{2}\right)}{\Gamma\left(\lambda-\frac{\nu}{2}\right)}
$$
$$
\times\frac{\left(r^2+r'^2\right)^{\lambda-\nu/2}}{(r_>^2-r_<^2)^{(1-d-\nu)/2}}\left(2rr'\right)^{-\lambda+(1-d)/2}Q_{\lambda+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right),
$$

which results in

$$
R_{\nu,\lambda}(r,r')=\frac{e^{i\pi(\nu+d-1)/2}(\lambda+\frac{d}{2}-1)\Gamma\left(\frac{d-2}{2}\right)}{\sqrt{\pi}\Gamma\left(-\frac{\nu}{2}\right)}\frac{(r_>^2-r_<^2)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}}Q_{\lambda+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right).
$$

Inserting this expression into (6.17) and using (6.4) we have

$$
\|\mathbf{x}-\mathbf{x}'\|^\nu=\frac{e^{i\pi(\nu+d-1)/2}\Gamma\left(\frac{d-2}{2}\right)}{\sqrt{\pi}\Gamma\left(-\frac{\nu}{2}\right)}\frac{(r_>^2-r_<^2)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}}
$$
$$
\times\sum_{\lambda=0}^{\infty}\left(\lambda+\frac{d}{2}-1\right)Q_{\lambda+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_\lambda^{d/2-1}(\cos\gamma). \tag{6.34}
$$

This is seen to be a generalization of Laplace's expansion in $\mathbf{R}^3$

$$
\frac{1}{\|\mathbf{x}-\mathbf{x}'\|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma),
$$

which is demonstrated by simplifying the associated Legendre function of the second kind through (2.59) and utilizing (2.97). The equation given by (6.34) is a further generalization, to arbitrary dimensions ($d\geq 3$), of the result presented in Sack (1964) [84].

Notice that (6.34) is ill-defined for $d=2$. However, the angular harmonics in $d=2$ are well-known, being

$$T_m(\cos(\phi-\phi')) = \cos(m(\phi-\phi')),$$

so we may directly derive the corresponding expansion formula in $\mathbf{R}^2$. In $\mathbf{R}^2$ let us write

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \sum_{m=0}^{\infty} R_{\nu,m}(r,r')\cos(m(\phi-\phi')).$$

Through standard Fourier theory, the Fourier coefficients are given as follows

$$R_{\nu,m}(r,r') = \frac{\epsilon_m(2rr')^{\nu/2}}{\pi}\int_0^\pi \frac{\cos(n\psi)d\psi}{\left(\dfrac{r^2+r'^2}{2rr'}-\cos\psi\right)^{-\nu/2}}.$$

This integral is just the definite integral given by (5.12), so the result is

$$R_{\nu,m}(r,r') = \frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}\frac{(r_>^2-r_<^2)^{(\nu+1)/2}}{\sqrt{\pi rr'}}Q_{m-1/2}^{-(\nu+1)/2}\left(\frac{r^2+r'^2}{2rr'}\right),$$

and therefore the full expansion in $d=2$ is given by

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}\frac{(r_>^2-r_<^2)^{(\nu+1)/2}}{\sqrt{\pi rr'}}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}\left(\frac{r^2+r'^2}{2rr'}\right), \quad (6.35)$$

where $\nu\in\mathbf{C}\setminus\{0,2,4,\ldots\}$. However using (5.13), we find that for $d=2$ we have

$$\|\mathbf{x}-\mathbf{x}'\|^{2p} = i(-1)^{p+1}\frac{(r_>^2-r_<^2)^{p+1/2}}{\sqrt{\pi rr'}}\sum_{m=0}^{\infty}\epsilon_m\cos[m(\phi-\phi')]\frac{(-p)_m}{(p+m)!}Q_{m-1/2}^{p+1/2}\left(\frac{r^2+r'^2}{2rr'}\right), \quad (6.36)$$

where $p\in\mathbf{N}_0$.

Now we use (6.34) to find the logarithmic expansion formula mentioned in (6.18). Notice that

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\|\mathbf{x}-\mathbf{x}'\|^{\nu+2p} = \|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\|, \quad (6.37)$$

where $p\in\mathbf{N}_0$. Let's apply this procedure to both sides of (6.34). Let's consider the right-hand side of (6.34). Collecting the terms which contain ν, we have

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{e^{i\pi\nu/2}(r_>^2-r_<^2)^{\nu/2}}{\Gamma\left(-\frac{\nu}{2}-p\right)}Q_{\lambda+(d-3)/2}^{(1-\nu-2p-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right) = \frac{1}{2}(-1)^{p+1}p!\,Q_{\lambda+(d-3)/2}^{(1-2p-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right),$$

since

$$\lim_{\nu\to 0}\frac{1}{\Gamma\left(-\frac{\nu}{2}-p\right)}=0, \tag{6.38}$$

and

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\left[\Gamma\left(-\frac{\nu}{2}-p\right)\right]^{-1}=\frac{1}{2}\lim_{\nu\to 0}\frac{\psi\left(-\frac{\nu}{2}-p\right)}{\Gamma\left(-\frac{\nu}{2}-p\right)}=\frac{1}{2}(-1)^{p+1}p!, \tag{6.39}$$

which can be established using the reflection formulae for gamma functions (2.5) and digamma functions (2.17). The final resulting expression for $d\in\{3,5,\ldots\}$ is

$$\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\|=\frac{(-1)^{p+1}p!\Gamma\left(\frac{d-2}{2}\right)e^{i\pi(d-1)/2}}{2\sqrt{\pi}}\frac{(r_>^2-r_<^2)^{p+(d-1)/2}}{(rr')^{(d-1)/2}}$$
$$\times\sum_{\lambda=0}^{\infty}\left(\lambda+\frac{d}{2}-1\right)Q_{\lambda+(d-3)/2}^{(1-2p-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_\lambda^{d/2-1}(\cos\gamma). \tag{6.40}$$

Unfortunately, this formula is not valid for d even because the corresponding associated Legendre functions are not defined (see §2.6.3). In order to remedy this fact we return to the $d=2$ case.

First we show that these methods work for the $p=0$ case. The main difference here is that the associated Legendre function of the second kind for $m=0$, $Q_{-1/2}^{-1/2}(z)$ is not defined, so we must treat the $m=0$ case separately. We must compute the limit derivative

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{e^{i\pi\nu/2}(r_>^2-r_<^2)^{\nu/2}}{\Gamma(-\nu/2)}Q_{-1/2}^{-(\nu+1)/2}\left(\frac{r^2+r'^2}{2rr'}\right). \tag{6.41}$$

This is facilitated by use of the Whipple formulae (2.57) which when applied to our specific associated Legendre function yields

$$Q_{-1/2}^{-(\nu+1)/2}\left(\frac{r^2+r'^2}{2rr'}\right)=\sqrt{\frac{\pi}{2}}\Gamma\left(-\frac{\nu}{2}\right)e^{-i\pi(\nu+1)/2}\sqrt{\frac{2rr'}{r_>^2-r_<^2}}P_{\nu/2}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right).$$

By taking advantage of the linearity of the derivative, (6.41) reduces to

$$-\frac{i}{2}\sqrt{\frac{\pi rr'}{r_>^2-r_<^2}}\lim_{\nu\to 0}\frac{\partial}{\partial\nu}(r_>^2-r_<^2)^{\nu}P_\nu\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right).$$

The limit for the derivative of the associated Legendre function of the first kind is given in

(2.70), and therefore using (2.36) we obtain

$$\left[\frac{\partial}{\partial\nu}P_\nu\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\right]_{\nu=0}=\log\left(\frac{r_>^2}{r_>^2-r_<^2}\right).$$

We have

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{e^{i\pi\nu/2}(r_>^2-r_<^2)^{\nu/2}}{\Gamma(-\nu/2)}Q_{-1/2}^{-(\nu+1)/2}\left(\frac{r^2+r'^2}{2rr'}\right)=-i\sqrt{\frac{\pi rr'}{r_>^2-r_<^2}}\log r_>, \tag{6.42}$$

and therefore

$$\left[\log\|\mathbf{x}-\mathbf{x}'\|\right]_{m=0}=\log r_>. \tag{6.43}$$

The components of $\log\|\mathbf{x}-\mathbf{x}'\|$ for $m\ge 1$ are straightforwardly computed using (6.38) and (6.39) plus the definition of the $-\frac{1}{2}$ order associated Legendre functions of the second kind (2.60) combined with (6.43), which yields

$$\log\|\mathbf{x}-\mathbf{x}'\|=\log r_>-\sum_{m=1}^\infty\frac{\cos(m(\phi-\phi'))}{m}\left(\frac{r_<}{r_>}\right)^m.$$

This formula exactly matches (5.25) with $e^\eta=r_>/r_<$. Having demonstrated the viability of our method, we now consider the general case for $d=2$.

We compute the Chebyshev expansion for unnormalized fundamental solutions for general powers $k\in\mathbf{N}$ ($p\in\mathbf{N}_0$) of the Laplacian in $d=2$. First let us determine the contribution for $p\in\mathbf{N}$. We would like to match our results to the computations in §5.3, which clearly demonstrate different behaviours for the two regimes, $0\le m\le p$ and $m\ge p+1$. By starting with (6.35) and applying (6.37), we find after using the negative-order condition for associated Legendre functions of the second kind (2.54),

$$\begin{aligned}\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\|=&\frac{i(-1)^{p+1}(r_>^2-r_<^2)^{p+1/2}}{\sqrt{\pi rr'}}\sum_{m=0}^{p}\epsilon_m\cos[m(\phi-\phi')]\\&\times\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{e^{-i\pi\nu/2}(r_>^2-r_<^2)^{\nu/2}\left(-p-\frac{\nu}{2}\right)_m}{\Gamma\left(m+p+\frac{\nu}{2}+1\right)}Q_{m-1/2}^{\nu/2+p+1/2}\left(\frac{r^2+r'^2}{2rr'}\right)\\&+\frac{2i(-1)^p}{\sqrt{\pi rr'}}(r_>^2-r_<^2)^{p+1/2}\sum_{m=p+1}^\infty\cos[m(\phi-\phi')]\\&\times\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{e^{i\pi\nu/2}(r_<^2-r_<^2)^{\nu/2}}{\Gamma\left(-\frac{\nu}{2}-p\right)}Q_{m-1/2}^{-(\nu+2p+1)/2}\left(\frac{r^2+r'^2}{2rr'}\right)\end{aligned} \tag{6.44}$$

The associated Legendre function of the second kind in the first term can be re-written as an associated Legendre function of the first kind using the Whipple formulae (2.57) and the negative-order condition for associated Legendre functions of the first kind (2.53)

$$Q_{m-1/2}^{\nu/2+p+1/2}\left(\frac{r^2+r'^2}{2rr'}\right)=\sqrt{\frac{\pi rr'}{r_>^2-r_<^2}}\,i(-1)^p e^{i\pi\nu/2}\Gamma\left(\frac{\nu}{2}+p-m+1\right)P_{\nu/2+p}^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right).$$

The limit derivative in the second term can be computed using (6.38) and (6.39) and the negative-order condition for associated Legendre functions of the second kind (2.54). If we perform this evaluation and use the Whipple formulae (2.57), this converts (6.44) to

$$\begin{aligned}\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| &= (r_>^2-r_<^2)^p\sum_{m=0}^{p}\epsilon_m\cos[m(\phi-\phi')]\\ &\quad\times\lim_{\nu\to0}\frac{\partial}{\partial\nu}\frac{(r_>^2-r_<^2)^{\nu/2}\Gamma\left(\frac{\nu}{2}+p-m+1\right)(-p-\nu/2)_m}{\Gamma\left(\frac{\nu}{2}+p+m+1\right)}P_{\nu/2+p}^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\\ &+\frac{ip!(r_>^2-r_<^2)^{p+1/2}}{\sqrt{\pi rr'}}\sum_{m=p+1}^{\infty}\cos[m(\phi-\phi')]\frac{(m-p-1)!}{(m+p)!}Q_{m-1/2}^{p+1/2}\left(\frac{r^2+r'^2}{2rr'}\right).\end{aligned}$$

Now we are in a position to perform the limit derivative in the above expression. The limit derivative of the power-law results in a logarithm, the limit derivatives of the gamma functions and Pochhammer symbols result in products of gamma functions and Pochhammer symbols with digamma functions, and the limit derivative of the associated Legendre function of the first kind is expressible using the results in §2.6.4. The result is

$$\begin{aligned}&\lim_{\nu\to0}\frac{\partial}{\partial\nu}\frac{(r_>^2-r_<^2)^{\nu/2}\Gamma\left(\frac{\nu}{2}+p-m+1\right)(-p-\nu/2)_m}{\Gamma\left(\frac{\nu}{2}+p+m+1\right)}P_{\nu/2+p}^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)=\\ &\qquad(-p)_m\frac{(p-m)!}{(p+m)!}\Big[\log r_>+\psi(2p+1)-\psi(p+m+1)\Big]P_p^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\\ &\qquad+(-p)_m\frac{(p-m)!}{(p+m)!}\frac{(-1)^{p+m}}{2}\sum_{j=0}^{p-m-1}(-1)^j\frac{2j+2m+1}{(p-m-j)(p+m+j+1)}\\ &\qquad\quad\times\left[1+\frac{j!(p+m)!}{(j+2m)!(p-m)!}\right]P_{j+m}^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\\ &\qquad+(-p)_m\frac{(-1)^p}{2}\sum_{j=0}^{m-1}(-1)^j\frac{2j+1}{(p-j)(p+j+1)}P_j^{-m}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right),\end{aligned}\tag{6.45}$$

for $0 \le m \le p$. Therefore in $d = 2$ we have derived

$$\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| = (r_>^2 - r_<^2)^p \sum_{m=0}^{p} \epsilon_m \cos[m(\phi-\phi')]$$
$$\times \Bigg\{ (-p)_m \frac{(p-m)!}{(p+m)!}\Big[\log r_> + \psi(2p+1) - \psi(p+m+1)\Big] P_p^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)$$
$$+(-p)_m \frac{(p-m)!}{(p+m)!}\frac{(-1)^{p+m}}{2}\sum_{j=0}^{p-m-1}(-1)^j \frac{2j+2m+1}{(p-m-j)(p+m+j+1)}$$
$$\times\left[1+\frac{j!(p+m)!}{(j+2m)!(p-m)!}\right]P_{j+m}^m\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)$$
$$+(-p)_m\frac{(-1)^p}{2}\sum_{j=0}^{m-1}(-1)^j\frac{2j+1}{(p-j)(p+j+1)}P_j^{-m}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\Bigg\}$$
$$+\frac{ip!(r_>^2-r_<^2)^{p+1/2}}{\sqrt{\pi r r'}}\sum_{m=p+1}^{\infty}\cos[m(\phi-\phi')]\frac{(m-p-1)!}{(m+p)!}Q_{m-1/2}^{p+1/2}\left(\frac{r^2+r'^2}{2rr'}\right). \tag{6.46}$$

As a consequence of the function in the above formula, we now have a proof of the correspondence (5.36) for the "ending" function $\Re_{n,q-1}$ mentioned in §5.3. Not only that, we now have closed-form expressions for the finite terms given by (5.38), (5.35) and (5.32) in terms of associated Legendre functions of the first kind. These associated Legendre functions can also be converted to associated Legendre functions of the second kind using the Whipple formulae (2.57). The integer-order, integer-degree associated Legendre functions of the first kind have well established closed-form expressions (see (2.97) in §2.7), so we could easily leave these expressions as they are. However to be consistent, we will convert back to associated Legendre functions of the second kind.

Now we compute the logarithmic Gegenbauer expansion of unnormalized fundamental solutions for powers ($k \ge d/2$) of the Laplacian in $\mathbf{R}^d$ for $d \in \{4, 6, \ldots\}$. By starting with (6.34) and applying (6.37), we find that

$$\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| = \frac{e^{i\pi(d-1)/2}(-1)^p\left(\frac{d}{2}-2\right)!(r_>^2-r_<^2)^{p+(d-1)/2}}{\sqrt{\pi}(rr')^{(d-1)/2}}$$
$$\times\sum_{\lambda=0}^{\infty}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma)$$
$$\times\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{e^{i\pi\nu/2}(r_>^2-r_<^2)^{\nu/2}}{\Gamma\left(-\frac{\nu}{2}-p\right)}Q_{\lambda+(d-3)/2}^{-(\nu+2p+d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right). \tag{6.47}$$

Just like we did for the logarithmic contribution in $d=2$, we break the sum into two pieces, one for $0 \le \lambda \le p$ and for $\lambda \ge p+1$. For $0 \le \lambda \le p$ we use the Whipple formulae (2.57) (to convert the associated Legendre function of the second kind to an associated Legendre function of the first kind), followed by the negative-degree (2.51) and negative-order (2.53) conditions for associated Legendre functions of the first kind. In the regime $\lambda \ge p+1$, the associated Legendre functions of the second kind are well-defined, so all that is needed is to use (6.38), (6.39), and the negative order condition for associated Legendre functions of the second kind (2.54) for aesthetic purposes. This converts (6.47) to

$$\begin{aligned}
\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| &= \left(\frac{d}{2}-2\right)!\frac{(r_>^2-r_<^2)^{p+d/2-1}}{(rr')^{(d-1)/2}}\sum_{\lambda=0}^{p}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma)\\
&\quad\times\lim_{\nu\to0}\frac{\partial}{\partial\nu}\frac{(r_>^2-r_<^2)^{\nu/2}\left(-\frac{\nu}{2}-p\right)_\lambda\Gamma\left(\frac{\nu}{2}+p-\lambda+1\right)}{\Gamma\left(\frac{\nu}{2}+p+\lambda+d-1\right)}P_{\nu/2+p+d/2-1}^{\lambda+d/2-1}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\\
&\quad+\frac{e^{i\pi(d-1)/2}\left(\frac{d}{2}-2\right)!(r_>^2-r_<^2)^{p+(d-1)/2}}{2\sqrt{\pi}(rr')^{(d-1)/2}}\\
&\quad\times\sum_{\lambda=p+1}^{\infty}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma)\frac{(\lambda-p-1)!}{(\lambda+p+d-2)!}Q_{\lambda+(d-3)/2}^{p+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right). \qquad (6.48)
\end{aligned}$$

Now we are again in a position to perform the limit derivative in the above expression. Again, the limit derivative of the power-law results in a logarithm, the limit derivatives of the gamma functions and Pochhammer symbols result in products of gamma functions and Pochhammer symbols with digamma functions, and the limit derivative of the associated Legendre function of the first kind is expressible using the results in §2.6.4. Therefore (6.48) becomes

$$\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| = \left(\frac{d}{2}-2\right)!\frac{(r_>^2-r_<^2)^{p+d/2-1}}{(rr')^{d/2-1}}\sum_{\lambda=0}^{p}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma)$$

$$\times\left\{\frac{(-p)_\lambda(p-\lambda)!}{(p+\lambda+d-2)!}\Big[2\log r_> + 2\psi(2p+d-1)+\psi(p+1)-\psi\left(p+\frac{d}{2}\right)\right.$$

$$-\psi(p+\lambda+d-1)-\psi(p-\lambda+1)\Big]P_{p+d/2-1}^{\lambda+d/2-1}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)$$

$$+\frac{(-p)_\lambda(p-\lambda)!}{2(p+\lambda+d-2)!}(-1)^{p+\lambda}\sum_{j=0}^{p-\lambda-1}\frac{(-1)^j(2j+2\lambda+d-1)}{(p-\lambda-j)(p+\lambda+j+d-1)}$$

$$\times\left[1+\frac{j!(p+\lambda+d-2)!}{(j+2\lambda+d-2)!(p-\lambda)!}\right]P_{j+\lambda+d/2-1}^{\lambda+d/2-1}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)$$

$$\left.+(-p)_\lambda\frac{(-1)^{p+d/2-1}}{2}\sum_{j=0}^{\lambda+d/2-2}\frac{(-1)^j(2j+1)}{(p-j+\frac{d}{2}-1)(p+j+\frac{d}{2})}P_j^{-\lambda-d/2+1}\left(\frac{r_>^2+r_<^2}{r_>^2-r_<^2}\right)\right\}$$

$$+\frac{e^{i\pi(d-1)/2}\left(\frac{d}{2}-2\right)!(r_>^2-r_<^2)^{p+(d-1)/2}}{2\sqrt{\pi}(rr')^{(d-1)/2}}$$

$$\times\sum_{\lambda=p+1}^{\infty}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma)\frac{(\lambda-p-1)!}{(\lambda+p+d-2)!}Q_{\lambda+(d-3)/2}^{p+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right). \quad (6.49)$$

Using the Whipple formulae (2.57), the negative-order (2.54) and negative-degree (2.52) conditions for associated Legendre functions of the second kind, and the reflection formula for gamma functions (2.5) to (6.49) yields

$$
\begin{aligned}
\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| &= \frac{i(-1)^{p+d/2}\left(\frac{d}{2}-2\right)!(r_>^2-r_<^2)^{p+(d-1)/2}}{\sqrt{\pi}(rr')^{(d-1)/2}} \\
&\quad\times\sum_{\lambda=0}^{p}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma) \\
&\quad\times\Bigg\{\frac{(-p)_\lambda}{(p+\lambda+d-2)!}\Big[2\log r_> + 2\psi(2p+d-1)+\psi(p+1) \\
&\qquad -\psi\left(p+\frac{d}{2}\right)-\psi(p+\lambda+d-1)-\psi(p-\lambda+1)\Big]Q_{\lambda+(d-3)/2}^{p+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right) \\
&\qquad +\frac{(-p)_\lambda(p-\lambda)!}{2(p+\lambda+d-2)!}\sum_{j=0}^{p-\lambda-1}\frac{2j+2\lambda+d-1}{j!(p-\lambda-j)(p+\lambda+j+d-1)} \\
&\qquad\times\left[1+\frac{j!(p+\lambda+d-2)!}{(j+2\lambda+d-2)!(p-\lambda)!}\right]Q_{\lambda+(d-3)/2}^{j+\lambda+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right) \\
&\qquad +\frac{(-p)_\lambda}{2}\sum_{j=0}^{\lambda+d/2-2}\frac{2j+1}{(p-j+\frac{d}{2}-1)(p+j+\frac{d}{2})(\lambda+\frac{d}{2}+j-1)!}Q_{\lambda+(d-3)/2}^{j+1/2}\left(\frac{r^2+r'^2}{2rr'}\right)\Bigg\} \\
&\quad +\frac{e^{i\pi(d-1)/2}\left(\frac{d}{2}-2\right)!(r_>^2-r_<^2)^{p+(d-1)/2}}{2\sqrt{\pi}(rr')^{(d-1)/2}} \\
&\quad\times\sum_{\lambda=p+1}^{\infty}\left(\lambda+\frac{d}{2}-1\right)C_\lambda^{d/2-1}(\cos\gamma)\frac{(\lambda-p-1)!}{(\lambda+p+d-2)!}Q_{\lambda+(d-3)/2}^{p+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right).
\end{aligned}
\tag{6.50}
$$

In order to structure things nicely let us define the following functions

$$M_{\lambda,d,p}(r,r') = \frac{(-p)_\lambda}{(p+\lambda+d-2)!}\Big[2\log r_> + 2\psi(2p+d-1) + \psi(p+1)$$
$$-\psi\left(p+\frac{d}{2}\right) - \psi(p+\lambda+d-1) - \psi(p-\lambda+1)\Big] Q_{\lambda+(d-3)/2}^{p+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right)$$
$$+\frac{(-p)_\lambda(p-\lambda)!}{2(p+\lambda+d-2)!}\sum_{j=0}^{p-\lambda-1}\frac{2j+2\lambda+d-1}{j!(p-\lambda-j)(p+\lambda+j+d-1)}$$
$$\times\left[1+\frac{j!(p+\lambda+d-2)!}{(j+2\lambda+d-2)!(p-\lambda)!}\right] Q_{\lambda+(d-3)/2}^{j+\lambda+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right)$$
$$+\frac{(-p)_\lambda}{2}\sum_{j=0}^{\lambda+d/2-2}\frac{(2j+1)Q_{\lambda+(d-3)/2}^{j+1/2}\left(\frac{r^2+r'^2}{2rr'}\right)}{(p-j+\frac{d}{2}-1)(p+j+\frac{d}{2})(\lambda+\frac{d}{2}+j-1)!}$$

and

$$N_{\lambda,d,p}(r,r') = \frac{(\lambda-p-1)!}{(\lambda+p+d-2)!} Q_{\lambda+(d-3)/2}^{p+(d-1)/2}\left(\frac{r^2+r'^2}{2rr'}\right).$$

If we define

$$\mathbf{Q}_{\lambda,d,p}(r,r') = \begin{cases} M_{\lambda,d,p}(r,r') & \text{if } 0 \le \lambda \le p, \\ N_{\lambda,d,p}(r,r') & \text{if } \lambda \ge p+1, \end{cases}$$

then we can write

$$\|\mathbf{x}-\mathbf{x}'\|^{2p}\log\|\mathbf{x}-\mathbf{x}'\| = \frac{i(-1)^{p+d/2}\left(\frac{d}{2}-2\right)!(r_>^2-r_<^2)^{p+(d-1)/2}}{\sqrt{\pi}(rr')^{(d-1)/2}}$$
$$\times\sum_{\lambda=0}^{\infty}\left(\lambda+\frac{d}{2}-1\right)\mathbf{Q}_{\lambda,d,p}(r,r')C_\lambda^{d/2-1}(\cos\gamma). \tag{6.51}$$

6.4 Multi-summation addition theorems

By comparing the Fourier expansions for unnormalized fundamental solutions in arbitrary dimension with the eigenfunction expansions of fundamental solutions in the appropriate coordinate system, one can construct multi-summation and multi-integration addition theorems. We have already constructed some of these addition theorems in $\mathbf{R}^3$ (see Cohl et al. (2000) [27] and Cohl et al. (2001) [25]). In Cohl et al. (2000) [27], a determination addition theorems derived, in this fashion, for fundamental solutions for the Laplacian is presented in cylindrical, oblate spheroidal, prolate spheroidal, parabolic, bispherical and toroidal coordinates. In Cohl et al. (2001) [25] a determination of the addition theorems derived in spherical coordinates is presented. In circular cylindrical and parabolic coordinates, the derived addition theorems are given in terms of definite integrals. In oblate spheroidal, prolate spheroidal, bispherical, toroidal, and spherical coordinates, the derived addition theorems are given in terms of infinite sums. As is indicated by the above presentation, one may construct addition theorems in this manner in any rotationally invariant coordinate system which yields solutions through separation of variables for the Laplace equation. In a similar setting, addition theorems may be generated for other inhomogeneous linear partial differential equations, such as for the Helmholtz, wave and heat equations in arbitrary dimensions. Furthermore an extension of this concept is possible when working with linear partial differential operators on smooth manifolds, such as for the Laplace-Beltrami operator. Once a Fourier expansion for a fundamental solution is obtained, for that partial differential operator, on that particular manifold, one must construct eigenfunction expansions in the solution space for a fundamental solution corresponding to that operator and identify those nested multi-summation and multi-integration eigenfunction expansions which correspond to the Fourier coefficient for that operator.

This problem must be attacked in two different ways. First one must construct Fourier expansions of fundamental solutions for the operator and second, one must perform eigenfunction expansions for the operator in a particular rotationally invariant coordinate system which yields solutions to the equation through separation of variables. As one might imagine, the construction of the eigenfunction expansions in particular separable coordinate systems is a tedious process in general, especially in higher dimensional spaces.

In this thesis, we have focused upon eigenfunction expansions for the Laplacian in hyperspherical coordinates, as these are perhaps the most studied rotationally invariant coordinate systems in existence for this operator. We have succeeded in generating Fourier expansions of fundamental solutions for this operator and in this chapter we have succeeded in a construction of the eigenfunction (hyperspherical harmonic) expansions for a fundamental solution of the Laplacian.

In what follows in the rest of this chapter, we give a small number of examples of the types of addition theorems that one is able to generate. We will give some examples of the addition theorems in lower dimensions (i.e. $d \in 3, 4$) and we will attempt to give some examples of addition theorems in arbitrary dimensions. There are an infinitude of possibilities, especially as you increase dimensions. We have alluded to the large numbers of possibilities of hyperspherical coordinate systems in arbitrary dimensions in §4.1.2. Of course that large number increases multi-fold when one allows the possibility of non-subgroup type coordinates (see for instance Kalnins (1986) [62]). If we are able to give the reader just a small taste of the possibilities for choices of addition theorems which are derivable in the current manner, then we feel that this thesis has accomplished its goal. To give a framework for future researchers to further explore the possibilities in the field of special functions and fundamental solutions for linear partial differential operators.

Suppose one adopts for the vector space $\mathbf{R}^d$, a transformation using standard hyperspherical coordinates (4.31). Then new addition theorems can be generated by re-arranging the relevant sums over the hyperspherical harmonics. This works as follows

$$\sum_{l_1=0}^{l_{1\max}} \sum_{\boldsymbol{\mu}} = \sum_{m=-l_{1\max}}^{l_{1\max}} \sum_{l_{d-2}=|m|}^{l_{1\max}} \sum_{l_{d-3}=l_{d-2}}^{l_{1\max}} \cdots \sum_{l_2=l_3}^{l_{1\max}} \sum_{l_1=l_2}^{l_{1\max}} . \tag{6.52}$$

In (6.52), one must sum over all allowed quantum numbers $\boldsymbol{\lambda} := \{l_1\} \cup \boldsymbol{\mu}$. This is accomplished by taking the limit on both sides of (6.52) as $l_{1\max} \to \infty$. The forward sum (6.16) is now to be interpreted *in a reverse fashion* as (6.52). Take for instance the limit in the final sum, i.e. $l_2 \le l_1 < \infty$. This gives a new space of quantum numbers to be summed over, where λ has now been eliminated and all other quantum numbers remain. Then for all possible combinations of the quantum numbers, one sums over all remaining values of $l_3 \le l_2 < \infty$. This eliminates l_2 from the series. One continues this process until we have eliminated the contribution from all quantum numbers except m and finally we are left with a series which only depends on m. Then we sum over all contributions to m. Since we have already identified the azimuthal Fourier coefficients in Chapter 5, i.e. the terms only corresponding to m with the other quantum numbers completely summed over, then we can see that we can construct a countable number of multi-index addition theorems in any particular general hyperspherical coordinate system. Now we proceed to demonstrate a sampling of the variety of addition theorems that one may compute using the methods of §6.3.

6.4.1 Power-law addition theorems in $\mathbf{R}^3$

In $d = 3$ there are two ways to construct pure hyperspherical coordinates, with trees of type **b′a** and **ba**. This is the addition theorem derived for type **ba** hyperspherical coordinates using the methods described above

$$Q_{m-1/2}^{-(\nu+1)/2}(\chi) = i\sqrt{\pi}2^{-(\nu+3)/2}(\sin\theta\sin\theta')^{-\nu/2}(\chi^2-1)^{-(\nu+1)/4}\left(\frac{r_>^2-r_<^2}{rr'}\right)^{(\nu+2)/2}$$

$$\times \sum_{l=|m|}^{\infty}(2l+1)\frac{(l-m)!}{(l+m)!}Q_l^{-(\nu+2)/2}\left(\frac{r^2+r'^2}{2rr'}\right)P_l^m(\cos\theta)P_l^m(\cos\theta'), \tag{6.53}$$

where

$$\chi = \frac{r^2 + r'^2 - 2rr'\cos\theta\cos\theta'}{2rr'\sin\theta\sin\theta'}.$$

Equation (6.53) is a generalization of one of the main results of Cohl et al. (2001) [25], namely that which you obtain if you substitute $\nu = -1$ in (6.53)

$$Q_{m-1/2}(\chi) = \pi\sqrt{\sin\theta\sin\theta'} \sum_{l=|m|}^{\infty} \frac{(l-m)!}{(l+m)!} \left(\frac{r_<}{r_>}\right)^{l+1/2} P_l^m(\cos\theta) P_l^m(\cos\theta').$$

The addition theorem derived for type $\mathbf{b'a}$ hyperspherical coordinates using the methods described above is

$$Q_{m-1/2}^{-(\nu+1)/2}(\chi) = i\sqrt{\pi}2^{-(\nu+3)/2}(\cos\theta\cos\theta')^{-\nu/2}(\chi^2-1)^{-(\nu+1)/4}\left(\frac{r_>^2 - r_<^2}{rr'}\right)^{(\nu+2)/2}$$
$$\times \sum_{l=|m|}^{\infty}(2l+1)\frac{(l-m)!}{(l+m)!} Q_l^{-(\nu+2)/2}\left(\frac{r^2+r'^2}{2rr'}\right) P_l^m(\sin\theta) P_l^m(\sin\theta'), \qquad (6.54)$$

where

$$\chi = \frac{r^2 + r'^2 - 2rr'\sin\theta\sin\theta'}{2rr'\cos\theta\cos\theta'}.$$

Equation (6.54) is a generalization of one of the main results of Cohl et al. (2001) [25], namely that which you obtain if you substitute $\nu = -1$ in (6.54)

$$Q_{m-1/2}(\chi) = \pi\sqrt{\cos\theta\cos\theta'} \sum_{l=|m|}^{\infty} \frac{(l-m)!}{(l+m)!} \left(\frac{r_<}{r_>}\right)^{l+1/2} P_l^m(\sin\theta) P_l^m(\sin\theta').$$

6.4.2 Power-law addition theorems in $\mathbf{R}^4$

In type $\mathbf{b^2a}$ coordinates, this is the addition theorem derived using the methods described above

$$
\begin{aligned}
Q_{m-1/2}^{-(\nu+1)/2}(\chi) =& -2^{-(\nu+1)/2}\left(\frac{r_>^2-r_<^2}{rr'}\right)^{(\nu+3)/2}\left(\chi^2-1\right)^{-(\nu+1)/4}(\sin\theta_1\sin\theta_1'\sin\theta_2\sin\theta_2')^{-\nu/2}\\
&\times\sum_{l_2=|m|}^{\infty}\frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!}(\sin\theta_1\sin\theta_1')^{l_2}P_{l_2}^m(\cos\theta_2)P_{l_2}^m(\cos\theta_2')\\
&\times\sum_{l_1=l_2}^{\infty}\frac{(l_1+1)(l_1-l_2)!}{(l_1+l_2+1)!}Q_{l_1+1/2}^{-(\nu+3)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_{l_1-l_2}^{l_2+1}(\cos\theta_1)C_{l_1-l_2}^{l_2+1}(\cos\theta_1'),
\end{aligned}
\tag{6.55}
$$

where

$$
\chi=\frac{r^2+r'^2-2rr'\cos\theta_1\cos\theta_1'-2rr'\sin\theta_1\sin\theta_1'\cos\theta_2\cos\theta_2'}{2rr'\sin\theta_1\sin\theta_1'\sin\theta_2\sin\theta_2'}. \tag{6.56}
$$

If you substitute $\nu=-2$ (an unnormalized fundamental solution for the Laplacian in $\mathbf{R}^4$) in (6.55) then you obtain the following

$$
\begin{aligned}
Q_{m-1/2}^{1/2}(\chi) =&\sqrt{2\pi}i\left(\chi^2-1\right)^{1/4}\sin\theta_1\sin\theta_1'\sin\theta_2\sin\theta_2'\\
&\times\sum_{l_2=|m|}^{\infty}\frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!}(\sin\theta_1\sin\theta_1')^{l_2}P_{l_2}^m(\cos\theta_2)P_{l_2}^m(\cos\theta_2')\\
&\times\sum_{l_1=l_2}^{\infty}\frac{(l_1-l_2)!}{(l_1+l_2+1)!}\left(\frac{r_<}{r_>}\right)^{l+1+1}C_{l_1-l_2}^{l_2+1}(\cos\theta_1)C_{l_1-l_2}^{l_2+1}(\cos\theta_1').
\end{aligned}
$$

In type $\mathbf{b}'^2\mathbf{a}$ coordinates, this is the addition theorem derived using the methods described above

$$
\begin{aligned}
Q_{m-1/2}^{-(\nu+1)/2}(\chi) =& -2^{-(\nu+1)/2}\left(\frac{r_>^2-r_<^2}{rr'}\right)^{(\nu+3)/2}\left(\chi^2-1\right)^{-(\nu+1)/4}(\cos\theta_1\cos\theta_1'\cos\theta_2\cos\theta_2')^{-\nu/2}\\
&\times\sum_{l_2=|m|}^{\infty}\frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!}(\cos\theta_1\cos\theta_1')^{l_2}P_{l_2}^m(\sin\theta_2)P_{l_2}^m(\sin\theta_2')\\
&\times\sum_{l_1=l_2}^{\infty}\frac{(l_1+1)(l_1-l_2)!}{(l_1+l_2+1)!}Q_{l_1+1/2}^{-(\nu+3)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_{l_1-l_2}^{l_2+1}(\sin\theta_1)C_{l_1-l_2}^{l_2+1}(\sin\theta_1'),
\end{aligned}
\tag{6.57}
$$

where

$$\chi = \frac{r^2 + r'^2 - 2rr' \sin\theta_1 \sin\theta_1' - 2rr' \cos\theta_1 \cos\theta_1' \sin\theta_2 \sin\theta_2'}{2rr' \cos\theta_1 \cos\theta_1' \cos\theta_2 \cos\theta_2'}.$$

If you substitute $\nu = -2$ in (6.57) then you obtain the following

$$\begin{aligned} Q_{m-1/2}^{1/2}(\chi) =& \sqrt{2\pi} i \left(\chi^2 - 1\right)^{1/4} \cos\theta_1 \cos\theta_1' \cos\theta_2 \cos\theta_2' \\ & \times \sum_{l_2=|m|}^{\infty} \frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!} (\cos\theta_1 \cos\theta_1')^{l_2} P_{l_2}^m(\sin\theta_2) P_{l_2}^m(\sin\theta_2') \\ & \times \sum_{l_1=l_2}^{\infty} \frac{(l_1-l_2)!}{(l_1+l_2+1)!} \left(\frac{r_<}{r_>}\right)^{l+1+1} C_{l_1-l_2}^{l_2+1}(\sin\theta_1) C_{l_1-l_2}^{l_2+1}(\sin\theta_1'). \end{aligned}$$

In type **bb′a** coordinates, this is the addition theorem derived using the methods described above

$$\begin{aligned} Q_{m-1/2}^{-(\nu+1)/2}(\chi) =& -2^{-(\nu+1)/2} \left(\frac{r_>^2 - r_<^2}{rr'}\right)^{(\nu+3)/2} \left(\chi^2 - 1\right)^{-(\nu+1)/4} (\sin\theta_1 \sin\theta_1' \cos\theta_2 \cos\theta_2')^{-\nu/2} \\ & \times \sum_{l_2=|m|}^{\infty} \frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!} (\sin\theta_1 \sin\theta_1')^{l_2} P_{l_2}^m(\sin\theta_2) P_{l_2}^m(\sin\theta_2') \\ & \times \sum_{l_1=l_2}^{\infty} \frac{(l_1+1)(l_1-l_2)!}{(l_1+l_2+1)!} Q_{l_1+1/2}^{-(\nu+3)/2}\left(\frac{r^2+r'^2}{2rr'}\right) C_{l_1-l_2}^{l_2+1}(\cos\theta_1) C_{l_1-l_2}^{l_2+1}(\cos\theta_1'), \end{aligned} \tag{6.58}$$

where

$$\chi = \frac{r^2 + r'^2 - 2rr' \cos\theta_1 \cos\theta_1' - 2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2'}{2rr' \sin\theta_1 \sin\theta_1' \cos\theta_2 \cos\theta_2'}.$$

If you substitute $\nu = -2$ in (6.58) then you obtain the following

$$\begin{aligned} Q_{m-1/2}^{1/2}(\chi) =& \sqrt{2\pi} i \left(\chi^2 - 1\right)^{1/4} \sin\theta_1 \sin\theta_1' \cos\theta_2 \cos\theta_2' \\ & \times \sum_{l_2=|m|}^{\infty} \frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!} (\sin\theta_1 \sin\theta_1')^{l_2} P_{l_2}^m(\sin\theta_2) P_{l_2}^m(\sin\theta_2') \\ & \times \sum_{l_1=l_2}^{\infty} \frac{(l_1-l_2)!}{(l_1+l_2+1)!} \left(\frac{r_<}{r_>}\right)^{l+1+1} C_{l_1-l_2}^{l_2+1}(\cos\theta_1) C_{l_1-l_2}^{l_2+1}(\cos\theta_1'). \end{aligned}$$

In type $\mathbf{b'ba}$ coordinates, this is the addition theorem derived using the methods described above

$$Q_{m-1/2}^{-(\nu+1)/2}(\chi) = -\,2^{-(\nu+1)/2}\left(\frac{r_>^2-r_<^2}{rr'}\right)^{(\nu+3)/2}\left(\chi^2-1\right)^{-(\nu+1)/4}(\cos\theta_1\cos\theta_1'\sin\theta_2\sin\theta_2')^{-\nu/2}$$
$$\times\sum_{l_2=|m|}^{\infty}\frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!}(\cos\theta_1\cos\theta_1')^{l_2}P_{l_2}^m(\cos\theta_2)P_{l_2}^m(\cos\theta_2')$$
$$\times\sum_{l_1=l_2}^{\infty}\frac{(l_1+1)(l_1-l_2)!}{(l_1+l_2+1)!}Q_{l_1+1/2}^{-(\nu+3)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_{l_1-l_2}^{l_2+1}(\sin\theta_1)C_{l_1-l_2}^{l_2+1}(\sin\theta_1'), \qquad (6.59)$$

where

$$\chi=\frac{r^2+r'^2-2rr'\sin\theta_1\sin\theta_1'-2rr'\cos\theta_1\cos\theta_1'\cos\theta_2\cos\theta_2'}{2rr'\cos\theta_1\cos\theta_1'\sin\theta_2\sin\theta_2'}.$$

If you substitute $\nu=-2$ in (6.59) then you obtain the following

$$Q_{m-1/2}^{1/2}(\chi) = \sqrt{2\pi}i\left(\chi^2-1\right)^{1/4}\cos\theta_1\cos\theta_1'\sin\theta_2\sin\theta_2'$$
$$\times\sum_{l_2=|m|}^{\infty}\frac{2^{2l_2}(2l_2+1)(l_2!)^2(l_2-m)!}{(l_2+m)!}(\cos\theta_1\cos\theta_1')^{l_2}P_{l_2}^m(\cos\theta_2)P_{l_2}^m(\cos\theta_2')$$
$$\times\sum_{l_1=l_2}^{\infty}\frac{(l_1-l_2)!}{(l_1+l_2+1)!}\left(\frac{r_<}{r_>}\right)^{l+1+1}C_{l_1-l_2}^{l_2+1}(\sin\theta_1)C_{l_1-l_2}^{l_2+1}(\sin\theta_1').$$

In type $\mathbf{ca}^2$ coordinates, this is the addition theorem derived using the methods described above

$$Q_{m_1-1/2}^{-(\nu+1)/2}(\chi) = -\,2^{-(\nu-|m_1|-2)/2}\pi\left(\frac{r_>^2-r_<^2}{rr'}\right)^{(\nu+3)/2}\left(\chi^2-1\right)^{-(\nu+1)/4}(\cos\vartheta\cos\vartheta')^{|m_1|-\nu/2}$$
$$\times\sum_{m_2=-\infty}^{\infty}e^{im_2(\phi_2-\phi_2')}2^{|m_2|/2}(\sin\vartheta\sin\vartheta')^{|m_2|}\sum_{n=0}^{\infty}\frac{n!(|m_1|+|m_2|+2n+1)(|m_1|+|m_2|+n)!}{(|m_1|+n)!(|m_2|+n)!}$$
$$\times\,Q_{|m_1|+|m_2|+2n+1/2}^{-(\nu+3)/2}\left(\frac{r^2+r'^2}{2rr'}\right)P_n^{(|m_2|,|m_1|)}(\cos 2\vartheta)P_n^{(|m_2|,|m_1|)}(\cos 2\vartheta'), \qquad (6.60)$$

where

$$\chi = \frac{r^2 + r'^2 - 2rr' \sin\vartheta \sin\vartheta' \cos(\phi_2 - \phi_2')}{2rr' \cos\vartheta_1 \cos\vartheta'}.$$

If you substitute $\nu = -2$ in (6.60) then you obtain the following

$$\begin{aligned} Q^{1/2}_{m_1-1/2}(\chi) &= 2^{|m_1|/2+2} i\pi^{3/2} \left(\chi^2 - 1\right)^{1/4} (\cos\vartheta\cos\vartheta')^{|m_1|+1} \\ &\times \sum_{m_2=-\infty}^{\infty} e^{im_2(\phi_2-\phi_2')} 2^{|m_2|/2} (\sin\vartheta\sin\vartheta')^{|m_2|} \sum_{n=0}^{\infty} \frac{n!(|m_1|+|m_2|+n)!}{(|m_1|+n)!(|m_2|+n)!} \left(\frac{r_<}{r_>}\right)^{|m_1|+|m_2|+2n+1} \\ &\quad \times P_n^{(|m_2|,|m_1|)}(\cos 2\vartheta) P_n^{(|m_2|,|m_1|)}(\cos 2\vartheta'). \end{aligned} \tag{6.61}$$

Note that in the addition theorems (6.60) and (6.61), that if you make the map $\vartheta \mapsto \vartheta - \frac{\pi}{2}$, then this transformation preservers the addition theorem such that $m_1 \leftrightarrow m_2$, and it is equivalent to swapping position of ϕ_1 and ϕ_2 for the tree in Figure 4.9.

6.4.3 Logarithmic addition theorem in $\mathbf{R}^4$ for type $\mathbf{b^2a}$ coordinates

Now let's give an example of how to construct addition theorems in the even dimensions using the logarithmic Fourier expansion of a fundamental solution. We take $d = 4$ and expand an unnormalized fundamental solution, $\|\mathbf{x} - \mathbf{x}'\|^{2k-4} \log \|\mathbf{x} - \mathbf{x}'\|$ for $k \geq 2$. We examine type $\mathbf{b^2a}$ hyperspherical coordinates. In these coordinates we have

$$\begin{aligned} &\|\mathbf{x} - \mathbf{x}'\|^{2k-4} \log \|\mathbf{x} - \mathbf{x}'\| = \\ &\quad \frac{1}{2} \log(2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2')(2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2')^{k-2} \left[\chi - \cos(\phi - \phi')\right]^{k-2} \\ &\quad + \frac{1}{2}(2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2')^{k-2} \left[\chi - \cos(\phi - \phi')\right]^{k-2} \log\left[\chi - \cos(\phi - \phi')\right], \end{aligned}$$

where χ is given by (6.56). With the methods of Chapter 5 we are now in a position to Fourier expand this fundamental solution. Using (5.15) we know that

$$\left[\chi - \cos(\phi - \phi')\right]^{k-2} = i\sqrt{\frac{2}{\pi}}(-1)^{k+1}(\chi^2 - 1)^{k/2-3/4} \sum_{n=0}^{k-2} \cos[n(\phi - \phi')] \frac{\epsilon_n (2-k)_n}{(k+n-2)!} Q^{k-3/2}_{n-1/2}(\chi),$$

and from

(5.35) we know that

$$[\chi - \cos(\phi - \phi')]^{k-2} \log(\cosh \eta - \cos \psi) = \sum_{m=0}^{\infty} \cos[m(\phi - \phi')] \mathfrak{Q}_{m,k-2}(\chi),$$

so therefore

$$\|\mathbf{x} - \mathbf{x}'\|^{2k-4} \log \|\mathbf{x} - \mathbf{x}'\| = \sum_{m=0}^{\infty} \cos[m(\phi - \phi')] \Bigg\{ \frac{1}{2}(2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2')^{k-2} \mathfrak{Q}_{m,k-2}(\chi)$$
$$+ \frac{1}{2} \log(2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2')(2rr' \sin\theta_1 \sin\theta_1' \sin\theta_2 \sin\theta_2')^{k-2}$$
$$\times i\sqrt{\frac{2}{\pi}} \frac{\epsilon_m (2-k)_m (-1)^{k+1}}{(k+m-2)!} (\chi^2 - 1)^{k/2 - 3/4} Q_{m-1/2}^{k-3/2}(\chi) \Bigg\}. \tag{6.62}$$

Now we utilize the Gegenbauer polynomial expansion for a logarithmic unnormalized fundamental solution in $d = 4$, namely (6.51). In $d = 4$, we use the appropriate p, i.e. $p = k - 2$, which gives us

$$\|\mathbf{x} - \mathbf{x}'\|^{2k-4} \log \|\mathbf{x} - \mathbf{x}'\| = \frac{i(-1)^k (r_>^2 - r_<^2)^{k-1/2}}{\sqrt{\pi}(rr')^{3/2}}$$
$$\times \sum_{l_1=0}^{\infty} (l_1 + 1) \mathbf{Q}_{l_1,4,k-2}(r, r') C_{l_1}^1(\cos\gamma). \tag{6.63}$$

Using the addition theorem for hyperspherical harmonics (6.2) we see that

$$C_{l_1}^1(\cos\gamma) = \frac{2\pi^2}{l_1 + 1} \sum_{l_2=0}^{l_1} \sum_{m=-l_2}^{l_2} Y_{l_1,l_2,m}(\theta_1, \theta_2, \phi) Y_{l_1,l_2,m}^*(\theta_1', \theta_2', \phi'), \tag{6.64}$$

where $Y_{l_1,l_2,m}(\theta_1, \theta_2, \phi)$ is the appropriate normalized hyperspherical harmonic, namely (6.11). Substituting (6.11) in (6.64), inserting the result in (6.63), and reversing the order of the summations as in (6.52) gives us in type $\mathbf{b}^2\mathbf{a}$ pure hyperspherical coordinates

$$\|\mathbf{x}-\mathbf{x}'\|^{2k-4}\log\|\mathbf{x}-\mathbf{x}'\| = \sum_{m=0}^{\infty}\cos[m(\phi-\phi')]\frac{\epsilon_m\sqrt{\pi}}{4}i(-1)^k\frac{(r_>^2-r_<^2)^{k-1/2}}{(rr')^{3/2}}$$
$$\times\sum_{l_2=m}^{\infty}\frac{[(2l_2+1)!]^2(2l_2+1)(l_2-m)!}{2^{2l_2}\left[\Gamma\left(l_2+\frac{3}{2}\right)\right]^2}(\sin\theta_1\sin\theta_1')^{l_2}P_{l_2}^m(\cos\theta_1)P_{l_2}^m(\cos\theta_1')$$
$$\times\sum_{l_1=l_2}^{\infty}\frac{(l_1+1)(l_1-l_2)!}{(l_1+l_2+1)!}\mathbf{Q}_{l_1,4,k-2}(r,r')C_{l_1-l_2}^{l_2+1}(\cos\theta_1)C_{l_1-l_2}^{l_2+1}(\cos\theta_1') \tag{6.65}$$

Now by equating the Fourier coefficients of (6.62) and (6.65) we obtain the following addition theorem which is

$$\mathfrak{Q}_{m,k-2}(\chi) = -i\sqrt{\frac{2}{\pi}}\frac{(2-k)_m(-1)^{k+1}}{(k+m+2)!}\left(\chi^2-1\right)^{k/2-3/4}Q_{m-1/2}^{k-3/2}(\chi)$$
$$\times\log\left(2rr'\sin\theta_1\sin\theta_1'\sin\theta_2\sin\theta_2'\right)$$
$$+\frac{i\epsilon_m(-1)^k\sqrt{\pi}}{2}(2rr'\sin\theta_1\sin\theta_1'\sin\theta_2\sin\theta_2')^{2-k}\frac{(r_>^2-r_<^2)^{k-1/2}}{(rr')^{3/2}}$$
$$\times\sum_{l_2=m}^{\infty}\frac{[(2l_2+1)!]^2(2l_2+1)(l_2-m)!}{2^{2l_2}\left[\Gamma\left(l_2+\frac{3}{2}\right)\right]^2}(\sin\theta_1\sin\theta_1')^{l_2}P_{l_2}^m(\cos\theta_1)P_{l_2}^m(\cos\theta_1')$$
$$\times\sum_{l_1=l_2}^{\infty}\frac{(l_1+1)(l_1-l_2)!}{(l_1+l_2+1)!}C_{l_1-l_2}^{l_2+1}(\cos\theta_1)C_{l_1-l_2}^{l_2+1}(\cos\theta_1')\mathbf{Q}_{l_1,4,k-2}(r,r'), \tag{6.66}$$

for $k\in\mathbf{N}$.

6.4.4 A power-law addition theorem in $\mathbf{R}^d$ for $d\in\{3,4,\ldots\}$

If we adopt standard hyperspherical coordinates (4.31), then we have derived the expression for the harmonics, namely (6.12). We can use these harmonics in combination with the addition theorem for hyperspherical harmonics (6.2), the Gegenbauer expansion for powers of the distance (6.34), and the Fourier expansion for the powers of the distance (5.39) to generate a multi-summation addition theorem in Euclidean space $\mathbf{R}^d$ valid for $d\in\{3,4,\ldots\}$.

The Fourier expansion (5.39) gives us

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \sqrt{\frac{\pi}{2}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma\left(-\nu/2\right)}\left(2rr'\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i{}'\right)^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}$$

$$\times\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi_d^d), \tag{6.67}$$

where

$$\chi_d^d = \frac{r^2+r'^2-2rr'\sum_{i=1}^{d-2}\cos\theta_i\cos\theta_i{}'\prod_{j=1}^{i-1}\sin\theta_j\sin\theta_j{}'}{2rr'\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i{}'}. \tag{6.68}$$

We can now use the Gegenbauer expansion for powers of the distance (6.34) and then insert the appropriate Gegenbauer polynomial using the addition theorem for hyperspherical harmonics (6.2). The result is

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \frac{4e^{i\pi(\nu+d-1)/2}\pi^{(d-1)/2}}{\Gamma\left(-\frac{\nu}{2}\right)}\frac{\left(r_>^2-r_<^2\right)^{(\nu+d-1)/2}}{\left(rr'\right)^{(d-1)/2}}$$

$$\times\sum_{\boldsymbol{\lambda}}Q_{l_1+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right)Y_{l_1,\mu}(\widehat{\mathbf{x}})Y_{l_1,\mu}^*(\widehat{\mathbf{x}}'). \tag{6.69}$$

Now if we expand the product of hyperspherical harmonics in (6.69) with (6.12) (6.52), we obtain

$$
\begin{aligned}
\|\mathbf{x}-\mathbf{x}'\|^{\nu} = \sum_{m=0}^{\infty} \cos[m(\phi-\phi')] & \frac{\pi^{(d-3)/2} e^{i\pi(\nu+d-1)/2}\epsilon_m}{\Gamma\left(-\frac{\nu}{2}\right)} \frac{\left(r_>^2-r_<^2\right)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}} \\
& \times \sum_{l_{d-2}=m}^{\infty} \frac{(2l_{d-2}+1)(l_{d-2}-m)!}{(l_{d-2}+m)!} P_{l_{d-2}}^{m}\left(\cos\theta_{d-2}\right) P_{l_{d-2}}^{m}\left(\cos\theta_{d-2}'\right) \\
& \times \sum_{l_{d-3}=l_{d-2}}^{\infty} \Theta\left(l_{d-3}, l_{d-2}; \theta_{d-3}\right) \Theta\left(l_{d-3}, l_{d-2}; \theta_{d-3}'\right) \\
& \vdots \\
& \times \sum_{l_2=l_3}^{\infty} \Theta\left(l_2, l_3; \theta_2\right) \Theta\left(l_2, l_3; \theta_2'\right) \\
& \times \sum_{l_1=l_2}^{\infty} \Theta\left(l_1, l_2; \theta_1\right) \Theta\left(l_1, l_2; \theta_1'\right) Q_{l_1+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right),
\end{aligned}
\tag{6.70}
$$

where $\Theta(l_j, l_{j+1}; \theta)$ is defined in (6.14). If we compare the Fourier coefficients for (6.70) and (6.67), we derive the following addition theorem for associated Legendre functions of the second kind

$$
\begin{aligned}
Q_{m-1/2}^{-(\nu+1)/2}(\chi_d^d) = \sqrt{2} e^{i\pi(d-2)/2} \pi^{(d-4)/2} & \left(2rr' \prod_{i=1}^{d-2} \sin\theta_i \sin\theta_i{}'\right)^{-\nu/2} \\
& \times \left(\chi^2-1\right)^{-(\nu+1)/4} \frac{\left(r_>^2-r_<^2\right)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}} \\
& \times \sum_{l_{d-2}=m}^{\infty} \frac{(2l_{d-2}+1)(l_{d-2}-m)!}{(l_{d-2}+m)!} P_{l_{d-2}}^{m}\left(\cos\theta_{d-2}\right) P_{l_{d-2}}^{m}\left(\cos\theta_{d-2}'\right) \\
& \times \sum_{l_{d-3}=l_{d-2}}^{\infty} \Theta\left(l_{d-3}, l_{d-2}; \theta_{d-3}\right) \Theta\left(l_{d-3}, l_{d-2}; \theta_{d-3}'\right) \\
& \vdots \\
& \times \sum_{l_2=l_3}^{\infty} \Theta\left(l_2, l_3; \theta_2\right) \Theta\left(l_2, l_3; \theta_2'\right) \\
& \times \sum_{l_1=l_2}^{\infty} \Theta\left(l_1, l_2; \theta_1\right) \Theta\left(l_1, l_2; \theta_1'\right) Q_{l_1+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right).
\end{aligned}
\tag{6.71}
$$

This is just one example of a derived multi-summation addition theorem for arbitrary dimension. There are an unlimited numbers of such straightforward examples to generate. One must only take the time to construct them.

7

Normalized fundamental solutions for the Laplacian in $\mathbf{H}^d$

In this chapter we start with a literature review, namely §7.1 on hyperbolic geometry which concerns hyperbolic space and its models, §7.2 on the hyperboloid model of hyperbolic geometry and §7.3 on subgroup type coordinate systems which parametrize the points on the hyperboloid. Additional background material can be found in Vilenkin (1968) [100], Thurston (1997) [96], Lee (1997) [65] and Pogosyan & Winternitz (2002) [79].

This background material is used to develop results which, as far as the author is aware, have not yet appeared in the literature. The material in §7.4, on radial harmonics for a general hyperbolic hyperspherical coordinate system, is joint work with Ernie Kalnins. In §7.5 we compute an unnormalized fundamental solution for the Laplacian on the d-dimensional hyperboloid, and in §7.6 we compute Fourier expansions for an unnormalized fundamental solution of the Laplacian in $d \in \{2, 3\}$. The results of §7.5 and §7.6 are new.

7.1 Hyperbolic space

The hyperbolic d-space, denoted by $\mathbf{H}^d$, is a fundamental example of a space exhibiting hyperbolic geometry. It was developed independently by Lobachevsky and Bolyai around 1830 (see Trudeau (1987) [98]). It is a geometry analogous to Euclidean geometry, but such that Euclid's parallel postulate is no longer assumed to hold. It is a maximally symmetric, simply connected, d-dimensional Riemannian manifold with negative-constant sectional curvature (Vilenkin (1968) [100]), whereas Euclidean space $\mathbf{R}^d$ equipped with the Pythagorean norm, is a space with zero sectional curvature. The unit hyper-sphere $\mathbf{S}^d$, is an example of a space (submanifold) with positive constant sectional curvature.

There are several models of d-dimensional hyperbolic space $\mathbf{H}^d$, including the Klein (see Figure 7.1)), Poincaré (see Figure 7.2), hyperboloid, upper-half space and hemisphere models (see Thurston (1997) [96]).

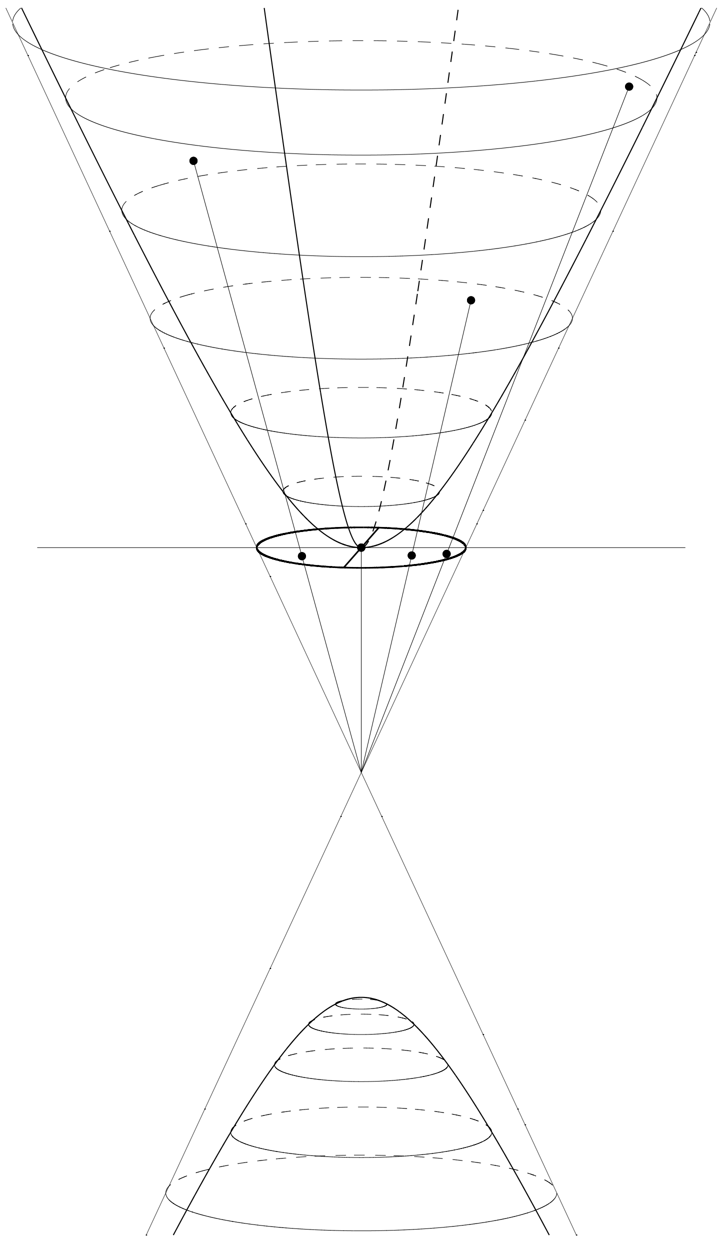

Figure 7.1: This figure depicts the stereographic projection from the hyperboloid model to the Klein model of hyperbolic geometry.

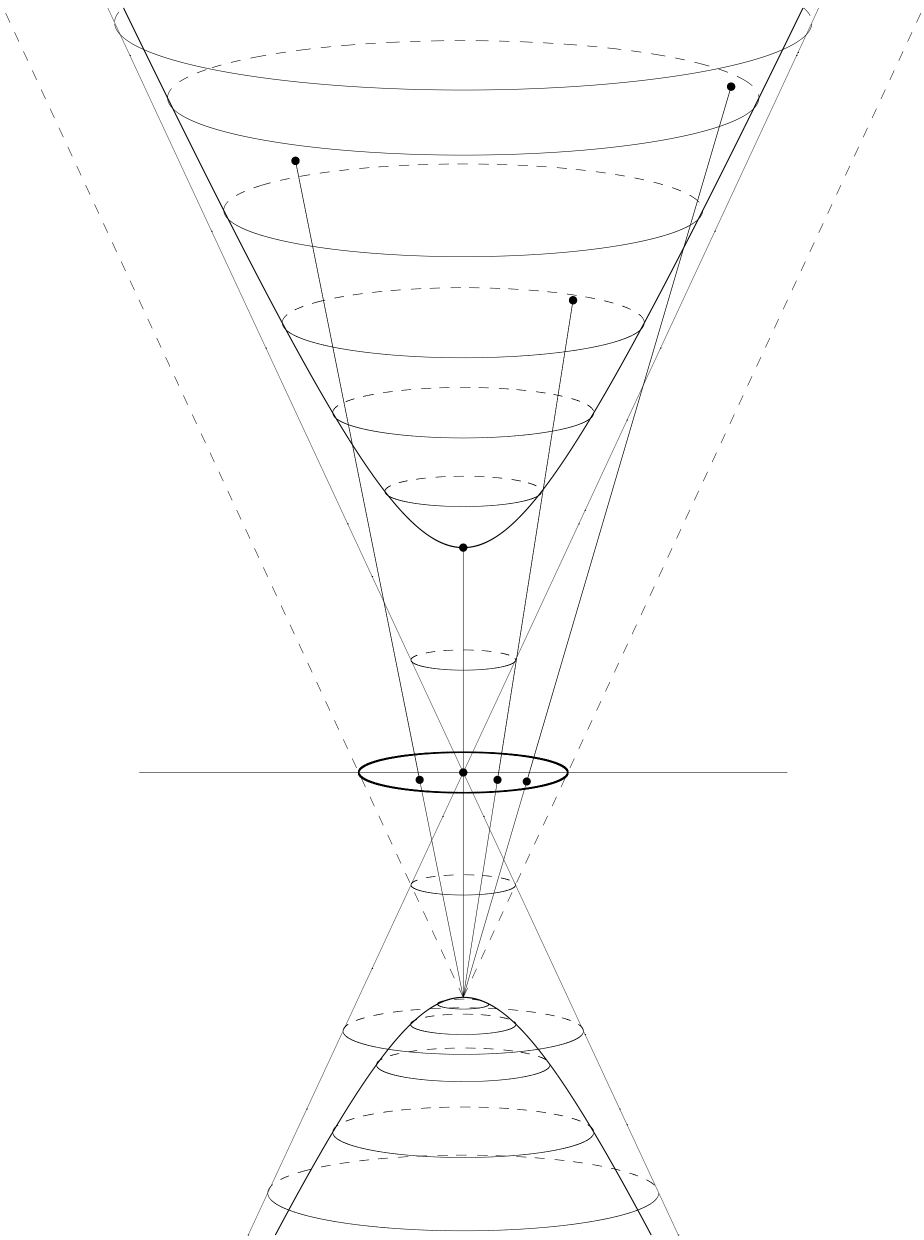

Figure 7.2: This figure depicts the stereographic projection from the hyperboloid model to the Poincaré model of hyperbolic geometry.

7.2 The hyperboloid model of $\mathbf{H}^d$

The hyperboloid model for d-dimensional hyperbolic space is closely related to the Klein and Poincaré models: each can be obtained projectively from the others. The upper-half space and hemisphere models can be obtained from one another by inversions with the Poincaré model (see §2.2 in Thurston (1997) [96]). The model we will be focusing on in this chapter is the hyperboloid model.

The hyperboloid model, also known as the Minkowski or Lorentz models, are models of d-dimensional hyperbolic geometry in which points are represented by the upper sheet (submanifold) S^+ of a two-sheeted hyperboloid embedded in the Minkowski space $\mathbf{R}^{d,1}$. The Minkowski space is a $(d+1)$-dimensional pseudo-Riemannian manifold which is a real finite-dimensional vector space, with coordinates given by $\mathbf{x} = (x_0, x_1, \ldots, x_d)$. It is equipped with a nondegenerate, symmetric bilinear form, the Minkowski inner product

$$[\mathbf{x}, \mathbf{y}] = x_0 y_0 - x_1 y_1 - \ldots - x_d y_d.$$

The above bilinear form is symmetric, but not positive-definite, so it is not a true inner product. It is defined analogously with the Euclidean inner product (cf. (4.15)) for $\mathbf{R}^{d+1}$

$$(\mathbf{x}, \mathbf{y}) = x_0 y_0 + x_1 y_1 + \ldots + x_d y_d.$$

The variety $[\mathbf{x}, \mathbf{x}] = R^2$ defines a pseudo-sphere with radius R. Points on the pseudo-sphere with zero radius coincide with a cone. Points on the pseudo-sphere with radius greater than zero lie within this cone, and points on the pseudo-sphere with purely imaginary radius lie outside the cone. In this discussion of the hyperboloid model of $\mathbf{H}^d$, we focus on the unit pseudo-sphere, i.e. the corresponding submanifold with $[\mathbf{x}, \mathbf{x}] = 1$.

The isometry group of this space is the pseudo-orthogonal group $SO(d, 1)$, the Lorentz group in $d + 1$ dimensions. Hyperbolic space $\mathbf{H}^d$, can be identified with the quotient space $SO(d, 1)/SO(d)$. The isometry group acts transitively on $\mathbf{H}^d$. That is, any point on the hyperboloid can be carried, with the help of a Euclidean rotation of $SO(d-1)$, to the point $(\cosh\alpha, \sinh\alpha, 0, \ldots, 0)$ (See Figure 7.3), and a hyperbolic rotation

$$\left.\begin{array}{rcl} x_0' &=& -x_1 \sinh\alpha + x_0 \cosh\alpha \\ x_1' &=& -x_1 \cosh\alpha - x_0 \sinh\alpha \end{array}\right\}$$

maps that point to the origin $(1, 0, \ldots, 0)$ of the space. In order to study normalized fundamental solutions on the hyperboloid, we need to describe how one computes distances in this space.

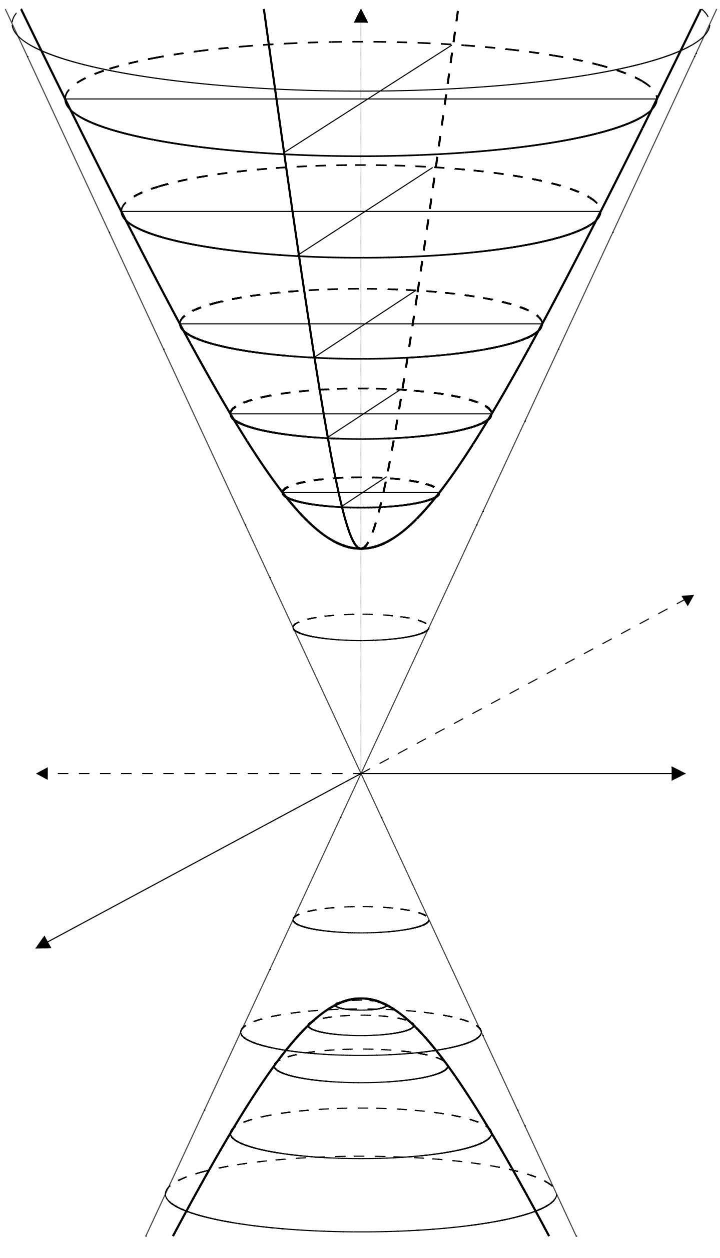

Figure 7.3: This figure depicts the transitivity of the isometry group on $\mathbf{H}^d$.

One can see how this works by analogy with the unit hyper-sphere. Distances on the unit hyper-sphere are simply given by arc lengths, angles between two arbitrary vectors, from the origin, in the ambient Euclidean space. We consider the unit d-dimensional hyper-sphere embedded in $\mathbf{R}^{d+1}$. Points on the unit hyper-sphere can be parametrized using a general hyperspherical coordinate system (these are not the only valid parametrizations).

Any parametrization of the unit hyper-sphere $\mathbf{S}^d$, must have $(\mathbf{x},\mathbf{x})=1$. The distance between two points on the unit hyper-sphere is given by

$$d(\mathbf{x},\mathbf{x}')=\gamma=\cos^{-1}\left(\frac{(\mathbf{x},\mathbf{x}')}{(\mathbf{x},\mathbf{x})(\mathbf{x}',\mathbf{x}')}\right)=\cos^{-1}\big((\mathbf{x},\mathbf{x}')\big). \tag{7.1}$$

This is evident from the fact that the geodesics on $\mathbf{S}^d$ are great circles (i.e. intersections of $\mathbf{S}^d$ with planes through the origin) with constant speed parametrizations (see Lee (1997) [65], p. 82). Therefore (7.1) actually represents the general formula for computing geodesic distances on the unit hyper-sphere $\mathbf{S}^d$.

Accordingly, we now look at the geodesic distance function on the unit d-dimensional pseudo-sphere $\mathbf{H}^d$. Distances between two points in the unit pseudo-sphere are given by the hyperangle between two arbitrary vectors, from the origin, in the ambient Minkowski space. The pseudo-sphere can be parametrized through standard hyperbolic hyperspherical coordinates (7.12). Of course there are many coordinates upon which one may parametrize the unit pseudo-sphere (see Olevskiĭ (1950) [73]), but none equivalent to Cartesian coordinates. Any parametrization of the unit hyperboloid $\mathbf{H}^d$, must have $[\mathbf{x},\mathbf{x}]=1$. The distance between two points in $\mathbf{H}^d$ is given by

$$d(\mathbf{x},\mathbf{x}')=\cosh^{-1}\left(\frac{[\mathbf{x},\mathbf{x}']}{[\mathbf{x},\mathbf{x}][\mathbf{x}',\mathbf{x}']}\right)=\cosh^{-1}\big([\mathbf{x},\mathbf{x}']\big), \tag{7.2}$$

where the inverse hyperbolic cosine with argument $x\in(1,\infty)$ is given by (2.4). The general formula for computing geodesic distances on the unit pseudo-sphere $\mathbf{H}^d$ is given by (7.2). This is clear from the fact that the geodesics on $\mathbf{H}^d$ are great hyperbolas (i.e. intersections of $\mathbf{H}^d$ with planes through the origin) with constant speed parametrizations (see Lee (1997) [65], p. 84).

7.3 Coordinate systems and the Laplacian on the hyperboloid

Parametrizations of a submanifold embedded in either a Euclidean or Minkowski space is given in terms of coordinate systems whose coordinates are curvilinear. These are coordinates based on some transformation that converts the standard Cartesian coordinates in the ambient space to a coordinate system with the same number of coordinates as the submanifold in which the coordinate lines are curved.

The Laplace-Beltrami operator (Laplacian) in curvilinear coordinates $\xi=(\xi^1,\ldots,\xi^d)$ on the Riemannian manifold $\mathbf{H}^d$ is given by

$$\Delta = \Delta_{LB} = \sum_{i,j=1}^{d} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial \xi^j} \right), \tag{7.3}$$

where the metric is

$$ds^2 = \sum_{i,j=1}^{d} g_{ij} d\xi^i d\xi^j, \;\; g = |\det(g_{ij})|, \;\; \sum_{i=1}^{d} g_{ki} g^{ij} = \delta_k^j. \tag{7.4}$$

The relation between the metric tensor $G_{ij} = \mathrm{diag}(1, -1, \ldots, -1)$ in the ambient space and g_{ij} of (7.3) and (7.4) is

$$g_{ij}(\xi) = \sum_{k,l=0}^{d} G_{kl} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^l}{\partial \xi^j}.$$

7.3.1 Subgroup type coordinates in $\mathbf{H}^d$

The class of coordinate systems which allow separation of variables for Laplace's equation on the hyperboloid, just like in Euclidean space, can be broken up into two types, those which are subgroup type coordinates and those which are not. Again, subgroup type coordinates are coordinates which can be written in terms of (maximal) subgroup chains and the powerful "method of trees," (a graphical method for constructing coordinate transformations for subgroup type coordinates) can be adopted (see §4.1).

Take for instance the proper subgroups of $O(2,1)$

$$O(2,1) \supset O(2), \tag{7.5}$$

$$O(2,1) \supset O(1,1), \quad \text{and,}$$

$$O(2,1) \supset E(1), \tag{7.6}$$

or for instance the proper subgroups of $O(3,1)$

$$O(3,1) \supset O(3) \supset O(2), \tag{7.7}$$

$$O(3,1) \supset O(2,1) \supset O(2), \tag{7.8}$$

$$O(3,1) \supset O(2,1) \supset O(1,1),$$

$$O(3,1) \supset O(2,1) \supset E(1), \tag{7.9}$$

$$O(3,1) \supset E(2) \supset O(2), \tag{7.10}$$

$$O(3,1) \supset E(2) \supset E(1) \otimes E(1), \quad \text{and,} \tag{7.11}$$

$$O(3,1) \supset O(2) \otimes O(1,1).$$

Corresponding to each of these subgroup chains is a particular tree and therefore coordinate system. See Pogosyan & Winternitz (2002) [79] for explicit parametrizations for the corresponding coordinate systems and trees. Examples of subgroup type coordinates on the hyperboloid include hyperspherical coordinates in which the subgroup chain contains a copy (or copies) of $O(p)$ for $p \in \{2, \ldots, d\}$, such as in (7.5), (7.7), (7.8), and (7.10). Other examples include cylindrical coordinates which contain p copies of the Euclidean group, such as in (7.6), (7.9) and (7.11). Non-subgroup type coordinates include those which are analogous to ellipsoidal coordinates, parabolic coordinates, etc.

At each link in the subgroup chain, each indicated subgroup on the right is maximal in the group on the left. The possible subgroup chain links can be of the following types: (see Pogosyan & Winternitz (2002) [79] and references therein for a detailed explanation of why we restrict to these particular types of subgroups) (1) for the pseudo-orthogonal group

$$O(p,1) \supset O(p),$$

$$O(p,1) \supset O(p-1,1),$$

$$O(p,1) \supset E(p-1),$$

$$O(p,1) \supset O(p_1,1) \otimes O(p_2),\ \ p_1 + p_2 = p,\ \ p_1 \geq 1, p_2 \geq 2,$$

(2) for the orthogonal group

$$O(p) \supset O(p-1),$$

$$O(p) \supset O(p_1) \otimes O(p_2),\ \ p_1 + p_2 = p,\ \ p_1 \geq p_2 \geq 1,$$

and (3) for the Euclidean group

$$E(p) \supset O(p),$$
$$E(p) \supset E(p_1) \otimes E(p_2),\ p_1 + p_2 = p,\ p_1 \geq p_2 \geq 1.$$

To simplify matters for our computation, we introduce standard hyperbolic hyperspherical coordinates, similar to spherical coordinates in Euclidean space

$$\left.\begin{array}{rcl} x_0 &=& \cosh r \\ x_1 &=& \sinh r \cos\theta_1 \\ x_2 &=& \sinh r \sin\theta_1 \cos\theta_2 \\ &\vdots& \\ x_{d-2} &=& \sinh r \sin\theta_1 \cdots \cos\theta_{d-2} \\ x_{d-1} &=& \sinh r \sin\theta_1 \cdots \sin\theta_{d-2} \cos\phi \\ x_d &=& \sinh r \sin\theta_1 \cdots \sin\theta_{d-2} \sin\phi, \end{array}\right\} \tag{7.12}$$

where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $\theta_i \in [0, \pi]$ for $i \in \{1, \ldots, d-2\}$. Standard hyperbolic hyperspherical coordinates corresponds to the subgroup chain given by $O(d,1) \supset O(d) \supset O(d-1) \supset \cdots \supset O(2)$.

7.3.2 General hyperbolic hyperspherical coordinates in $\mathbf{H}^d$

The set of all general hyperbolic hyperspherical coordinate systems corresponds to the many ways one can put coordinates on a hyperbolic hyper-sphere, namely those which correspond to subgroup chains starting with $O(d,1) \supset O(d) \supset \cdots$, with standard hyperbolic hyperspherical coordinates given by (7.12) being only one of them. They all share the property that they are described by $d+1$ variables: $r \in [0, \infty)$ plus d angles each being given by the values $[0, 2\pi)$, $[0, \pi]$, $[-\pi/2, \pi/2]$ or $[0, \pi/2]$ (see Izmest'ev et al. (1999, 2001) [58, 59]). The possibilities for subgroup chains for these coordinates are described in Chapter 4.

In any of the general hyperbolic hyperspherical coordinate systems, the geodesic distance between two points on the submanifold is given by

$$d(\mathbf{x}, \mathbf{x}') = \cosh^{-1}([\mathbf{x}, \mathbf{x}']) = \cosh^{-1}\big(\cosh r \cosh r' - \sinh r \sinh r' \cos\gamma\big), \tag{7.13}$$

where γ is the unique separation angle given in each hyperspherical coordinate system. For instance, the separation angle in standard hyperbolic hyperspherical coordinates is given by the formula

$$\cos\,\gamma = \cos(\phi-\phi')\prod_{i=1}^{d-2}\sin\theta_i\sin{\theta_i}' + \sum_{i=1}^{d-2}\cos\theta_i\cos{\theta_i}'\prod_{j=1}^{i-1}\sin\theta_j\sin{\theta_j}'. \tag{7.14}$$

where these coordinates are defined in Transformation (7.12).

Corresponding formulae for any general hyperbolic hyperspherical coordinate system can be computed using (7.1), (7.2), and the associated formulae for the appropriate inner-products. The Riemannian metric in a general hyperbolic hyperspherical coordinate system on this submanifold is

$$ds^2 = dr^2 + \sinh^2 r\ d\gamma^2, \tag{7.15}$$

where the appropriate expression for $\cos\gamma$ is chosen. If one combines (7.3), (7.12), (7.14) and (7.15), then in a particular general hyperbolic hyperspherical coordinate system, Laplace's equation on $\mathbf{H}^d$ is

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + (d-1)\coth r\frac{\partial f}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbf{S}^{d-1}} f = 0, \tag{7.16}$$

where $\Delta_{\mathbf{S}^{d-1}}$ is the corresponding Laplace-Beltrami operator on $\mathbf{S}^{d-1}$.

7.4 Harmonics in standard hyperbolic hyperspherical coordinates

General hyperbolic hyperspherical coordinate systems partition $\mathbf{H}^d$ into a family of concentric $(d-1)$-dimensional hyper-spheres, each with a radius $r \in (0,\infty)$, on which all possible hyperspherical coordinate systems for $\mathbf{S}^{d-1}$ may be used (see for instance, in Vilenkin (1968) [100]). One then must also consider the limiting case for $r=0$ to fill out all of $\mathbf{H}^d$. In standard hyperbolic hyperspherical coordinates one can compute the normalized hyperspherical harmonics in this space by solving the Laplace equation using separation of variables which results in a general procedure which is given explicitly in Izmest'ev et al. (1999, 2001) [58, 59]). These angular harmonics are given as general expressions involving complex exponentials, Gegenbauer polynomials and Jacobi polynomials.

The harmonics in general hyperbolic hyperspherical coordinate systems are given in terms of a radial solution multiplied by the angular harmonics. The angular harmonics are eigenfunctions of the Laplace-Beltrami operator on $\mathbf{S}^{d-1}$ which satisfy the following eigenvalue problem

$$\Delta_{\mathbf{S}^{d-1}} Y_{l,\mu}(\widehat{\mathbf{x}}) = l(l+d-2)Y_{l,\mu}(\widehat{\mathbf{x}}),$$

where $Y_{l,\mu}(\widehat{\mathbf{x}})$ are the normalized hyperspherical harmonics given as a function of the angular coordinates chosen to parametrize $\mathbf{S}^{d-1}$, $l \in \mathbf{N}_0$ is the angular momentum quantum number, and $\mu \in \boldsymbol{\mu}$.

Since the angular solutions are well-known, we will now focus on the radial solutions, which are therefore satisfied by the following ordinary differential equation

$$\frac{d^2u}{dr^2} + (d-1)\coth r\frac{du}{dr} - \frac{l(l+d-2)}{\sinh^2 r}u = 0.$$

Solutions to this ordinary differential equation are

$$u_{1\pm}^{d,l}(\cosh r) = \frac{1}{(\sinh r)^{d/2-1}} P_{d/2-1}^{\pm(d/2-1+l)}(\cosh r),$$

and

$$u_{2\pm}^{d,l}(\cosh r) = \frac{1}{(\sinh r)^{d/2-1}} Q_{d/2-1}^{\pm(d/2-1+l)}(\cosh r),$$

where P_ν^μ and Q_ν^μ are the associated Legendre functions of the first and second kind respectively with degree ν, order μ, and argument z (see §2.6).

Due to the fact that the space $\mathbf{H}^d$ is homogeneous with respect to its isometry group $SO(d,1)$, and therefore an isotropic manifold, we expect that there exists a normalized fundamental solution on this space with spherically symmetric dependence. We specifically expect these solutions to be given in terms of associated Legendre functions of the second kind with argument given by $\cosh r$. This associated Legendre function naturally fits our requirements because it is singular at $r = 0$ and vanishes at infinity, whereas the associated Legendre functions of the first kind, with the same argument, are regular at $r = 0$ and singular as infinity.

7.5 Normalized fundamental solution for the Laplacian in $\mathbf{H}^d$

In computing a normalized fundamental solution for the Laplacian in $\mathbf{H}^d$, we know that

$$\Delta\mathcal{H}^d(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}').$$

In general since we can add any harmonic function to a fundamental solution for the Laplacian and still have a fundamental solution, we will use this freedom to make our fundamental solution as simple as possible.

It is reasonable to expect that there exists a particular normalized spherically symmetric fundamental solution $\mathcal{H}^d(\mathbf{x}, \mathbf{x}')$ on the hyperboloid with pure radial $r := d(\mathbf{x}, \mathbf{x}')$ and constant angular (invariant under rotations centered about the origin) dependence due to the influence of the point-like nature of the Dirac delta function. For a spherically symmetric solution to the Laplace equation, the corresponding $\Delta_{\mathbf{S}^{d-1}}$ term vanishes since only the $l = 0$ term survives. In other words we expect there to exist a fundamental solution such that $\mathcal{H}^d(\mathbf{x}, \mathbf{x}') = f(r)$. We have proven that on the hyperboloid $\mathbf{H}^d$, a normalized Green's function for the Laplace operator (normalized fundamental solution for the Laplacian) can be given as follows.

Theorem 7.5.1. *Let $d \in \{2, 3, \ldots\}$. Define*

$$I_d(r) := \frac{1}{(d-1)(\cosh r)^{1-d}}\,{}_2F_1\left(\frac{d}{2}; \frac{d-1}{2}; \frac{d+1}{2}; \frac{1}{\cosh^2 r}\right),$$

and

$$\mathcal{H}^d(\mathbf{x}, \mathbf{x}') := \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}} I_d(r),$$

where $r := \cosh^{-1}([\mathbf{x}, \mathbf{x}'])$ is the geodesic distance between $\mathbf{x}$ and $\mathbf{x}'$ on the hyperboloid $\mathbf{H}^d$, then $\mathcal{H}^d$ is a normalized fundamental solution for $-\Delta$, where Δ is the Laplace-Beltrami operator on $\mathbf{H}^d$. Moreover,

$$I_d(r) = \begin{cases} (-1)^{d/2}\dfrac{(d-3)!!}{(d-2)!!}\left[\log\tanh\dfrac{r}{2} + \cosh r \displaystyle\sum_{k=1}^{d/2-1} \dfrac{(-1)^{k+1}2^{k-1}(k-1)!}{(2k-1)!!\sinh^{2k} r}\right] & \text{if } d \text{ even,} \\ (-1)^{(d-1)/2}\left[\dfrac{(d-3)!!}{(d-2)!!} + \left(\dfrac{d-3}{2}\right)! \displaystyle\sum_{k=1}^{(d-1)/2} \dfrac{(-1)^k \coth^{2k-1} r}{(2k-1)(k-1)!((d-2k-1)/2)!}\right] & \text{if } d \text{ odd.} \end{cases}$$

$$= -ie^{id\pi/2}\frac{(d-3)!!}{(d-2)!!}a_d - e^{id\pi/2}\cosh r\ {}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right)$$

$$= \frac{b_d\ e^{id\pi/2}}{(d-2)!!(\sinh r)^{d/2-1}} Q^{d/2-1}_{d/2-1}(\cosh r),$$

where

$$a_d = \begin{cases} \dfrac{\pi}{2} & \text{if } d \text{ even,} \\ 1 & \text{if } d \text{ odd,} \end{cases}$$

and

$$b_d = \begin{cases} -1 & \text{if } d \text{ even,} \\ \sqrt{\dfrac{2}{\pi}} & \text{if } d \text{ odd.} \end{cases}$$

In the rest of this section, we develop the material in order to prove this theorem. Since a spherically symmetric choice for a normalized fundamental solution satisfies Laplace's equation everywhere except at the origin, we may first set $g = f'$ in (7.16) and solve the first-order equation

$$g' + (d-1)\coth r\ g = 0,$$

which is integrable and clearly has the general solution

$$g(r) = \frac{df}{dr} = c_0 (\sinh r)^{1-d}, \tag{7.17}$$

where $c_0 \in \mathbf{R}$ is a constant which depends on d. Now we integrate (7.17) to obtain a normalized fundamental solution for the Laplacian in $\mathbf{H}^d$

$$\mathcal{H}^d(\mathbf{x}, \mathbf{x}') = c_0 I_d(r) + c_1, \tag{7.18}$$

where

$$I_d(r) := \int_r^\infty \frac{dx}{\sinh^{d-1} x}, \tag{7.19}$$

and $c_0, c_1 \in \mathbf{R}$ are constants which depend on d. Notice that we can add any harmonic function to (7.18) and still have a fundamental solution of the Laplacian since a fundamental solution of the Laplacian must satisfy

$$\int (-\Delta\varphi)(\mathbf{x}')\ \mathcal{H}^d(\mathbf{x}, \mathbf{x}') d\mathbf{x}' = \varphi(\mathbf{x}),$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^d)$, where $\mathcal{D}$ is the space of test functions. In particular, we notice that from our definition of I_d (7.19) we can see that

$$\lim_{r\to\infty} I_d(r) = 0, \tag{7.20}$$

therefore it is convenient to set $c_1 = 0$ leaving us with

$$\mathcal{H}^d(\mathbf{x}, \mathbf{x}') = c_0 I_d(r). \tag{7.21}$$

The hyperboloid, being a Riemannian manifold, must behave locally like Euclidean space, $\mathbf{R}^d$. Therefore for small r we have $e^r \simeq 1 + r$ and $e^{-r} \simeq 1 - r$ and in that limiting regime

$$I_d(r) \approx \int_r^1 \frac{dx}{x^{d-1}} \simeq \begin{cases} -\log r & \text{if } d = 2, \\ \dfrac{1}{r^{d-2}} & \text{if } d \geq 3, \end{cases}$$

which has exactly the same singularity as a normalized Euclidean fundamental solution (see Theorem 3.1.1). Therefore the proportionality constant c_0 is obtained by matching locally to a normalized Euclidean fundamental solution

$$\mathcal{H}^d = c_0 I_d \simeq \mathcal{G}^d, \tag{7.22}$$

at the singularity $\mathbf{x} = \mathbf{x}'$.

We have shown how to compute a normalized fundamental solution of the Laplace-Beltrami operator on the hyperboloid in terms of an improper integral (7.19). We would now like to express this integral in terms of well-known special functions.

An unnormalized fundamental solution I_d can be computed using elementary methods through its definition, (7.19). In $d = 2$ we have

$$I_2(r) = \int_r^\infty \frac{dx}{\sinh x} = -\log\tanh\frac{r}{2} = \frac{1}{2}\log\frac{\cosh r + 1}{\cosh r - 1},$$

and in $d = 3$ we have

$$I_3(r) = \int_r^\infty \frac{dx}{\sinh^2 x} = \coth r - 1 = \frac{e^{-r}}{\sinh r},$$

which exactly matches up to that given by (3.27) in Hostler (1955) [53]. In $d \in \{4, 5, 6, 7\}$ we have

$$\begin{aligned}
I_4(r) &= \frac{\cosh r}{2\sinh^2 r} + \frac{1}{2}\log\tanh\frac{r}{2},\\
I_5(r) &= \frac{1}{3}(\coth^3 r - 1) - (\coth r - 1),\\
I_6(r) &= \frac{\cosh r}{4\sinh^4 r} - \frac{3\cosh r}{8\sinh^2 r} - \frac{3}{8}\log\tanh\frac{r}{2}, \quad \text{and}\\
I_7(r) &= \frac{1}{5}(\coth^5 r - 1) - \frac{2}{3}(\coth^3 r - 1) + \coth r - 1.
\end{aligned}$$

In fact, using Gradshteyn & Ryzhik (2007) ([48], (2.416.2–3)) we obtain the following finite summation expressions for $I_d(r)$

$$I_d(r) = \begin{cases} (-1)^{d/2}\dfrac{(d-3)!!}{(d-2)!!}\left[\log\tanh\dfrac{r}{2} + \cosh r \displaystyle\sum_{k=1}^{d/2-1}\frac{(-1)^{k+1}2^{k-1}(k-1)!}{(2k-1)!!\sinh^{2k} r}\right] & \text{if } d \text{ even},\\[2ex] (-1)^{(d-1)/2}\left[\dfrac{(d-3)!!}{(d-2)!!} + \left(\dfrac{d-3}{2}\right)! \displaystyle\sum_{k=1}^{(d-1)/2}\frac{(-1)^k\coth^{2k-1} r}{(2k-1)(k-1)!((d-2k-1)/2)!}\right] & \\ & \text{if } d \text{ odd}. \end{cases} \tag{7.23}$$

The antiderivative (indefinite integral) can be given in terms of the Gauss hypergeometric function as

$$\int \frac{dr}{(\sinh r)^{d-1}} = e^{id\pi/2}\cosh r\,{}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right) + C,$$

where $C \in \mathbf{R}$.

This is verified as follows. By using (2.31) and the chain rule, we show

$$\frac{d}{dr} e^{id\pi/2} \cosh r \, {}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right) = e^{id\pi/2} \sinh r$$

$$\times \left[{}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right) + \frac{d}{3} \cosh^2 r \, {}_2F_1\left(\frac{3}{2}, \frac{d+2}{2}; \frac{5}{2}; \cosh^2 r\right)\right].$$

The second hypergeometric function can be simplified using Gauss' relations for contiguous hypergeometric functions, namely (2.32) and (2.33). By doing this, the term with the hypergeometric function cancels leaving only a term which is proportional to a binomial through (2.35) which reduces to $1/(\sinh r)^{d-1}$. Using this antiderivative, we can write an expression for $I_d(r)$. Applying Pfaff's transformation (2.25) to the Gauss hypergeometric function produces

$$\cosh r \; {}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right) = -i \coth r \; {}_2F_1\left(\frac{1}{2}, \frac{3-d}{2}; \frac{3}{2}; \coth^2 r\right). \tag{7.24}$$

By using the appropriate integration interval in (7.19), the above transformation, and using the Gauss summation formula (2.24) combined with (2.8) and (2.6) we obtain

$$I_d(r) = -ie^{id\pi/2} \frac{(d-3)!!}{(d-2)!!} a_d - e^{id\pi/2} \cosh r \; {}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right), \tag{7.25}$$

where $a_d \in \mathbf{R}$ is given by

$$a_d = \begin{cases} \dfrac{\pi}{2}, & \text{if } d \text{ even;} \\ 1, & \text{if } d \text{ odd.} \end{cases}$$

It is natural to ask how one might interpret the Gauss hypergeometric function

$${}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right),$$

since its argument lies on the cut of the Gauss hypergeometric function. We have used analytic continuation to extend the Gauss hypergeometric function for $|z| > 1$ and then used the principal branch of the hypergeometric function. More specifically, for d even, ${}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right)$ is analytically continued directly through (2.29) with $\arg(-z) = \pi$, since $(1-d)/2 \notin \mathbf{Z}$. For d odd ${}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right)$ is interpreted by first using Pfaff's transformation (7.24) and then through (2.29) with $\arg(-z) = \pi$, since $(d-2)/2 \notin \mathbf{Z}$.

We can derive a simpler alternative expression for $I_d(r)$ in terms of the Gauss hypergeometric function as follows. Using (2.29) we can show that

$$
{}_2F_1\left(\frac{1}{2},\frac{d}{2};\frac{3}{2},\cosh^2 r\right) = -\,i\frac{\pi}{2}\frac{(d-3)!!}{(d-2)!!}\frac{1}{\cosh r} - \frac{e^{-i\pi d/2}(\cosh r)^{-d}}{d-1}\,{}_2F_1\left(\frac{d}{2},\frac{d-1}{2};\frac{d+1}{2},\frac{1}{\cosh^2 r}\right),
$$

therefore our antiderivative is also given by

$$
\int \frac{dr}{(\sinh r)^{d-1}} = \frac{(\cosh r)^{1-d}}{1-d}\,{}_2F_1\left(\frac{d}{2};\frac{d-1}{2};\frac{d+1}{2};\frac{1}{\cosh^2 r}\right) + C,
$$

where $C \in \mathbf{R}$. By taking the appropriate interval of integration in (7.19) we have

$$
I_d(r) = \frac{1}{(d-1)(\cosh r)^{1-d}}\,{}_2F_1\left(\frac{d}{2};\frac{d-1}{2};\frac{d+1}{2};\frac{1}{\cosh^2 r}\right).
$$

Our derivation for an unnormalized fundamental solution in terms of associated Legendre functions of the second kind is as follows. We will break the problem into two cases, first d odd and then d even.

For $d \in \{3,5,7,\ldots\}$, we start with (2.46) and take $\nu = d/2-1$, $\mu = 1-d/2$ and use (2.54) to obtain

$$
\frac{1}{(\sinh r)^{d/2-1}}Q_{d/2-1}^{d/2-1}(\cosh r) = -i\sqrt{\frac{\pi}{2}}(d-3)!! + \frac{\sqrt{\pi}i(d-2)!}{2^{(d-2)/2}\left(\frac{d-3}{2}\right)!}\coth r\;{}_2F_1\left(\frac{1}{2},\frac{3-d}{2};\frac{3}{2};\coth^2 r\right). \tag{7.26}
$$

Notice that for our choice of ν and μ the $\mp$ signs for the complex exponentials in (2.46) go away since $\exp(\mp i\pi n) = (-1)^n$ for $n \in \mathbf{Z}$. After utilizing (2.19) and the properties of factorials, double factorials, the material in §2.2.1, the material on Pochhammer symbols from §2.2.2, and taking $z = \cosh r$ for $r \in (0,\infty)$, we obtain

$$
\frac{1}{(\sinh r)^{d/2-1}}Q_{d/2-1}^{d/2-1}(\cosh r) = -i\sqrt{\frac{\pi}{2}}(d-2)!! \times \left[\frac{(d-3)!!}{(d-2)!!} + \left(\frac{d-3}{2}\right)!\sum_{k=1}^{(d-1)/2}\frac{(-1)^k\coth^{2k-1} r}{(2k-1)(k-1)!((d-2k-1)/2)!}\right].
$$

We have just shown via (7.23) that

$$I_d(r) = \sqrt{\frac{2}{\pi}} \frac{e^{i\pi d/2}}{(d-2)!!} \frac{1}{(\sinh r)^{d/2-1}} Q_{d/2-1}^{d/2-1}(\cosh r),$$

for d odd. Using (7.24) we can see that we also have

$$I_d(r) = -ie^{i\pi d/2} \frac{(d-3)!!}{(d-2)!!} - e^{id\pi/2} \cosh r \; {}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right),$$

for d odd.

Now we examine the case $d \in \{2, 4, 6, \ldots\}$. For consistency, we demonstrate the finite summation expression (7.23) is consistent with our Gauss hypergeometric representation (7.25). We adopt the principal branch of the natural logarithm, perform a Taylor expansion, and take $z = \cosh r$ for $r \in (0, \infty)$, which establishes

$$\log \tanh \frac{r}{2} = -\frac{1}{2} \log \frac{\cosh r + 1}{\cosh r - 1} = -\frac{i\pi}{2} - \cosh r \; {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \cosh^2 r\right).$$

By using (2.30) we can establish

$${}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; z^2\right) = -\frac{i\pi}{2z} \frac{(d-3)!!}{(d-2)!!} - \frac{(1-z^2)^{-d/2}}{d-1} {}_2F_1\left(1, \frac{d}{2}; \frac{d+1}{2}; \frac{-1}{z^2-1}\right)$$

and

$${}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; z^2\right) = -\frac{i\pi}{2z} + (z^2-1)^{-1} {}_2F_1\left(1, 1; \frac{3}{2}; \frac{-1}{z^2-1}\right),$$

which establishes the consistency, since

$$\sum_{k=1}^{d/2-1} \frac{(-1)^k 2^k (k-1)!}{(2k-1)!! (\sinh r)^{2k}} = -\frac{-2}{\sinh^2 r} \sum_{k=0}^{d/2-2} \frac{(1)_k (1)_k}{\left(\frac{3}{2}\right)_k k!} \frac{(-1)^k}{(\sinh r)^{2k}},$$

and

$$\sum_{k=0}^{d/2-2} \frac{(1)_k (1)_k}{\left(\frac{3}{2}\right)_k k!} \frac{(-1)^k}{(\sinh r)^{2k}} = {}_2F_1\left(1, 1; \frac{3}{2}; \frac{-1}{\sinh^2 r}\right) + \frac{(d-2)!!(-1)^{d/2}}{(d-1)!!(\sinh r)^{d-2}} {}_2F_1\left(1, \frac{d}{2}; \frac{d+1}{2}; \frac{-1}{\sinh^2 r}\right).$$

By using (2.45) and taking $\nu = d/2-1$ and $\mu = 1-d/2$, we use (2.54) to obtain

$$Q_{d/2-1}^{d/2-1}(z) = \mp i\frac{\pi}{2}(d-3)!!(z^2-1)^{d/4-1/2} + \frac{(-1)^{d/2-1}(d-2)!!z}{(z^2-1)^{(d-2)/4}}\,{}_2F_1\left(\frac{3-d}{2},1;\frac{3}{2};z^2\right). \quad (7.27)$$

We see that for our choice of ν and μ the $\mp$ signs for the complex exponentials in (2.45) do not go away as $\exp(\mp i\pi/2) = \mp i$. If we applying Pfaff's transformation (2.25) twice to the hypergeometric function in (7.27) we derive

$${}_2F_1\left(\frac{3-d}{2},1;\frac{3}{2};z^2\right) = (1-z^2)^{(d-2)/2}\,{}_2F_1\left(\frac{1}{2},1;\frac{d}{2};\frac{3}{2},z^2\right).$$

Inserting this identity into (7.27) results in

$$Q_{d/2-1}^{d/2-1}(z) = (z^2-1)^{(d-2)/4}\left[\mp i\frac{\pi}{2}(d-3)!! + (d-2)!!\,z\,{}_2F_1\left(\frac{1}{2},\frac{d}{2};\frac{3}{2};z^2\right)\right]. \quad (7.28)$$

As is stated in §2.6, the $\mp$ sign in the complex exponential is appropriate for $\mathrm{Im}\,z \gtrless 0$, and since associated Legendre functions are continuous at $z \in (1,\infty)$, we would like to perform the limit of the hypergeometric function in (7.28) as z approaches this real interval from above and below. Applying (2.29), (2.27), (2.9) and (2.8) yields

$${}_2F_1\left(\frac{1}{2},\frac{d}{2};\frac{3}{2};z^2\right) = \frac{\pi}{2}\frac{(d-3)!!}{(d-2)!!}(-z^2)^{-1/2} - \frac{(-z^2)^{-d/2}}{d-1}\,{}_2F_1\left(\frac{d}{2},\frac{d-1}{2};\frac{d+1}{2};\frac{1}{z^2}\right). \quad (7.29)$$

Now we insert $z = x + i\epsilon$ for $x \in (1,\infty)$ and take the limit as ϵ approaches zero in (7.29). Using the properties of the complex logarithm we see that

$$(-z^2)^{-1/2} \to \frac{\pm i}{x},$$

and

$$(-z^2)^{-d/2} \to \frac{(-1)^{d/2}}{x^d},$$

and since the hypergeometric function is holomorphic on an open neighbourhood of x

$${}_2F_1\left(\frac{d}{2},\frac{d-1}{2};\frac{d+1}{2};\frac{1}{z^2}\right) \to {}_2F_1\left(\frac{d}{2},\frac{d-1}{2};\frac{d+1}{2};\frac{1}{x^2}\right).$$

Applying this limit to both sides of (7.29) gives

$${}_2F_1\left(\frac{1}{2},\frac{d}{2};\frac{3}{2};x^2\right) = \pm i\frac{\pi}{2x}\frac{(d-3)!!}{(d-2)!!} - \frac{(-1)^{d/2}}{x^d(d-1)}\,{}_2F_1\left(\frac{d}{2},\frac{d-1}{2};\frac{d+1}{2};\frac{1}{x^2}\right). \quad (7.30)$$

For d even $(z^2-1)^{(d-2)/4} \to (x^2-1)^{(d-2)/4}$, so if we substitute this limit in (7.28), the imaginary parts cancel and we obtain

$$Q_{d/2-1}^{d/2-1}(x) = \frac{(-1)^{d/2-1}(d-2)!!}{d-1} \frac{(x^2-1)^{(d-2)/4}}{x^{d-1}} {}_2F_1\left(\frac{d}{2}, \frac{d-1}{2}; \frac{d+1}{2}; \frac{1}{x^2}\right), \tag{7.31}$$

for d even. Solving (7.30) for ${}_2F_1(d/2,(d-1)/2;(d+1)/2;1/x^2)$, substituting this in (7.31), and replacing $x = \cosh r$ yields

$$\frac{1}{(\sinh r)^{d/2-1}} Q_{d/2-1}^{d/2-1}(\cosh r) = (d-2)!! \cosh r \; {}_2F_1\left(\frac{1}{2}, \frac{d}{2}; \frac{3}{2}; \cosh^2 r\right) + i\frac{\pi}{2}(d-3)!!.$$

Therefore using the results from §7.4 and §7.5, we can compute the integral in (7.19) to obtain

$$I_d(r) = \frac{b_d \; e^{id\pi/2}}{(d-2)!!(\sinh r)^{d/2-1}} Q_{d/2-1}^{d/2-1}(\cosh r),$$

where $b_d \in \mathbf{R}$ is given by

$$b_d = \begin{cases} -1 & \text{if } d \text{ even,} \\ \sqrt{\dfrac{2}{\pi}} & \text{if } d \text{ odd.} \end{cases}$$

Notice that our chosen fundamental solutions of the Laplacian on the hyperboloid have the property that they tend towards zero at infinity (even for the $d = 2$ case, unlike Euclidean fundamental solutions of the Laplacian).

The relevant associated Legendre functions for $d \in \{2,3,4,5,6,7\}$, which can be obtained using recurrence relation (2.47) with (2.67), (2.68) and (2.69) are given by

$$Q_0^0(\cosh r) = -\log \tanh \frac{r}{2},$$

$$\frac{1}{(\sinh r)^{1/2}} Q_{1/2}^{1/2}(\cosh r) = i\sqrt{\frac{\pi}{2}}(\coth r - 1),$$

$$\frac{1}{\sinh r} Q_1^1(\cosh r) = -\log \tanh \frac{r}{2} - \frac{\cosh r}{\sinh^2 r},$$

$$\frac{1}{(\sinh r)^{3/2}} Q_{3/2}^{3/2}(\cosh r) = 3i\sqrt{\frac{\pi}{2}}\left(-\frac{1}{3}\coth^3 r + \coth r - \frac{2}{3}\right),$$

$$\frac{1}{(\sinh r)^2} Q_2^2(\cosh r) = -3\log \tanh \frac{r}{2} - 2\frac{\cosh r}{(\sinh r)^4} - 3\frac{\cosh r}{(\sinh r)^2}, \quad \text{and}$$

$$\frac{1}{(\sinh r)^{5/2}} Q_{5/2}^{5/2}(\cosh r) = 15i\sqrt{\frac{\pi}{2}}\left(\frac{1}{15}\coth^5 r - \frac{2}{3}\coth^3 r + \coth r - \frac{8}{15}\right).$$

The constant c_0 in a normalized fundamental solution for the Laplace operator on the hyperboloid (7.21) is computed by locally matching up the singularity to a normalized fundamental solution for the Laplace operator in Euclidean space, Theorem 3.1.1. The coefficient c_0 depends on d. It is determined as follows. For $d \geq 3$ we take the asymptotic expansion for $c_0 I_d(r)$ as r approaches zero and match this to a normalized fundamental solution for Euclidean space given in Theorem 3.1.1. This yields

$$c_0 = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}}. \tag{7.32}$$

For $d = 2$ we take the asymptotic expansion for

$$c_0 I_2(r) = -c_0 \log\tanh\frac{r}{2} \simeq c_0 \log\|\mathbf{x}-\mathbf{x}'\|^{-1}$$

as r approaches zero, and match this to $\mathcal{G}^2(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\log\|\mathbf{x}-\mathbf{x}'\|^{-1}$, therefore $c_0 = \frac{1}{2\pi}$. This exactly matches (7.32) for $d = 2$. The derivation that $I_d(r)$ is an unnormalized fundamental solution of the Laplace operator on the hyperboloid $\mathbf{H}^d$ and the functions for $I_d(r)$ are computed above.

The proof of Theorem 7.5.1 is complete.

7.6 Fourier expansions for unnormalized fundamental solutions in $\mathbf{H}^d$

Now we compute Fourier expansions for unnormalized fundamental solutions $\mathfrak{h}^d$ in $\mathbf{H}^d$, for $d \in \{2, 3\}$.

7.6.1 Fourier expansion for a fundamental solution in $\mathbf{H}^2$

If we start with the generating function for Chebyshev polynomials of the first kind (2.96) we have

$$\frac{\sinh\eta}{\cosh\eta - \cos\psi} = \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) e^{-n\eta}.$$

Integrating both sides with respect to η, we obtain the following formula (see for instance Magnus, Oberhettinger & Soni (1966) [67], p. 259)

$$\log\left(1 + z^2 - 2z\cos\psi\right) = -2\sum_{n=1}^{\infty} \frac{\cos(n\psi)}{n} z^n.$$

Therefore if we take $z = \dfrac{r_<}{r_>}$, then we can derive

$$\mathcal{G}^2 \simeq \log \|\mathbf{x}-\mathbf{x}'\| = \log r_> - \sum_{n=1}^{\infty} \frac{\cos(n(\phi-\phi'))}{n}\left(\frac{r_<}{r_>}\right)^n, \tag{7.33}$$

the Fourier expansion for an unnormalized fundamental solution in Euclidean space for $d=2$.

On the hyperboloid for $d=2$ we have

$$\mathcal{H}^2 \simeq \mathfrak{h}^2 := \log\tanh\frac{1}{2}d(\mathbf{x},\mathbf{x}') = \frac{1}{2}\log\frac{\cosh d(\mathbf{x},\mathbf{x}')+1}{\cosh d(\mathbf{x},\mathbf{x}')-1}.$$

In standard hyperbolic 1-spherical coordinates, $\cos\gamma = \cos(\phi-\phi')$, and (7.13) produces

$$\cosh d(\mathbf{x},\mathbf{x}') = \cosh r\cosh r' - \sinh r \sinh r' \cos(\phi-\phi'),$$

therefore

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r\cosh r' + 1 - \sinh r\sinh r'\cos(\phi-\phi')}{\cosh r\cosh r' - 1 - \sinh r\sinh r'\cos(\phi-\phi')}.$$

Replacing $\psi = \phi-\phi'$ and rearranging the logarithms yield

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r\cosh r'+1}{\cosh r\cosh r'-1} + \frac{1}{2}\log\left(1-z_+\cos\psi\right) - \frac{1}{2}\log\left(1-z_-\cos\psi\right),$$

where

$$z_\pm := \frac{\sinh r\sinh r'}{\cosh r\cosh r'\pm 1}.$$

Note that $z_\pm \in (0,1)$ for $r, r' \in (0,\infty)$. We have the following MacLaurin series

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

where $x \in [-1,1)$. Therefore away from the singularity at $\mathbf{x}=\mathbf{x}'$ we have

$$\lambda_\pm := \log\left(1-z_\pm\cos\psi\right) = -\sum_{k=1}^{\infty}\frac{z_\pm^k}{k}\cos^k\psi. \tag{7.34}$$

We can expand the powers of cosine using the following trigonometric identity

$$\cos^k\psi = \frac{1}{2^k}\sum_{n=0}^{k}\binom{k}{n}\cos[(2n-k)\psi],$$

which is the standard expansion for powers using Chebyshev polynomials (see for instance

p. 52 in Fox & Parker (1968) [41]). Inserting this expression in (7.34), we obtain the following double-summation expression

$$\lambda_\pm = -\sum_{k=1}^{\infty}\sum_{n=0}^{k}\frac{z_\pm^k}{2^k k}\binom{k}{n}\cos[(2n-k)\psi]. \tag{7.35}$$

Now we perform a double-index replacement in (7.35). We break this sum into two separate sums, one for $k \le 2n$ and another for $k \ge 2n$. There is an overlap when both sums satisfy the equality, and in that situation we must halve after we sum over both sums. If $k \le 2n$, make the substitution $k' = k-n$ and $n' = 2n-k$. It follows that $k = 2k'+n'$ and $n = n'+k'$, therefore

$$\binom{k}{n} = \binom{2k'+n'}{n'+k'} = \binom{2k'+n'}{k'}.$$

If $k \ge 2n$ make the substitution $k' = n$ and $n' = k-2n$. Then $k = 2k'+n'$ and $n = k'$, therefore

$$\binom{k}{n} = \binom{2k'+n'}{k'} = \binom{2k'+n'}{k'+n'},$$

where the equalities of the binomial coefficients are confirmed using (2.14). To take into account the double-counting which occurs when $k = 2n$ (which occurs when $n' = 0$), we introduce a factor of $\epsilon_{n'}/2$ into the expression (and relabel $k' \mapsto k$ and $n' \mapsto n$). We are left with

$$\lambda_\pm = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{z_\pm^{2k}}{2^k k}\binom{2k}{k} - 2\sum_{n=1}^{\infty}\cos(n\psi)\sum_{k=0}^{\infty}\frac{z_\pm^{2k+n}}{2^{2k+n}(2k+n)}\binom{2k+n}{k}.$$

We can substitute

$$\binom{2k}{k} = \frac{2^{2k}\left(\frac{1}{2}\right)_k}{k!}$$

into the first term, which reduces to

$$I_\pm := -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)_k z_\pm^{2k}}{k!k} = -\int_0^{z_\pm}\frac{dz_\pm'}{z_\pm'}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)_k {z_\pm'}^{2k}}{k!} = -\int_0^{z_\pm}\frac{dz_\pm'}{z_\pm'}\left[\frac{1}{\sqrt{1-{z_\pm'}^2}}-1\right].$$

We are left with

$$I_\pm = -\log 2 + \log\left(1+\sqrt{1-z_\pm^2}\right) = -\log 2 + \log\left(\frac{(\cosh r_> \pm 1)(\cosh r_< + 1)}{\cosh r\cosh r' \pm 1}\right).$$

If we substitute

$$\binom{2k+n}{k} = \frac{2^{2k}\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}$$

into the second term, the Fourier coefficient reduces to

$$J_\pm := \frac{1}{2^{n-1}} \sum_{k=0}^{\infty} \frac{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k}{k!(n+1)_k} \frac{z_\pm^{2k+n}}{2k+n}$$

$$= \frac{1}{2^{n-1}} \int_0^{z_\pm} dz'_\pm {z'_\pm}^{n-1} \sum_{k=0}^{\infty} \frac{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k}{k!(n+1)_k} {z'_\pm}^{2k}.$$

The series in the integrand is a Gauss hypergeometric function which can be given as

$$\sum_{k=0}^{\infty} \frac{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k}{k!(n+1)_k} z^{2k} = \frac{2^n n!}{z^n\sqrt{1-z^2}} P_0^{-n}\left(\sqrt{1-z^2}\right),$$

where P_0^{-n} is an associated Legendre function of the first kind with vanishing degree and order given by $-n$. This is a consequence of the quadratic transformation of the hypergeometric function satisfied by (2.44). Therefore the Fourier coefficient is then given through (2.43) by

$$J_\pm = 2\int_{\sqrt{1-z_\pm^2}}^{1} \frac{dz'_\pm}{1-{z'_\pm}^2} \left(\frac{1-z'_\pm}{1+z'_\pm}\right)^{n/2} = \frac{2}{n}\left[\frac{1-\sqrt{1-z_\pm^2}}{1+\sqrt{1-z_\pm^2}}\right]^{n/2}.$$

Finally we have

$$\lambda_\pm = -\log 2 + \log\left(\frac{(\cosh r_> \pm 1)(\cosh r_< + 1)}{\cosh r \cosh r' \pm 1}\right) - 2\sum_{n=1}^{\infty} \frac{\cos(n\psi)}{n}\left[\frac{(\cosh r_> \mp 1)(\cosh r_< - 1)}{(\cosh r_> \pm 1)(\cosh r_< + 1)}\right]^{n/2}.$$

Therefore the Fourier expansion for an unnormalized fundamental solution of Laplace's equation on the $d = 2$ hyperboloid is given by

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r_> + 1}{\cosh r_> - 1}$$
$$+ \sum_{n=1}^{\infty} \frac{\cos(n(\phi-\phi'))}{n}\left[\frac{\cosh r_< - 1}{\cosh r_< + 1}\right]^{n/2} \left\{\left[\frac{\cosh r_> + 1}{\cosh r_> - 1}\right]^{n/2} - \left[\frac{\cosh r_> - 1}{\cosh r_> + 1}\right]^{n/2}\right\},$$

which exactly matches up to the Euclidean Fourier expansion for $d = 2$ (7.33) as $r, r' \to 0^+$.

7.6.2 Fourier expansion for a fundamental solution in $\mathbf{H}^3$

The result for the Fourier expansion for the three-dimensional Euclidean space (here given in standard spherical coordinates $\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, is well-known Cohl & Tohline (1999) [26]

$$\mathfrak{g}^3 := \mathfrak{g}_1^3 = \frac{1}{\pi\sqrt{rr'\sin\theta\sin\theta'}}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}Q_{m-1/2}\left(\frac{r^2+r'^2-2rr'\cos\theta\cos\theta'}{2rr'\sin\theta\sin\theta'}\right), \tag{7.36}$$

where $Q_{m-1/2}$ is an associated Legendre function of the second kind with odd-half-integer degree and vanishing order. These associated Legendre functions, toroidal harmonics, are given in terms of complete elliptic integrals of the first and second kind. Since $Q_{-1/2}(z)$ is given in (2.64), the $m=0$ component of $\mathfrak{g}^3$ is given by

$$\mathfrak{g}^3\big|_{m=0} = \frac{2}{\pi\sqrt{r^2+r'^2-2rr'\cos(\theta+\theta')}}K\left(\sqrt{\frac{4rr'\sin\theta\sin\theta'}{r^2+r'^2-2rr'\cos(\theta+\theta')}}\right), \tag{7.37}$$

where K is Legendre's complete elliptic integral of the first kind.

An unnormalized fundamental solution for the 3-hyperboloid in a general hyperbolic 2-spherical coordinate system is given by

$$\mathcal{H}^3 \simeq \mathfrak{h}^3 = \coth d(\mathbf{x},\mathbf{x}') = \frac{\cosh d(\mathbf{x},\mathbf{x}')}{\sqrt{\cosh^2 d(\mathbf{x},\mathbf{x}')-1}} = \frac{\cosh r\cosh r' - \sinh r\sinh r'\cos\gamma}{\sqrt{(\cosh r\cosh r' - \sinh r\sinh r'\cos\gamma)^2-1}}.$$

In standard hyperbolic 2-spherical coordinates

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi').$$

Replacing $\psi = \phi - \phi'$ and defining

$$A := \cosh r\cosh r' - \sinh r\sinh r'\cos\theta\cos\theta',$$

and

$$B := \sinh r\sinh r'\sin\theta\sin\theta',$$

we have in the standard manner, the Fourier coefficients of the expansion

$$A_m(r,r',\theta,\theta') = \frac{\epsilon_m}{\pi}\int_0^\pi \frac{(A/B-\cos\psi)\cos(m\psi)d\psi}{\sqrt{\left(\cos\psi-\frac{A+1}{B}\right)\left(\cos\psi-\frac{A-1}{B}\right)}},$$

so that

$$\mathfrak{h}^3 = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} A_m(r,r',\theta,\theta').$$

If we make the substitution $x = \cos\psi$, this integral can be converted to

$$A_m(r,r',\theta,\theta') = \frac{\epsilon_m}{\pi}\int_{-1}^{1} \frac{(A/B - x)\,T_m(x)dx}{\sqrt{(1-x)(1+x)\left(x-\frac{A+1}{B}\right)\left(x-\frac{A-1}{B}\right)}}, \tag{7.38}$$

where T_m is the Chebyshev polynomial of the first kind. Since $T_m(x)$ is expressible as a finite sum over powers of x, (7.38) involves the square root of a quartic multiplied by a rational function of x. Therefore by definition, this integral is an elliptic integral (see §2.5) and we can directly compute it using Byrd & Friedman (1954) ([19], (253.11)).

We have

$$d = -1,\ y = -1,\ c = 1,\ b = \frac{A-1}{B},\ a = \frac{A+1}{B}, \tag{7.39}$$

and clearly $d \le y < c < b < a$. We have thus converted the Fourier coefficient (7.38), as a sum of two integrals, each of the form (see Byrd & Friedman (1954) [19], (253.11))

$$\int_y^c \frac{x^m dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = c^m g\int_0^{u_1}\left[\frac{1-\alpha_1^2\mathrm{sn}^2 u}{1-\alpha^2\mathrm{sn}^2 u}\right]^m du. \tag{7.40}$$

In this expression sn is a Jacobi elliptic function (see §2.5),

$$\alpha^2 = \frac{c-d}{b-d} < 1,$$

$$\alpha_1^2 = \frac{b(c-d)}{c(b-d)},$$

$$g = \frac{2}{\sqrt{(a-c)(b-d)}},$$

$$u_1 = F(\varphi,k),$$

$$\varphi = \sin^{-1}\sqrt{\frac{(b-d)(c-y)}{(c-d)(b-y)}},$$

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)},$$

with $k^2 < \alpha^2$, and $F(\varphi,k)$ is Legendre's incomplete elliptic integral of the first kind (see

§2.5). For our specific choices in (7.39), we have

$$\alpha^2 = \frac{2B}{A+B-1},$$

$$\alpha_1^2 = \frac{2(A-1)}{A+B-1},$$

$$g = \frac{2B}{\sqrt{(A+B-1)(A-B+1)}},$$

$$k^2 = \frac{4B}{(A+B-1)(A-B+1)},$$

$$\varphi = \frac{\pi}{2},$$

$$u_1 = F\left(\frac{\pi}{2}, k\right) =: K(k),$$

and

$$\Pi\left(\frac{\pi}{2}, \alpha, k\right) =: \Pi(\alpha, k),$$

where $\Pi(\varphi, \alpha, k)$ is Legendre's incomplete elliptic integral of the third kind, $K(k)$ and $\Pi(\alpha, k)$ are Legendre's complete elliptic integrals of the first and third kind respectively (see §2.5). Specific cases include

$$\int_y^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = gK(k)$$

(Byrd & Friedman (1954) [19], (340.00)) and

$$\int_y^c \frac{xdx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{cg}{\alpha^2}\left[\alpha_1^2 K(k) + (\alpha^2 - \alpha_1^2)\Pi(\alpha, k)\right]$$

(Bryd & Friedman (1954) [19], (340.01)).

Byrd & Friedman (1954) [19] give a procedure for computing all values of (7.40), and they are all given in terms of complete elliptic integrals of the first three kinds (see the discussion in Byrd & Friedman (1954) [19], p. 201, 204, and p. 205). In general we have

$$\int_y^c \frac{x^m dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{c^m g \alpha_1^{2m} m!}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(\alpha^2 - \alpha_1^2)^j}{\alpha_1^{2j} j!(m-j)!} V_j$$

(Byrd & Friedman (1954) [19], (340.04)), where

$$V_0 = K(k),$$

$$V_1 = \Pi(\alpha, k),$$

$$V_2 = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\left[\alpha^2 E(k) + (k^2-\alpha^2)K(k) + (2\alpha^2k^2+2\alpha^2-\alpha^4-3k^2)\Pi(\alpha,k)\right],$$

$E(k) := E\left(\frac{\pi}{2}, k\right)$, where $E(\varphi, k)$ is Legendre's incomplete elliptic integral of the first kind, $E(k)$ is Legendre's complete elliptic integral of the second kind, and larger values of V_j can be computed using the following recurrence relation

$$\begin{aligned}V_{m+3} = {} & \frac{1}{2(m+2)(1-\alpha^2)(k^2-\alpha^2)} \\ & \times\big[(2m+1)k^2V_m + 2(m+1)(\alpha^2k^2+\alpha^2-3k^2)V_{m+1} \\ & \quad + (2m+3)(\alpha^4-2\alpha^2k^2-2\alpha^2+3k^2)V_{m+2}\big]\end{aligned}$$

(see Byrd & Friedman (1954) [19], (336.00–03)). For instance,

$$\int_y^c \frac{x^2\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{c^2g}{\alpha^4}\left[\alpha_1^4K(k) + 2\alpha_1^2(\alpha^2-\alpha_1^2)V_1 + (\alpha^2-\alpha_1^2)^2V_2\right]$$

(see Byrd & Friedman (1954) [19], (340.02)).

To demonstrate the behaviour of the Fourier coefficients, let's directly compute the $m=0$ component of $\mathfrak{h}^3$. For $m=0$, (7.38) reduces to

$$A_0(r,r',\theta,\theta') = \frac{1}{\pi}\int_{-1}^{1} \frac{(A/B - x)\,dx}{\sqrt{(1-x)(1+x)\left(x-\frac{A+1}{B}\right)\left(x-\frac{A-1}{B}\right)}}.$$

Therefore using the above formulae, we have

$$\begin{aligned}&\mathfrak{h}_3|_{m=0} = A_0(r,r',\theta,\theta') = \\ &= \frac{2K(k)}{\pi\sqrt{(A-B+1)(A+B-1)}} + \frac{2(A-B-1)\Pi(\alpha,k)}{\pi\sqrt{(A-B+1)(A+B-1)}} \\ &= \frac{2\left[K(k) + (\cosh r\cosh r' - \sinh r\sinh r'\cos(\theta-\theta') - 1)\,\Pi(\alpha,k)\right]}{\pi\sqrt{(\cosh r\cosh r' - \sinh r\sinh r'\cos(\theta-\theta')+1)\,(\cosh r\cosh r' - \sinh r\sinh r'\cos(\theta+\theta')-1)}}.\end{aligned} \tag{7.41}$$

Note that the Fourier coefficients

$$\mathfrak{h}^3|_{m=0} \to \mathfrak{g}^3|_{m=0},$$

in the limit as $r, r' \to 0^+$, where $\mathfrak{g}^3|_{m=0}$ is given in (7.37). This is expected since $\mathbf{H}^3$ is a Riemannian manifold.

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