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Double summation addition theorems for Jacobi functions of the first and second kind

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Dedicated to Dick Askey whose favorite function was the Jacobi polynomial.

Abstract. In this paper, we review and derive hyperbolic and trigonometric double summation addition theorems for Jacobi functions of the first and second kind. In connection with these addition theorems, we perform a full analysis of the relation between (i) Jacobi functions with symmetric, antisymmetric, and half odd integer parameter values, and (ii) certain Gauss hypergeometric functions that satisfy a quadratic transformation, including associated Legendre, Gegenbauer and Ferrers functions of the first and second kind. We also introduce Olver normalizations of the Jacobi functions, which are particularly useful in the derivation of expansion formulas when the parameters are integers. We apply the addition theorems for Jacobi functions of the second kind to separated eigenfunction expansions of fundamental solutions of Laplace–Beltrami operators on compact and noncompact rank-one symmetric spaces.

1. Introduction

Jacobi polynomials (hypergeometric polynomials) were introduced by the German mathematician Carl Gustav Jacob Jacobi (1804–1851). These polynomials first appear in an article by Jacobi, which was published posthumously in 1859 by Heinrich Eduard Heine [**29**]. Jacobi polynomials, $P_n^{(\alpha,\beta)}$, which for $\Re\alpha, \Re\beta > -1$ are orthogonal on the real segment $[-1, 1]$ [**30**, (9.8.2)], and can be defined in terms of a terminating sum by

$$P_n^{(\alpha,\beta)}(\cos\theta) := \frac{\Gamma(\alpha+1+n)}{\Gamma(n+1)\,\Gamma(\alpha+\beta+1+n)} \times \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{\Gamma(\alpha+\beta+1+n+k)}{\Gamma(\alpha+1+k)} \sin^{2k}(\tfrac{1}{2}\theta), \tag{1.1}$$

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where $\Gamma(\cdot)$ is the gamma function [**11**, (5.2.1)] and $\binom{n}{k}$ is the binomial coefficient [**11**, (1.2.1)]. This definition of the Jacobi polynomial is equivalent to the Gauss hypergeometric representation [**11**, (18.5.7)]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right), \tag{1.2}$$

where $x = \cos\theta$. We will return to the notations used in (1.2) in the following section.

Ultraspherical polynomials, traditionally defined by [**11**, (18.7.1)]

$$C_n^\mu(\cos\theta) := \frac{\Gamma(\mu+\frac{1}{2})\Gamma(2\mu+n)}{\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+n)} P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(\cos\theta), \tag{1.3}$$

are symmetric ($\alpha = \beta$) Jacobi polynomials. These polynomials are commonly referred to as Gegenbauer polynomials after the Austrian mathematician Leopold Gegenbauer (1849–1903). However the Czech (Austrian) astronomer and mathematician Moriz Allé discovered and used many of their fundamental properties including their generating function *and* addition theorems [**1**] almost a decade prior to Gegenbauer [**19**, **20**] and Heine [**24**, p. 455]. See the nice discussion of the history of the addition theorem for ultraspherical polynomials by Koornwinder in [**39**, p. 383]. The addition theorem for ultraspherical polynomials is [**11**, (18.18.8)]

$$\begin{aligned} &C_n^\mu(\cos\theta_1\cos\theta_2 \pm \sin\theta_1\sin\theta_2\cos\phi) \\ &\quad = \frac{n!}{(2\mu)_n}\sum_{k=0}^n \frac{(\mp 1)^k(\mu)_k(2\mu)_{2k}}{(-n)_k(\mu-\frac{1}{2})_k(2\mu+n)_k} \\ &\qquad \times (\sin\theta_1\sin\theta_2)^k C_{n-k}^{\mu+k}(\cos\theta_1) C_{n-k}^{\mu+k}(\cos\theta_2) C_k^{\mu-\frac{1}{2}}(\cos\phi). \end{aligned} \tag{1.4}$$

This result is quite important by itself. In the special case $\mu = \frac{1}{2}$, it becomes one way of writing the addition theorem for spherical harmonics on the two-dimensional sphere (Legendre polynomials), namely

$$\begin{aligned} &P_n(\cos\theta_1\cos\theta_2 \pm \sin\theta_1\sin\theta_2\cos\phi) \\ &\quad = \sum_{k=-n}^n (\pm 1)^k \frac{(n-k)!}{(n+k)!} \mathsf{P}_n^k(\cos\theta_1)\mathsf{P}_n^k(\cos\theta_2)\cos(k\phi), \end{aligned} \tag{1.5}$$

where the P_n^k are Ferrers functions of the first kind [**11**, (14.3.1)]. See the foreword of Willard Miller (1977) [**45**] written by Richard Askey for a beautiful discussion (on pp. xix–xx) of addition theorems (see also [**28**, §2.7]).

Given the addition theorem for ultraspherical polynomials (1.4), it was a natural problem to extend it to Jacobi polynomials for $\alpha \neq \beta$. It was a good match when Richard Askey, on sabbatical at the Mathematical Centre in Amsterdam during 1969–1970, met Tom Koornwinder there, who had some experience with group-theoretical methods and was looking for a good subject for a Ph.D. thesis. Askey suggested to Koornwinder the problem of finding an addition theorem for Jacobi polynomials, and he also arranged that Koornwinder could attend a special year at the Mittag-Leffler Institute in Sweden. There Koornwinder obtained the desired result [**35**–**38**]. He later found that his group-theoretic method and the resulting addition theorem in a special case had been anticipated by two papers in Russian. Vilenkin and Šapiro [**51**] had realized that disk polynomials [**11**, (18.37.1)] and in particular the Jacobi polynomials $P_n^{(\alpha,0)}$, for α an integer, can be interpreted as spherical functions on the complex projective space $\mathrm{SU}(\alpha+2)/\mathrm{U}(\alpha+1)$ [**5**, **6**], or as spherical functions on the complex unit sphere $\mathrm{U}(\alpha+2)/\mathrm{U}(\alpha+1)$ in $\mathbb{C}^{\alpha+2}$, a homogeneous space of the unitary group $\mathrm{U}(\alpha+2)$. (See references by Ikeda, Kayama and Seto in [**35**, **36**].) Šapiro obtained from that observation the addition theorem for Jacobi polynomials in the $\beta=0$ case [**52**].

Koornwinder initially presented his addition theorem for Jacobi polynomials in a series of three papers in 1972 [**35**, **37**, **38**]. Koornwinder gave four different proofs of the addition formula for Jacobi polynomials. His first proof focused on the spherical functions of the Lie group $\mathrm{U}(d)/\mathrm{U}(d-1)$, $d \geqslant 2$ an integer, and appeared in [**37**, **38**]; his second proof, which used ordinary spherical harmonics, appeared in [**31**]; his third proof was an analytic proof and appeared in [**3**, **32**, **33**]; and a short fourth proof using orthogonal polynomials in three variables appeared in [**34**].

Let us consider the trigonometric context of Koornwinder's addition theorem for the Jacobi polynomials. Let $n \in \mathbb{N}_0$, $\alpha > \beta > -\frac{1}{2}$, $\cos\theta_1 = \frac{1}{2}(\mathrm{e}^{i\theta_1} + \mathrm{e}^{-i\theta_1})$, $\cos\theta_2 = \frac{1}{2}(\mathrm{e}^{i\theta_2} + \mathrm{e}^{-i\theta_2})$, and $w \in (-1,1)$, $\phi \in [0,\pi]$. Then Koornwinder's addition theorem for Jacobi polynomials is

$$
\begin{aligned}
&P_n^{(\alpha,\beta)}\left(2|\cos\theta_1\cos\theta_2 \pm \mathrm{e}^{i\phi} w \sin\theta_1 \sin\theta_1|^2 - 1\right) \\
&\quad = \frac{n!\,\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \sum_{k=0}^{n} \frac{(\alpha+1)_k(\alpha+\beta+n+1)_k}{(\alpha+k)(\beta+1)_k(-n)_k} \\
&\qquad \times \sum_{l=0}^{k} (\mp 1)^{k-l} \frac{(\alpha+k+l)(-\beta-n)_l}{(\alpha+n+1)_l} (\cos\theta_1\cos\theta_2)^{k-l} (\sin\theta_1\sin\theta_2)^{k+l} \\
&\qquad\quad \times P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos(2\theta_1))\, P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos(2\theta_2)) \\
&\qquad\quad \times w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^{\beta}(\cos\phi).
\end{aligned}
\tag{1.6}
$$

As we will see in Section 3 below, this addition theorem and its various counterparts for Jacobi functions of the first and second kind are deeply connected to a system of 2-variable orthogonal polynomials, sometimes referred to as parabolic biangle polynomials and denoted by $\mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi)$.

In the case of ultraspherical and Jacobi polynomials, the sum is terminating, as one would expect since the object of study is a polynomial. However, as Flensted-Jensen and Koornwinder realized [**17**], the addition theorem for Jacobi polynomials can be extended to Jacobi functions of the first kind by formally taking the outer sum limit to infinity. While Jacobi polynomials $P_n^{(\alpha,\beta)}$ have n discrete, the Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ have λ continuous (see (3.6) below). When one starts to consider Jacobi functions, many new questions arise which must be understood for a full theory of separated eigenfunction expansions. First of all, one must consider two separate contexts: the trigonometric context, where the arguments of the functions are analytically continued from the segment $(-1,1)$; and the hyperbolic context, where the arguments of the functions are analytically continued from the segment $(1,\infty)$. On top of that, one must consider the expansions of Jacobi functions of the second kind. Gegenbauer and Jacobi functions are solutions of second-order ordinary differential equations. Therefore there are two linearly independent solutions, namely the functions of the first kind and those of the second kind. The separated eigenfunction expansions of the Gegenbauer functions of the first and second kind were treated quite extensively in a paper by Durand, Fishbane and Simmons (1976) [**14**]. Since one should refer to ultraspherical polynomials as opposed to Gegenbauer polynomials, it would be prudent in future publications to refer instead to ultraspherical functions. Durand extended Koornwinder's addition theorem to Jacobi (and other) functions of the second kind in [**13**].

The study of multi-summation addition theorems for Jacobi functions of the first and second kind seems not to have moved forward since the advances by Durand and by Flensted-Jensen and Koornwinder. In the remainder of this paper, we give the full multi-summation expansions of Jacobi functions of the second kind and bring the full theory of the expansions of Jacobi functions to a circle.

Addition theorems, such as the ones for Jacobi polynomials, are intimately related to separated eigenfunction expansions of spherical functions (reproducing kernels) on Riemannian symmetric spaces. For instance, the argument on the left-hand side of Gegenbauer's addition theorem (1.4) is easily expressible in terms of the geodesic distance between two arbitrary points on the d-dimensional real hypersphere. One is often interested in eigenfunction expansions of a fundamental solution of Laplace's or Poisson's equation, because this allows one to perform a multipole expansion of arbitrarily shaped mass distributions. When one performs a global analysis of the Laplacian (Laplace–Beltrami operator) on rank-one symmetric spaces, one finds that a fundamental solution is given in terms of radial solutions of a second-order differential equation, and since the solutions are singular at the origin, the solutions are given in terms of functions of the second kind. For rank-one symmetric spaces beyond the real case (complex, quaternionic, and octonionic), the fundamental solutions are given in terms of Jacobi functions of the second kind. This is the motivation for the work in the present paper. We were originally motivated by the symmetric spaces. They show up in the solution to the problem we were trying to solve, and the project evolved from a study of the singular part which arises within the function of the second kind.

2. Preliminaries

Throughout this paper we adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and $\mathbb{C}$ represents the complex numbers. Jacobi functions (and their special cases such as Gegenbauer, associated Legendre and Ferrers functions) have representations given in terms of Gauss hypergeometric functions, which can be defined as infinite series of ratios of shifted factorials (Pochhammer symbols). The shifted factorial can be defined for $a \in \mathbb{C}$, $n \in \mathbb{N}_0$ by $(a)_n := (a)(a+1)\cdots(a+n-1)$ [**11**, (5.2.4), (5.2.5)]. For $a \in \mathbb{C}\setminus-\mathbb{N}_0$, it is related to a ratio of two gamma functions by [**11**, Chapter 5]

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.1}$$

which allows one to extend the definition to non-positive integer values of n. Some other properties of shifted factorial which we will use are (for $n, k \in \mathbb{N}_0$, $n \geqslant k$)

$$\Gamma(a-n) = \frac{(-1)^n\Gamma(a)}{(1-a)_n}, \tag{2.2}$$

and

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}. \tag{2.3}$$

One also has an expression for the generalized binomial coefficient for $z \in \mathbb{C}$, $n \in \mathbb{N}_0$ [**11**, (1.2.6)],

$$\binom{z}{n} = \frac{(-1)^n(-z)_n}{n!}. \tag{2.4}$$

Define the multisets $\mathbf{a} := \{a_1, \ldots, a_r\}$, $\mathbf{b} := \{b_1, \ldots, b_s\}$. We will adhere to the common notational product convention that when $a_l \in \mathbb{C}$, $l \in \mathbb{N}$, $r, k \in \mathbb{N}_0$,

$$(\mathbf{a})_k := (a_1, \ldots, a_r)_k := (a_1)_k(a_2)_k \cdots (a_r)_k, \tag{2.5}$$

$$\Gamma(\mathbf{a}) := \Gamma(a_1, \ldots, a_r) := \Gamma(a_1)\cdots\Gamma(a_r). \tag{2.6}$$

Also, in the multiset notation $\mathbf{a} + t := \{a_1 + t, \ldots, a_r + t\}$.

For any expression of the form $(z^2-1)^\alpha$, we fix the branch of the power functions so that

$$(z^2-1)^\alpha := (z+1)^\alpha(z-1)^\alpha,$$

for any fixed $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{-1, 1\}$. The generalized hypergeometric function [**11**, Chapter 16] is defined as the infinite series [**11**, (16.2.1)]

$${}_rF_s(\mathbf{a}; \mathbf{b}; z) := {}_rF_s\left({\mathbf{a} \atop \mathbf{b}}; z\right) := \sum_{k=0}^\infty \frac{(\mathbf{a})_k}{(\mathbf{b})_k}\frac{z^k}{k!}, \tag{2.7}$$

where $|z| < 1$, $b_j \notin -\mathbb{N}_0$, for $j \in \{1, \ldots, s\}$; and elsewhere by analytic continuation. The Olver-normalized (scaled or regularized) generalized hypergeometric series ${}_r\boldsymbol{F}_s(\mathbf{a}; \mathbf{b}; z)$ is defined by

$${}_r\boldsymbol{F}_s(\mathbf{a}; \mathbf{b}; z) := {}_r\boldsymbol{F}_s\left({\mathbf{a} \atop \mathbf{b}}; z\right) := \frac{1}{\Gamma(\mathbf{b})}{}_rF_s\left({\mathbf{a} \atop \mathbf{b}}; z\right) = \sum_{k=0}^\infty \frac{(a_1, \ldots, a_r)_k}{\Gamma(\mathbf{b}+k)}\frac{z^k}{k!}, \tag{2.8}$$

which is entire for all $a_l, b_j \in \mathbb{C}$, $l \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$. Both the generalized and Olver-normalized generalized hypergeometric series, if nonterminating, are entire if $r \leqslant s$, convergent for $|z| < 1$ if $r = s+1$, and divergent if $r \geqslant s+1$.

The special case of the generalized hypergeometric function with $r = 2$, $s = 1$ is referred to as the Gauss hypergeometric function [**11**, Chapter 15], or simply the hypergeometric function. It has many interesting properties, including linear transformations which were discovered by Euler and Pfaff. Euler's linear transformation is [**11**, (15.8.1)]

$${}_2\boldsymbol{F}_1\left({a, b \atop c}; z\right) = (1-z)^{c-a-b}\,{}_2\boldsymbol{F}_1\left({c-a, c-b \atop c}; z\right), \tag{2.9}$$

and Pfaff's linear transformation is [**11**, (15.8.1)]

(2.10)
$${}_2\boldsymbol{F}_1\left({a, b \atop c}; z\right) = (1-z)^{-a}\,{}_2\boldsymbol{F}_1\left({a, c-b \atop c}; \frac{z}{z-1}\right) = (1-z)^{-b}\,{}_2\boldsymbol{F}_1\left({b, c-a \atop c}; \frac{z}{z-1}\right).$$

2.1. The Gegenbauer and associated Legendre functions. The Gauss hypergeometric functions which satisfy quadratic transformations are Gegenbauer and associated Legendre functions of the first and second kind. As we will see, these functions correspond to Jacobi functions of the first and second kind with parameters satisfying certain relations. We now describe some of the properties of these functions, which have a deep and long history.

Let $n \in \mathbb{N}_0$. The Gegenbauer (ultraspherical) polynomial, which is an important specialization of the Jacobi polynomial for symmetric parameters values, is given in terms of a terminating Gauss hypergeometric series by [**11**, (18.7.1)]

$$C_n^\mu(z) = \frac{(2\mu)_n}{(\mu+\frac12)_n} P_n^{(\mu-\frac12,\mu-\frac12)}(z) = \frac{(2\mu)_n}{n!} {}_2F_1\left(\begin{matrix}-n, 2\mu+n\\ \mu+\frac12\end{matrix}; \frac{1-z}{2}\right). \tag{2.11}$$

Note that the ultraspherical polynomials satisfy the parity relation [**11**]

$$C_n^\mu(-z) = (-1)^n C_n^\mu(z). \tag{2.12}$$

Gegenbauer functions, which generalize ultraspherical polynomials in having arbitrary degrees $n = \lambda \in \mathbb{C}$, are solutions $w = w(z) = w_\lambda^\mu(z)$ of the Gegenbauer differential equation [**11**, Table 18.8.1]

$$(z^2-1)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} + (2\lambda+1)\, z\frac{\mathrm{d}w(z)}{\mathrm{d}z} - \lambda(\lambda+2\mu)w(z) = 0. \tag{2.13}$$

There are two linearly independent solutions of this second-order ordinary differential equation which are referred to as Gegenbauer functions of the first and second kind: C_λ^μ, D_λ^μ. A differential equation closely connected to the Gegenbauer differential equation (2.13) is the associated Legendre differential equation, which is [**11**, (14.2.2)]

$$(1-z^2)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}w(z)}{\mathrm{d}z} + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right) w(z) = 0. \tag{2.14}$$

Two linearly independent solutions of this equation are referred to as associated Legendre functions of the first and second kind: P_ν^μ, Q_ν^μ. In the following subsection we present the definitions of these important functions, which are Gauss hypergeometric functions that satisfy a quadratic transformation.

2.1.1. *Hypergeometric representations of the Gegenbauer and associated Legendre functions.* The Gegenbauer function of the first kind is defined by [**11**, (15.9.15)]

$$C_\lambda^\mu(z) := \frac{\sqrt{\pi}\,\Gamma(\lambda+2\mu)}{2^{2\mu-1}\Gamma(\mu)\Gamma(\lambda+1)}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}-\lambda, 2\mu+\lambda\\ \mu+\frac12\end{matrix}; \frac{1-z}{2}\right), \tag{2.15}$$

where $\lambda+2\mu \notin -\mathbb{N}_0$. It is a clear extension of the Gegenbauer polynomial, with the index allowed to be a complex number as well as a non-negative integer. Gegenbauer functions of the second kind, which will be useful to us in comparing to Jacobi functions of the second kind, are the following. They have two hypergeometric representations: for $\lambda+2\mu\not\in -\mathbb{N}_0$ [**14**, (2.3)]

(2.16)
$$D_\lambda^\mu(z) := \frac{\mathrm{e}^{i\pi\mu}\Gamma(\lambda+2\mu)}{\Gamma(\mu)(2z)^{\lambda+2\mu}}\,{}_2\boldsymbol{F}_1\left(\begin{array}{c}\frac12\lambda+\mu,\frac12\lambda+\mu+\frac12\\ \lambda+\mu+1\end{array};\frac{1}{z^2}\right)$$

$$(2.17)\qquad = \frac{\mathrm{e}^{i\pi\mu}2^\lambda\Gamma(\lambda+\mu+\frac12)\Gamma(\lambda+2\mu)}{\sqrt{\pi}\,\Gamma(\mu)(z-1)^{\lambda+\mu+\frac12}(z+1)^{\mu-\frac12}}\,{}_2\boldsymbol{F}_1\left(\begin{array}{c}\lambda+1,\lambda+\mu+\frac12\\ 2\lambda+2\mu+1\end{array};\frac{2}{1-z}\right),$$

and in the second representation $\lambda+\mu+\frac12\not\in-\mathbb{N}_0$. The equality of these representations follows from a quadratic transformation of the Gauss hypergeometric function from Group 3 to Group 1 in [**11**, Table 15.8.1].

The associated Legendre function of the first kind is defined as [**11**, (14.3.6) and §14.21(i)]

$$(2.18)\qquad P_\nu^\mu(z) := \left(\frac{z+1}{z-1}\right)^{\frac12\mu}{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\nu,\nu+1\\ 1-\mu\end{array};\frac{1-z}{2}\right),$$

for $|1-z|<2$, and elsewhere in z by analytic continuation. The associated Legendre function of the second kind $Q_\nu^\mu:\mathbb{C}\setminus(-\infty,1]\to\mathbb{C}$, $\nu+\mu\not\in-\mathbb{N}$, has the two single Gauss hypergeometric function representations (see [**11**, (14.3.7) and §14.21], [**44**, entry 24, p. 161])

(2.19)
$$Q_\nu^\mu(z) := \frac{\sqrt{\pi}\,\mathrm{e}^{i\pi\mu}\Gamma(\nu+\mu+1)(z^2-1)^{\frac12\mu}}{2^{\nu+1}z^{\nu+\mu+1}}\,{}_2\boldsymbol{F}_1\left(\begin{array}{c}\frac{\nu+\mu+1}{2},\frac{\nu+\mu+2}{2}\\ \nu+\frac32\end{array};\frac{1}{z^2}\right)$$

$$(2.20)\qquad = \frac{2^\nu\mathrm{e}^{i\pi\mu}\Gamma(\nu+1)\Gamma(\nu+\mu+1)(z+1)^{\frac12\mu}}{(z-1)^{\frac12\mu+\nu+1}}\,{}_2\boldsymbol{F}_1\left(\begin{array}{c}\nu+1,\nu+\mu+1\\ 2\nu+2\end{array};\frac{2}{1-z}\right),$$

where for the second representation, $\nu\not\in-\mathbb{N}$. These representations are convergent as a Gauss hypergeometric series for $|z|>1$, respectively $|z-1|>2$, and elsewhere in $z\in\mathbb{C}\setminus(-\infty,1]$ by analytic continuation of the Gauss hypergeometric function.

Remark 2.1. The associated Legendre functions of the first and second kind are related to the Gegenbauer functions of the first and second kind by [**11**, (14.3.22)]

$$(2.21)\qquad P_\nu^\mu(z) = \frac{\Gamma(\frac12-\mu)\Gamma(\nu+\mu+1)}{2^\mu\sqrt{\pi}\,\Gamma(\nu-\mu+1)(z^2-1)^{\frac12\mu}}C_{\nu+\mu}^{\frac12-\mu}(z),$$

$$(2.22)\qquad Q_\nu^\mu(z) = \frac{\mathrm{e}^{2\pi i(\mu-\frac14)}\sqrt{\pi}\,\Gamma(\frac12-\mu)\Gamma(\nu+\mu+1)}{2^\mu\Gamma(\nu-\mu+1)(z^2-1)^{\frac12\mu}}D_{\nu+\mu}^{\frac12-\mu}(z),$$

which are valid for $\mu\in\mathbb{C}\setminus\{\frac12,\frac32,\ldots\}$, $\nu+\mu\in\mathbb{C}\setminus-\mathbb{N}$. Equivalently, the inverse relationships are

$$(2.23)\qquad C_\lambda^\mu(z) = \frac{\sqrt{\pi}\,\Gamma(\lambda+2\mu)}{2^{\mu-\frac12}\Gamma(\mu)\Gamma(\lambda+1)(z^2-1)^{\frac{\mu}{2}-\frac14}}P_{\lambda+\mu-\frac12}^{\frac12-\mu}(z),$$

$$(2.24)\qquad D_\lambda^\mu(z) = \frac{\mathrm{e}^{2\pi i(\mu-\frac14)}\Gamma(\lambda+2\mu)}{\sqrt{\pi}\,2^{\mu-\frac12}\Gamma(\mu)\Gamma(\lambda+1)(z^2-1)^{\frac12\mu-\frac14}}Q_{\lambda+\mu-\frac12}^{\frac12-\mu}(z),$$

which are valid for $\lambda+2\mu\in\mathbb{C}\setminus-\mathbb{N}_0$.

REMARK 2.2. By comparing the Gauss hypergeometric representations of the various functions, one may express ${}_2\boldsymbol{F}_1(a,a+\frac12;c;z)$ in terms of associated Legendre functions of the first and second kind P^μ_ν, Q^μ_ν, and the Gegenbauer functions of the first and second kind C^μ_λ, D^μ_λ, by using the following very useful formulas. Let $z\in\mathbb{C}\setminus[1,\infty)$. Then

$$
{}_2\boldsymbol{F}_1\left(\begin{matrix}a,a+\frac12\\ c\end{matrix};z\right)=2^{c-1}z^{\frac12(1-c)}(1-z)^{\frac12 c-a-\frac12}P^{1-c}_{2a-c}\left(\frac{1}{\sqrt{1-z}}\right)
$$
$$
=\frac{2^{2c-2}\Gamma(c-\frac12)\Gamma(2(a-c+1))}{\sqrt{\pi}\,\Gamma(2a)(1-z)^a}C^{c-\frac12}_{2a-2c+1}\left(\frac{1}{\sqrt{1-z}}\right), \tag{2.25}
$$

where $2c\notin\{1,-1,-3,\ldots\}$, $2a-2c\notin\{-2,-3,\ldots\}$, and

$$
{}_2\boldsymbol{F}_1\left(\begin{matrix}a,a+\frac12\\ c\end{matrix};z\right)=\frac{\mathrm{e}^{i\pi(c-2a-\frac12)}2^{c-\frac12}(1-z)^{\frac12 c-a-\frac14}}{\sqrt{\pi}\,\Gamma(2a)z^{\frac12 c-\frac14}}Q^{2a-c+\frac12}_{c-\frac32}\left(\frac{1}{\sqrt{z}}\right)
$$
$$
=\frac{\mathrm{e}^{i\pi(2a-c)}2^{2c-2a-1}\Gamma(c-2a)(1-z)^{c-2a-\frac12}}{\Gamma(2c-2a-1)z^{c-a-\frac12}}D^{c-2a}_{2a-1}\left(\frac{1}{\sqrt{z}}\right), \tag{2.26}
$$

where $c, c-2a\notin-\mathbb{N}_0$.

2.1.2. *The Gegenbauer functions on the cut* $(-1,1)$ *and the Ferrers functions.* We will consider Jacobi functions of the second kind on-the-cut in Section 2.2.3. As we will see, for certain combinations of the parameters which we will describe below, the Jacobi functions of the first and second kind on-the-cut are related to the the Gegenbauer functions of the first and second kind on-the-cut and the associated Legendre functions of the first and second kind on-the-cut (Ferrers functions).

The Gegenbauer functions of the first and second kind on-the-cut are defined in terms of the Gegenbauer functions immediately above and below the segment $(-1,1)$ in the complex plane. These definition are [**12**, (3.3), (3.4)]

$$
\mathsf{C}^\mu_\lambda(x):=D^\mu_\lambda(x+i0)+\mathrm{e}^{-2\pi i\mu}D^\mu_\lambda(x-i0)=C^\mu_\lambda(x\pm i0),\quad x\in(-1,1], \tag{2.27}
$$
$$
\mathsf{D}^\mu_\lambda(x):=-iD^\mu_\lambda(x+i0)+i\mathrm{e}^{-2\pi i\mu}D^\mu_\lambda(x-i0),\quad x\in(-1,1). \tag{2.28}
$$

Note that $\mathsf{C}^\mu_\lambda(x)$ and D^μ_λ are real for real values of λ and μ.

The Ferrers functions of the first and second kind are defined as [**11**, (14.23.1), (14.23.2)]

$$
\mathsf{P}^\mu_\nu(x):=\mathrm{e}^{\pm i\pi\mu}P^\mu_\nu(x\pm i0)
$$
$$
=\frac{i\mathrm{e}^{-i\pi\mu}}{\pi}\left(\mathrm{e}^{-\frac12 i\pi\mu}Q^\mu_\nu(x+i0)-\mathrm{e}^{\frac12 i\pi\mu}Q^\mu_\nu(x-i0)\right), \tag{2.29}
$$
$$
\mathsf{Q}^\mu_\nu(x):=\frac{\mathrm{e}^{-i\pi\mu}}{2}\left(\mathrm{e}^{-\frac12 i\pi\mu}Q^\mu_\nu(x+i0)+\mathrm{e}^{\frac12 i\pi\mu}Q^\mu_\nu(x-i0)\right). \tag{2.30}
$$

Using the above definition one can readily obtain a single hypergeometric representation of the Gegenbauer function of the first kind on-the-cut, namely

$$
\mathsf{C}^\mu_\lambda(x)=\frac{\sqrt{\pi}\,\Gamma(2\mu+\lambda)}{2^{2\mu-1}\Gamma(\mu)\Gamma(\lambda+1)}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\lambda,2\mu+\lambda\\ \mu+\frac12\end{matrix};\frac{1-x}{2}\right), \tag{2.31}
$$

which is identical to the Gegenbauer function of the first kind defined by (2.15), because this function analytically continues to the segment $(-1,1)$; see (2.27). For the Gegenbauer function of the second kind on-the-cut, one can readily obtain a double hypergeometric representation by using the definition (2.28), using the interrelation between the Gegenbauer function of the second kind and the Legendre function of the second kind, and then comparing to the Ferrers function of the second kind through its definition (2.30).

However, first we will give hypergeometric representations of the Ferrers function of the first and second kind, which are easily found in the literature. The first author recently co-authored a paper with Park and Volkmer where all double hypergeometric representations of the Ferrers function of the second kind were computed [**10**]. Using (2.29) one can derive hypergeometric representations of the Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut) $\mathsf{P}_\nu^\mu\colon (-1,1)\to\mathbb{C}$. For instance, one has a single hypergeometric representation [**11**, (14.3.1)]

$$\mathsf{P}_\nu^\mu(x)=\left(\frac{1+x}{1-x}\right)^{\frac12\mu}{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\nu,\nu+1\\1-\mu\end{array};\frac{1-x}{2}\right). \tag{2.32}$$

Let $\nu\in\mathbb{C}$, $\mu\in\mathbb{C}\setminus\mathbb{Z}$, $\nu+\mu\notin-\mathbb{N}$; then a double hypergeometric representation of the Ferrers function of the second kind is [**11**, (14.3.2)]

$$\begin{aligned}\mathsf{Q}_\nu^\mu(x)=\frac{\pi}{2\sin(\pi\mu)}\Bigg(&\cos(\pi\mu)\left(\frac{1+x}{1-x}\right)^{\frac12\mu}{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\nu,\nu+1\\1-\mu\end{array};\frac{1-x}{2}\right)\\&-\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\frac12\mu}{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\nu,\nu+1\\1+\mu\end{array};\frac{1-x}{2}\right)\Bigg).\end{aligned} \tag{2.33}$$

Lemma 2.3. *Let $x\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $\lambda,\nu,\mu\in\mathbb{C}$. Then*

$$\mathsf{D}_\lambda^\mu(x)=\frac{\Gamma(\lambda+2\mu)}{2^{\mu-\frac32}\sqrt{\pi}\,\Gamma(\mu)\Gamma(\lambda+1)(1-x^2)^{\frac12\mu-\frac14}}\mathsf{Q}^{\frac12-\mu}_{\lambda+\mu-\frac12}(x) \tag{2.34}$$

when $\lambda+2\mu\notin-\mathbb{N}_0$, and

$$\mathsf{Q}_\nu^\mu(x)=\frac{\sqrt{\pi}\,\Gamma(\frac12-\mu)\Gamma(\nu+\mu+1)}{2^{\mu+1}\Gamma(\nu-\mu+1)(1-x^2)^{\frac12\mu}}\mathsf{D}^{\frac12-\mu}_{\nu+\mu}(x) \tag{2.35}$$

when $\mu\notin\{\frac12,\frac32,\ldots\}$ and $\nu+\mu\notin-\mathbb{N}$.

Proof. Start with the definition (2.28) and use the interrelation between the Gegenbauer function of the second kind on-the-cut and the Ferrers function of the second kind, given by (2.24). Applying this relation to the double hypergeometric representation completes the proof. ∎

Theorem 2.4. *Let $x\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, and $\lambda,\mu\in\mathbb{C}$, such that $\lambda+2\mu\notin-\mathbb{N}_0$. Then*

$$\begin{aligned}\mathsf{D}_\lambda^\mu(x)=&\frac{\sqrt{\pi}}{\cos(\pi\mu)2^{\mu-\frac12}\Gamma(\mu)}\\&\times\left(\frac{\sin(\pi\mu)\Gamma(\lambda+2\mu)}{\Gamma(\lambda+1)(1+x)^{\mu-\frac12}}{}_2\boldsymbol{F}_1\left(\begin{array}{c}\lambda+\mu+\frac12,\frac12-\lambda-\mu\\\frac12+\mu\end{array};\frac{1-x}{2}\right)\right.\\&\left.-\frac{1}{(1-x)^{\mu-\frac12}}{}_2\boldsymbol{F}_1\left(\begin{array}{c}\lambda+\mu+\frac12,\frac12-\lambda-\mu\\\frac32-\mu\end{array};\frac{1-x}{2}\right)\right).\end{aligned} \tag{2.36}$$

Proof. Start with the definition (2.28) and use the interrelation between the Gegenbauer function of the second kind and the Legendre function of the second kind, given by (2.24). Then comparing with the double hypergeometric representation (2.33) completes the proof. □

Note that we also have an interrelation between the Ferrers function of the first kind and the Gegenbauer function of the first kind on-the-cut [**11**, (14.3.21)]:

$$\mathsf{P}^{\mu}_{\nu}(x) = \frac{\Gamma(\frac12-\mu)\Gamma(\nu+\mu+1)}{2^{\mu}\sqrt{\pi}\,\Gamma(\nu-\mu+1)(1-x^2)^{\frac12\mu}}\mathsf{C}^{\frac12-\mu}_{\nu+\mu}(x), \tag{2.37}$$

where $\mu\not\in\{\frac12,\frac32,\ldots\}$, $\nu+\mu\not\in-\mathbb{N}$; or equivalently

$$\mathsf{C}^{\mu}_{\lambda}(x) = \frac{\sqrt{\pi}\,\Gamma(\lambda+2\mu)}{2^{\mu-\frac12}\Gamma(\mu)\Gamma(\lambda+1)(1-x^2)^{\frac12\mu-\frac14}}\mathsf{P}^{\frac12-\mu}_{\lambda+\mu-\frac12}(x). \tag{2.38}$$

Finally we should add that the Legendre polynomial (the associated Legendre function of the first kind P^{μ}_{ν} and the Ferrers function of the first kind P^{μ}_{ν} with $\mu=0$ and $\nu=n\in\mathbb{Z}$) is given by [**11**, (18.7.9)]

$$P_n(x) := P^0_n(x) = \mathsf{P}^0_n(x) = C^{\frac12}_n(x) = P^{(0,0)}_n(x),$$

which vanishes for n negative.

2.2. Brief introduction to Jacobi functions of the first and second kind. Now we will discuss fundamental properties and special values and limits for the Jacobi functions. Jacobi functions are complex solutions $w=w(z)=w^{(\alpha,\beta)}_{\gamma}(z)$ of the Jacobi differential equation [**11**, Table 18.8.1]

$$(1-z^2)\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+(\beta-\alpha-z(\alpha+\beta+2))\frac{\mathrm{d}w}{\mathrm{d}z}+\gamma(\alpha+\beta+\gamma+1)w=0 \tag{2.39}$$

which is a second-order linear homogeneous differential equation. Solutions of this equation satisfy the three-term recurrence relation [**15**, (10.8.11), p. 169]

$$B^{(\alpha,\beta)}_{\gamma}w^{(\alpha,\beta)}_{\gamma}(z)+A^{(\alpha,\beta)}_{\gamma}(z)w^{(\alpha,\beta)}_{\gamma+1}(z)+w^{(\alpha,\beta)}_{\gamma+2}(z)=0, \tag{2.40}$$

where

$$A^{(\alpha,\beta)}_{\gamma}(z) = -\frac{(\alpha+\beta+2\gamma+3)\left(\alpha^2-\beta^2+(\alpha+\beta+2\gamma+2)(\alpha+\beta+2\gamma+4)z\right)}{2(\gamma+2)(\alpha+\beta+\gamma+2)(\alpha+\beta+2\gamma+2)}, \tag{2.41}$$

$$B^{(\alpha,\beta)}_{\gamma} = \frac{(\alpha+\gamma+1)(\beta+\gamma+1)(\alpha+\beta+2\gamma+4)}{(\gamma+2)(\alpha+\beta+\gamma+2)(\alpha+\beta+2\gamma+2)}. \tag{2.42}$$

This three-term recurrence relation is very useful for deriving various solutions of (2.39) when solutions are known for values which have integer separations.

2.2.1. *The Jacobi function of the first kind.* The Jacobi function of the first kind is a generalization of the Jacobi polynomial (as given by (1.2)) when the degree is no longer restricted to be an integer. In the following material we derive properties of the Jacobi function of the first kind. In the following result we present the four single Gauss hypergeometric function representations of the Jacobi function of the first kind.

THEOREM 2.5. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha+\gamma \notin -\mathbb{N}$. Then, the Jacobi function of the first kind $P_\gamma^{(\alpha,\beta)} : \mathbb{C} \setminus (-\infty, -1] \to \mathbb{C}$ can be defined by*

$$P_\gamma^{(\alpha,\beta)}(z) := \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma, \alpha+\beta+\gamma+1\\ \alpha+1\end{matrix}; \frac{1-z}{2}\right) \tag{2.43}$$

$$= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{2}{z+1}\right)^{\beta} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\beta-\gamma, \alpha+\gamma+1\\ \alpha+1\end{matrix}; \frac{1-z}{2}\right) \tag{2.44}$$

$$= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{z+1}{2}\right)^{\gamma} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma, -\beta-\gamma\\ \alpha+1\end{matrix}; \frac{z-1}{z+1}\right) \tag{2.45}$$

$$= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{2}{z+1}\right)^{\alpha+\beta+\gamma+1} {}_2\boldsymbol{F}_1\left(\begin{matrix}\alpha+\gamma+1, \alpha+\beta+\gamma+1\\ \alpha+1\end{matrix}; \frac{z-1}{z+1}\right). \tag{2.46}$$

PROOF. Start with (1.2) and replace the shifted factorial by a ratio of gamma functions using (2.1) and $n! = \Gamma(n+1)$, and substitute $n \mapsto \gamma \in \mathbb{C}$, $x \mapsto z$. Application of Euler's transformation (2.9) and Pfaff's transformation (2.10) provides the other three single hypergeometric representations. This completes the proof. ☐

There exist double Gauss hypergeometric representations of the Jacobi function of the first kind which can be obtained by using the linear transformation formulas for the Gauss hypergeometric function based on $z \mapsto (1-z^{-1})^{-1}$, $z \mapsto z^{-1}$, $z \mapsto (1-z)^{-1}$, $z \mapsto 1-z$, $z \mapsto 1-z^{-1}$ [**11**, (15.8.1)–(15.8.5)], respectively. However, these in general will involve a sum of two Gauss hypergeometric functions. We will not present the double hypergeometric function representations of the Jacobi function of the first kind here.

One has the following connection relation for the Jacobi function of the first kind.

COROLLARY 2.5.1. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $z \in \mathbb{C} \setminus (-\infty, -1]$, $\gamma \notin -\mathbb{N}$, $\beta+\gamma \notin \mathbb{N}_0$. Then*

$$P_{-\gamma-\alpha-\beta-1}^{(\alpha,\beta)}(z) = \frac{\Gamma(-\beta-\gamma)\Gamma(\gamma+1)}{\Gamma(-\gamma-\alpha-\beta)\Gamma(\alpha+\gamma+1)} P_\gamma^{(\alpha,\beta)}(z). \tag{2.47}$$

PROOF. This relation can be derived from (2.43) by making the replacement $\gamma \mapsto -\gamma-\alpha-\beta-1$, which leaves the parameters and argument of the hypergeometric function unchanged. Comparing the prefactors completes the proof. ☐

REMARK 2.6. One of the consequences of the definition of the Jacobi function of the first kind is the special value

$$P_\gamma^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)}, \tag{2.48}$$

where $\alpha+\gamma \notin -\mathbb{N}$. For $\gamma = n \in \mathbb{Z}$ one has

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \qquad P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}, \tag{2.49}$$

which is consistent with (2.48) and the parity relation for Jacobi polynomials (see [**11**, Table 18.6.1]). From (2.43) we have

$$P_0^{(\alpha,\beta)}(z) = 1, \tag{2.50}$$

and $P_k^{(\alpha,\beta)}(z) = 0$ for all $k \in -\mathbb{N}$.

2.2.2. *The Jacobi function of the second kind.* The Jacobi function of the second kind $Q_\gamma^{(\alpha,\beta)}$, $\gamma \in \mathbb{C}$ is a generalization of the Jacobi function of the second kind $Q_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$ (as given by [**15**, (10.8.18)]), where the degree is no longer restricted to be an integer. In the following material we derive properties of the Jacobi function of the second kind. Below we give the four single Gauss hypergeometric function representations of the Jacobi function of the second kind.

THEOREM 2.7. *Let $\gamma, \alpha, \beta, z \in \mathbb{C}$ such that $z \in \mathbb{C} \setminus [-1,1]$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$. Then, the Jacobi function of the second kind has the Gauss hypergeometric representations*

$$Q_\gamma^{(\alpha,\beta)}(z) := \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha+\gamma+1}(z+1)^{\beta}}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}\gamma+1, \alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1-z}\right) \tag{2.51}$$

$$= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha+\beta+\gamma+1}}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}\beta+\gamma+1, \alpha+\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1-z}\right) \tag{2.52}$$

$$= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha}(z+1)^{\beta+\gamma+1}}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}\gamma+1, \beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1+z}\right) \tag{2.53}$$

$$= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z+1)^{\alpha+\beta+\gamma+1}}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}\alpha+\gamma+1, \alpha+\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1+z}\right). \tag{2.54}$$

PROOF. Start with [**15**, (10.8.18)] and let $n \mapsto \gamma \in \mathbb{C}$ and $x \mapsto z$. Application of Pfaff's ($z \mapsto z/(z-1)$) and Euler's ($z \mapsto z$) transformations, (2.9) and (2.10), provides the other three representations. This completes the proof. □

One has the following connection relation between Jacobi functions of the first kind and Jacobi functions of the second kind.

COROLLARY 2.7.1. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, $\alpha+\beta+2\gamma \notin \mathbb{Z}$. Then*

$$\begin{aligned} P_\gamma^{(\alpha,\beta)}(z) = \frac{-2\sin(\pi(\beta+\gamma))}{\pi\sin(\pi(\alpha+\beta+2\gamma+1))}\Big(&\sin(\pi\gamma) Q_\gamma^{(\alpha,\beta)}(z) \\ &- \sin(\pi(\alpha+\gamma))\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)} Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(z)\Big). \end{aligned} \tag{2.55}$$

PROOF. This can be derived by starting with (2.43), applying the linear transformation based on $z \mapsto z^{-1}$ [**11**, (15.8.2)], and then comparing twice with Theorem 2.7. □

REMARK 2.8. Using (2.55) one can see that for $\gamma = n \in \mathbb{N}_0$, $Q^{(\alpha,\beta)}_{-\alpha-\beta-\gamma-1}$ is a Jacobi polynomial, namely

(2.56)
$$Q^{(\alpha,\beta)}_{-\alpha-\beta-1-n}(z) = \frac{\Gamma(-\alpha)\Gamma(-\beta)}{2\Gamma(-\alpha-\beta)} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n} P^{(\alpha,\beta)}_n(z)$$
$$= -\frac{\pi}{2} \frac{\sin(\pi(\alpha+\beta))}{\sin(\pi\alpha)\sin(\pi\beta)} \frac{n!\,\Gamma(\alpha+\beta+1+n)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} P^{(\alpha,\beta)}_n(z).$$

REMARK 2.9. From Theorem 2.7 one can derive special values for $Q^{(\alpha,\beta)}_{-1}(z)$ and $Q^{(\alpha,\beta)}_0(z)$, namely

$$Q^{(\alpha,\beta)}_{-1}(z) = \frac{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)(z-1)^\alpha(z+1)^\beta}, \tag{2.57}$$

$$Q^{(\alpha,\beta)}_0(z) = \frac{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{(z+1)^{\alpha+\beta+1}} {}_2\boldsymbol{F}_1\left(\begin{matrix}\alpha+1, \alpha+\beta+1\\ \alpha+\beta+2\end{matrix}; \frac{2}{1+z}\right). \tag{2.58}$$

Using the three-term recurrence relation (2.40) one can derive values of the Jacobi function of the second kind at all negative integer values of the parameter γ. For instance, one can derive

$$Q^{(\alpha,\beta)}_{-2}(z) = \frac{2^{\alpha+\beta-2}\Gamma(\alpha-1)\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)(z-1)^\alpha(z+1)^\beta}\big(\alpha-\beta+(\alpha+\beta-2)z\big), \tag{2.59}$$

and expressions for Jacobi functions of the second kind with further negative integer values of γ.

If one examines the Gauss hypergeometric representations presented in Theorem 2.7 one can see that they are not defined for certain values of γ, α, β, since one must avoid $\alpha+\gamma$ and $\beta+\gamma$ being a negative integer. In fact, these singularities are removable and one is able to compute the values of these Jacobi functions. The Jacobi function of the second kind is evaluated when the parameters α, β and the degree γ are non-negative integers in the following result, which was inspired by the work in [**53**].

THEOREM 2.10. *Let $n, a, b \in \mathbb{N}_0$, $z \in \mathbb{C} \setminus [-1, 1]$. Then*

(2.60)
$$Q^{(a,b)}_n(z) = \frac{(-1)^{a+n}}{2^{n+1}} \sum_{\substack{k=0\\k\neq n}}^{a+b+2n} \frac{(-2)^k}{(n-k)} \big((z+1)^{n-k} - (z-1)^{n-k}\big) P^{(a+n-k,b+n-k)}_k(z)$$
$$+ \frac{(-1)^a}{2} \log\left(\frac{z+1}{z-1}\right) P^{(a,b)}_n(z).$$

PROOF. Start with the integral representation for the Jacobi function of the second kind [**49**, (4.61.1)]

$$Q^{(\alpha,\beta)}_\gamma(z) = \frac{1}{2^{\gamma+1}(z-1)^\alpha(z+1)^\beta} \int_{-1}^{1} \frac{(1-t)^{\alpha+\gamma}(1+t)^{\beta+\gamma}}{(z-t)^{\gamma+1}}\,\mathrm{d}t, \tag{2.61}$$

where $\Re(\alpha+\gamma)$, $\Re(\beta+\gamma) > -1$ [**53**, (2.5)], and identify $(\gamma,\alpha,\beta) = (n,a,b) \in \mathbb{N}_0^3$. Then consider

$$\begin{aligned}\mu_{n,k}^{(a,b)}(z) &:= \frac{\mathrm{d}^k}{\mathrm{d}z^k}(1-z)^{n+a}(1+z)^{n+b}\\ &= (-1)^k 2^k k!(1-z)^{a+n-k}(1+z)^{b+n-k}P_k^{(a+n-k,b+n-k)}(z),\end{aligned}$$

where we have used the Rodrigues-type formula for Jacobi polynomials (see [**11**, Table 18.5.1]). It is easy to show that

$$(1-t)^{n+a}(1+t)^{n+b} = \sum_{k=0}^{2n+a+b} \mu_{n,k}^{(a,b)}(z)\frac{(t-z)^k}{k!}, \tag{2.62}$$

and the right-hand side is valid for all $z \in \mathbb{C}$. Now start with (2.61) and insert (2.62) into the integrand and perform the integration over $t \in (-1,1)$, using

$$\int_{-1}^{1}(z-t)^{k-n-1}\,\mathrm{d}t = \begin{cases} \dfrac{(z+1)^{k-n}-(z-1)^{k-n}}{k-n} & \text{if } k \neq n,\\ \log\left(\dfrac{z+1}{z-1}\right) & \text{if } k = n,\end{cases}$$

which completes the proof. □

By using (2.53) we find that if $|z| \sim 1+\epsilon$, then as $\epsilon \to 0^+$ one has the behavior of the Jacobi function of the second kind near the singularity at $z=1$, as follows:

$$Q_\gamma^{(\alpha,\beta)}(1+\epsilon) \sim \frac{2^{\alpha-1}\Gamma(\alpha)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)\epsilon^\alpha}, \tag{2.63}$$

where $\Re\alpha > 0$, $\beta+\gamma \notin -\mathbb{N}$. By using (2.53) we see that as $|z| \to \infty$ one has

$$Q_\gamma^{(\alpha,\beta)}(z) \sim \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+2\gamma+2)z^{\alpha+\beta+\gamma+1}}, \tag{2.64}$$

where $\alpha+\gamma+1, \beta+\gamma \notin -\mathbb{N}$.

2.2.3. *Jacobi functions of the first and second kind on-the-cut.* We now refer to the real segment $(-1,1)$ as the cut and the Jacobi functions of the first and second kind on-the-cut as $\mathsf{P}_\gamma^{(\alpha,\beta)}, \mathsf{Q}_\gamma^{(\alpha,\beta)}$. The natural definitions of these Jacobi functions are due to Durand and can be found in [**12**, (2.3), (2.4)] (see also [**4**]). They are

$$\begin{aligned}\mathsf{P}_\gamma^{(\alpha,\beta)}(x) &:= \frac{i}{\pi}\left(\mathrm{e}^{i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x+i0) - \mathrm{e}^{-i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x-i0)\right)\\ &= P_\gamma^{(\alpha,\beta)}(x \pm i0)\end{aligned} \tag{2.65}$$

and

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) := \frac{1}{2}\left(\mathrm{e}^{i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x+i0) + \mathrm{e}^{-i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x-i0)\right). \tag{2.66}$$

Note that the Jacobi function of the first kind on-the-cut (2.65) is simply an analytic continuation of the Jacobi function of the first kind (see Theorem 2.5) since the complex-valued function is continuous across the real interval $(-1,1]$. On the other hand, the Jacobi function of the second kind on-the-cut is not an analytic continuation of the Jacobi function of the second kind (see Theorem 2.7). This is because $Q_\gamma^{(\alpha,\beta)}$ is not continuous across the real interval $(-1,1)$. Hence, an 'average' as shown in (2.66) must be taken of the function values with infinitesimal positive and negative arguments, to define it. Originally, in Szegő's book [**49**, §4.62.9] (see also [**15**, (10.8.22)]) the definition of the Jacobi function of the second kind on-the-cut was $\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) := \frac{1}{2}\big(Q_\gamma^{(\alpha,\beta)}(x+i0) + Q_\gamma^{(\alpha,\beta)}(x-i0)\big)$, but as is pointed out by Durand [**12**], Szegő's definition destroys the analogy between $\mathsf{P}_\gamma^{(\alpha,\beta)}(\cos\theta)$, $\mathsf{Q}_\gamma^{(\alpha,\beta)}(\cos\theta)$ and the trigonometric functions. Hence with the updated Durand definitions for the Jacobi functions of the first and second kind on-the-cut, namely (2.65) and (2.66), one has the asymptotics as $n\to\infty$ [**12**, p. 77]

$$Q_n^{(\alpha,\beta)}(\cos\theta\pm i0) \sim \frac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}} \big(\sin(\tfrac{1}{2}\theta)\big)^{-\alpha-\frac{1}{2}} \big(\cos(\tfrac{1}{2}\theta)\big)^{-\beta-\frac{1}{2}} \mathrm{e}^{\mp iN\theta\mp i\frac{\pi}{2}(\alpha+\frac{1}{2})},$$

where $N := n+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}$.

There are many double hypergeometric representations of the Jacobi function of the second kind on-the-cut $\mathsf{Q}_\gamma^{(\alpha,\beta)} : \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))\to\mathbb{C}$. These representations follow by applying the definition (2.66) to Theorem 2.7, which provides Gauss hypergeometric representations for the Jacobi function of the second kind. Applying (2.66) causes the argument of each Gauss hypergeometric function to be just above or below the ray $(1,\infty)$, on which it is known that the Gauss hypergeometric function is discontinuous. The values above and below this ray may be transformed into values in a region where the function is continuous, by utilizing the transformations which one can find in [**10**, Appendix B]. Each of these transformations expresses a Gauss hypergeometric function with argument $x\pm i0$ as a sum of two such functions, with a common argument which is one of x^{-1}, $1-x$, $1-x^{-1}$ or $(1-x)^{-1}$. Eight Gauss hypergeometric function representations of the Jacobi function of the second kind on-the-cut can be obtained by starting with (2.51)–(2.54), applying the transformation [**10**, Theorem B.1] $z\mapsto z^{-1}$, and utilizing the Euler (2.9) or the Pfaff (2.10) transformations as needed. There are certainly more Gauss hypergeometric representations that can be obtained for the Jacobi function of the second kind on-the-cut by applying [**10**, Theorems B.2–B.4], but the derivation of these representations must be left to a later publication. We will give two of these here for $\gamma,\alpha,\beta\in\mathbb{C}$ with $\alpha,\beta\notin\mathbb{Z}$, $\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$, namely (2.67)

$$\begin{aligned}
\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) &= \frac{\pi}{2\sin(\pi\alpha)}\Bigg(-\cos(\pi\alpha)\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,\alpha+\beta+\gamma+1\\ 1+\alpha\end{matrix};\frac{1-x}{2}\right)\\
&\quad+\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)}\left(\frac{2}{1-x}\right)^{\alpha}\left(\frac{2}{1+x}\right)^{\beta}{}_2\boldsymbol{F}_1\left(\begin{matrix}-\alpha-\beta-\gamma,\gamma+1\\ 1-\alpha\end{matrix};\frac{1-x}{2}\right)\Bigg),\\
&= \frac{\pi}{2^{\gamma+1}\sin(\pi\alpha)}\Bigg(-\cos(\pi\alpha)\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}(1+x)^{\gamma}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,-\beta-\gamma\\ 1+\alpha\end{matrix};\frac{x-1}{x+1}\right)\\
&\quad+\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)}\frac{(1+x)^{\alpha+\gamma}}{(1-x)^{\alpha}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\alpha-\beta-\gamma,-\alpha-\gamma\\ 1-\alpha\end{matrix};\frac{x-1}{x+1}\right)\Bigg).
\end{aligned}$$

Just as we were able to compute the values of the Jacobi function of the second kind with non-negative integer parameters and degree, the same evaluation can be accomplished for the Jacobi function of the second kind on-the-cut, which we present now.

THEOREM 2.11. *Let $n, a, b \in \mathbb{N}_0$, $x \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Then*
(2.68)
$$\mathsf{Q}_n^{(a,b)}(x) = \frac{(-1)^n}{2^{n+1}} \sum_{\substack{k=0\\k\neq n}}^{a+b+2n} \frac{(-2)^k}{(n-k)} \left((1+x)^{n-k} - (x-1)^{n-k}\right) P_k^{(a+n-k,b+n-k)}(x)$$
$$+ \frac{1}{2} \log\left(\frac{1+x}{1-x}\right) P_n^{(a,b)}(x).$$

PROOF. Start with Theorem 2.10 and use the definition (2.66), which completes the proof. □

Note that by setting $a = b$ in the above result we can obtain an interesting finite sum expression for the Ferrers functions of the second kind with non-negative integer degree and order, as a sum over ultraspherical polynomials.

COROLLARY 2.11.1. *Let $n, a \in \mathbb{N}_0$, Let $n, a \in \mathbb{N}_0$, $x \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Then*
(2.69)
$$\mathsf{Q}_n^a(x) = \frac{(-1)^a (1-x^2)^{\frac{1}{2}a}}{2\sqrt{\pi}} \Bigg((-1)^{n+a} 2^n (n+a)!$$
$$\times \sum_{\substack{k=0\\k\neq n-a}}^{2n} \frac{(-1)^k \Gamma(n-k+\frac{1}{2})}{2^k (2n-k)!(n-a-k)} \left((1+x)^{n-a-k} - (x-1)^{n-a-k}\right) C_k^{n-k+\frac{1}{2}}(x)$$
$$+ 2^a \Gamma(a+\tfrac{1}{2}) \log\left(\frac{1+x}{1-x}\right) C_{n-a}^{a+\frac{1}{2}}(x) \Bigg).$$

PROOF. Start with (2.11) and set $a = b$. Then utilizing (2.91) below with (2.71) completes the proof. □

By using (2.67) with $x = 1 - \epsilon$ one has that

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(1-\epsilon) \sim \frac{2^{\alpha-1}\Gamma(\alpha)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)\epsilon^\alpha} \qquad \text{as } \epsilon \to 0^+, \tag{2.70}$$

where $\beta + \gamma + 1 \notin -\mathbb{N}_0$ and $\Re\alpha > 0$.

2.3. Specializations to Gegenbauer, associated Legendre, and Ferrers functions. Here we discuss some limiting cases where the Jacobi functions reduce to more elementary functions such as Gegenbauer, associated Legendre, and Ferrers functions.

The following identities involve symmetric and antisymmetric Jacobi functions of the first kind. The relation between the symmetric Jacobi function of the first kind and the Gegenbauer function of the first kind for $z \in \mathbb{C} \setminus (-\infty, -1]$ is

$$P_\gamma^{(\alpha,\alpha)}(z) = \frac{\Gamma(2\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+\gamma+1)} C_\gamma^{\alpha+\frac{1}{2}}(z). \tag{2.71}$$

This follows by starting with (2.43) and comparing it to the Gauss hypergeometric representation of the Gegenbauer function of the first kind on the right-hand side, using (2.15).

Remark 2.12. The relation between the symmetric Jacobi function of the first kind and the Ferrers function of the first kind is

$$P_\gamma^{(\alpha,\alpha)}(x) = \frac{2^\alpha\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(1-x^2)^{\frac12\alpha}}\mathsf{P}_{\alpha+\gamma}^{-\alpha}(x), \tag{2.72}$$

where $x \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, and the relation between the symmetric Jacobi function of the first kind and the associated Legendre function of the first kind is

$$P_\gamma^{(\alpha,\alpha)}(z) = \frac{2^\alpha\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(z^2-1)^{\frac12\alpha}}P_{\alpha+\gamma}^{-\alpha}(z), \tag{2.73}$$

where $z \in \mathbb{C}\setminus(-\infty,1]$. These are easily obtained through [**11**, (18.7.2)] and [**11**, (14.3.21), (14.3.22)].

Remark 2.13. The relation between the antisymmetric Jacobi function of the first kind on-the-cut and the Ferrers function of the first kind and the Gegenbauer function of the first kind on-the-cut is

$$\begin{aligned}\mathsf{P}_\gamma^{(\alpha,-\alpha)}(x) &= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{1+x}{1-x}\right)^{\frac12\alpha}\mathsf{P}_\gamma^{-\alpha}(x)\\ &= \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^\alpha\Gamma(\gamma+1)\Gamma(\alpha+1)}(1+x)^\alpha\mathsf{C}_{\gamma-\alpha}^{\alpha+\frac12}(x),\end{aligned} \tag{2.74}$$

where $x \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, and the relation between the antisymmetric Jacobi function of the first kind and the associated Legendre and Gegenbauer function of the first kind is

$$\begin{aligned}P_\gamma^{(\alpha,-\alpha)}(z) &= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{z+1}{z-1}\right)^{\frac12\alpha}P_\gamma^{-\alpha}(z)\\ &= \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^\alpha\Gamma(\gamma+1)\Gamma(\alpha+1)}(z+1)^\alpha C_{\gamma-\alpha}^{\alpha+\frac12}(z),\end{aligned} \tag{2.75}$$

where $z \in \mathbb{C}\setminus(-\infty,-1]$. These are obtained by comparing (2.43) with (2.32) and (2.18).

Remark 2.14. One has the following quadratic transformations of the symmetric Jacobi functions of the first kind, which can be found in [**49**, Theorem 4.1]. Let $z \in \mathbb{C}\setminus(-\infty,-1]$, $\gamma,\alpha\in\mathbb{C}$, $\alpha+\gamma\notin-\mathbb{N}$. Then

$$P_{2\gamma}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha+2\gamma+1)}{2^{2\gamma}\Gamma(\gamma+\frac12)\Gamma(\alpha+\gamma+1)}P_\gamma^{(\alpha,-\frac12)}(2z^2-1), \tag{2.76}$$

where $\alpha+2\gamma\notin-\mathbb{N}$, $\gamma\notin-\mathbb{N}+\frac12$, and

$$P_{2\gamma+1}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha+2\gamma+2)z}{2^{2\gamma+1}\Gamma(\gamma+\frac32)\Gamma(\alpha+\gamma+1)}P_\gamma^{(\alpha,\frac12)}(2z^2-1), \tag{2.77}$$

where $\alpha+2\gamma+1\notin-\mathbb{N}$, $\gamma\notin-\mathbb{N}-\frac12$. The restrictions on the parameters come directly by applying the restrictions on the parameters in Theorem 2.5 to the Jacobi functions of the first kind on both sides of the relations.

Below we present some identities which involve symmetric and antisymmetric Jacobi functions of the second kind.

THEOREM 2.15. *Two equivalent relations between the symmetric Jacobi function of the second kind and the associated Legendre function of the second kind are*

$$Q_\gamma^{(\alpha,\alpha)}(z) = \frac{2^\alpha \mathrm{e}^{i\pi\alpha}\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(z^2-1)^{\frac{1}{2}\alpha}} Q_{\alpha+\gamma}^{-\alpha}(z), \tag{2.78}$$

$$Q_\gamma^{(\alpha,\alpha)}(z) = \frac{2^\alpha \mathrm{e}^{-i\pi\alpha}\Gamma(\alpha+\gamma+1)}{\Gamma(2\alpha+\gamma+1)(z^2-1)^{\frac{1}{2}\alpha}} Q_{\alpha+\gamma}^{\alpha}(z), \tag{2.79}$$

where $\alpha+\gamma\notin-\mathbb{N}$. *Also, two equivalent relations between antisymmetric Jacobi functions of the second kind and the associated Legendre function of the second kind are*

$$Q_\gamma^{(\alpha,-\alpha)}(z) = \frac{\mathrm{e}^{-i\pi\alpha}\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma+1)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\alpha} Q_\gamma^\alpha(z), \tag{2.80}$$

$$Q_\gamma^{(-\alpha,\alpha)}(z) = \frac{\mathrm{e}^{-i\pi\alpha}\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma+1)} \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}\alpha} Q_\gamma^\alpha(z), \tag{2.81}$$

where $\gamma-\alpha\notin-\mathbb{N}$.

PROOF. By comparing (2.51) and (2.54) with (2.20) and by using the Legendre duplication formula [**11**, (5.5.5)] one can obtain all these formulas in a straightforward way. $\square$

See [**8**, Section 3, (A.14)] for an interesting application of the symmetric relation for associated Legendre functions of the second kind.

REMARK 2.16. Observe that by identifying (2.78) and (2.79), for $z\in\mathbb{C}\setminus[-1,1]$ one has

$$Q_\gamma^{(\alpha,\alpha)}(z) = \frac{2^{2\alpha}}{(z^2-1)^\alpha} \begin{cases} Q_{\gamma+2\alpha}^{(-\alpha,-\alpha)}(z) & \text{if } z\in\mathbb{C}\setminus[-1,1] \\ & \quad\text{such that } \Re z\geqslant 0, \\ \mathrm{e}^{2\pi i\alpha} Q_{\gamma+2\alpha}^{(-\alpha,-\alpha)}(z) & \text{if } z\in\mathbb{C}\setminus[-1,1] \\ & \quad\text{such that } \Re z<0 \text{ and } \Im z<0, \\ \mathrm{e}^{-2\pi i\alpha} Q_{\gamma+2\alpha}^{(-\alpha,-\alpha)}(z) & \text{if } z\in\mathbb{C}\setminus[-1,1] \\ & \quad\text{such that } \Re z<0 \text{ and } \Im z\geqslant 0, \end{cases}$$

where the principal branches of complex powers are taken.

THEOREM 2.17. *Let* $\alpha,\gamma\in\mathbb{C}$, $z\in\mathbb{C}\setminus[-1,1]$, $\alpha+\gamma\notin-\mathbb{N}$. *Then the relations between the symmetric and antisymmetric Jacobi functions of the second kind to the Gegenbauer function of the second kind are*

$$Q_\gamma^{(\alpha,\alpha)}(z) = \mathrm{e}^{-i\pi(\alpha+\frac{1}{2})}\sqrt{\pi}\,2^{2\alpha}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+\gamma+1)}{\Gamma(2\alpha+\gamma+1)} D_\gamma^{\alpha+\frac{1}{2}}(z), \tag{2.82}$$

where $\alpha\in\mathbb{C}\setminus\{-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots\}$, *and*

$$Q_\gamma^{(\alpha,-\alpha)}(z) = \mathrm{e}^{i\pi(\alpha-\frac{1}{2})}2^{2\gamma-\alpha+1}\frac{\Gamma(\alpha+\gamma+1)\Gamma(\frac{1}{2}-\alpha)\Gamma(\gamma+\frac{3}{2})}{\Gamma(2\gamma+2)(z-1)^\alpha} D_{\alpha+\gamma}^{\frac{1}{2}-\alpha}(z), \tag{2.83}$$

where $\alpha\in\mathbb{C}\setminus\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots\}$, $\gamma\in\mathbb{C}\setminus\{-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},\ldots\}$.

PROOF. Start with the definition of the Jacobi function of the second kind (2.51) and take $\beta = \alpha$. Then comparing (2.17) using Euler's ($z \mapsto z$) transformation (2.9) produces (2.82). In order to produce (2.83), start with (2.51) and take $\beta = -\alpha$. Then compare (2.17) using Euler's transformation. This completes the proof. □

One has the following quadratic transformations of symmetric Jacobi functions of the second kind.

THEOREM 2.18. *Let $z \in \mathbb{C} \setminus [-1, 1]$, $\gamma, \alpha \in \mathbb{C}$, $\alpha + \gamma \notin -\mathbb{N}$. Then*

$$Q_{2\gamma}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha + 2\gamma + 1)}{2^{2\gamma}\Gamma(\gamma + \frac{1}{2})\Gamma(\alpha + \gamma + 1)} Q_\gamma^{(\alpha,-\frac{1}{2})}(2z^2 - 1), \tag{2.84}$$

where $\alpha + 2\gamma \notin -\mathbb{N}$, $\gamma \notin -\mathbb{N} + \frac{1}{2}$, and

$$Q_{2\gamma+1}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha + 2\gamma + 2)z}{2^{2\gamma+1}\Gamma(\gamma + \frac{3}{2})\Gamma(\alpha + \gamma + 1)} Q_\gamma^{(\alpha,\frac{1}{2})}(2z^2 - 1), \tag{2.85}$$

where $\alpha + 2\gamma + 1 \notin -\mathbb{N}$, $\gamma \notin -\mathbb{N} - \frac{1}{2}$.

PROOF. Starting with the left-hand sides of (2.84), (2.85) and using the Gauss hypergeometric representation (2.51) yields ${}_2F_1$'s with parameters $(a, b; c)$ satisfying $c = 2b$. Then for both equations one uses a quadratic transformation of the Gauss hypergeometric function [**11**, (15.8.14)]. This transforms the ${}_2F_1$'s to forms which are recognizable as the right-hand sides through (2.54) and (2.51), respectively. This completes the proof. The restrictions on the parameters come directly by applying the restrictions on the parameters in Theorem 2.7 to the Jacobi functions of the second kind on both sides of the relations. □

There is also an interesting alternative quadratic transformation of the Jacobi function of the second kind with $\alpha = \pm\frac{1}{2}$. Note that there does not seem to be a corresponding transformation formula for the Jacobi function of the first kind since in this case the left-hand side would be a sum of two Gauss hypergeometric functions.

THEOREM 2.19. *Let $x \in \mathbb{C}\setminus((-\infty, -1] \cup [1, \infty))$, $\beta, \gamma \in \mathbb{C}$ such that $\beta + \gamma + \frac{1}{2} \notin -\mathbb{N}_0$. Then*

$$C_{2\gamma+1}^\beta(x) = \frac{2^{2\gamma+2}\Gamma(\beta + \gamma + \frac{1}{2})(1 - x^2)^{-\beta-\gamma-\frac{1}{2}}}{\Gamma(-\gamma - \frac{1}{2})\Gamma(2\gamma + 2)\Gamma(\beta)} Q_{-\gamma-1}^{(-\frac{1}{2},\beta+2\gamma+1)}\left(\frac{1 + x^2}{1 - x^2}\right), \tag{2.86}$$

$$C_{2\gamma}^\beta(x) = \frac{2^{2\gamma+1}\Gamma(\beta + \gamma + \frac{1}{2})x(1 - x^2)^{-\beta-\gamma-\frac{1}{2}}}{\Gamma(-\gamma + \frac{1}{2})\Gamma(2\gamma + 1)\Gamma(\beta)} Q_{-\gamma-1}^{(\frac{1}{2},\beta+2\gamma)}\left(\frac{1 + x^2}{1 - x^2}\right). \tag{2.87}$$

PROOF. These results are easily verified by starting with (2.51), (2.53), substituting the related values in the Jacobi function of the second kind, comparing with associated Legendre functions of the first kind with argument $\sqrt{(z - 1)/(z + 1)}$, and utilizing a quadratic transformation of the Gauss hypergeometric function which relates the two. This completes the proof. □

REMARK 2.20. Note that in Theorem 2.19, if the argument of the Jacobi function of the second kind has modulus greater than unity then the argument of the Gegenbauer function of the first kind has modulus less than unity.

Corollary 2.20.1. *Let $z,\beta,\gamma\in\mathbb{C}$ such that $z\in\mathbb{C}\setminus[-1,1]$. Then*

(2.88)
$$Q_\gamma^{(\frac12,\beta)}(z)=\frac{2^{\beta+3\gamma+\frac52}\Gamma(-2\gamma-1)\Gamma(\gamma+\frac32)\Gamma(\beta+2\gamma+2)}{\Gamma(\beta+\gamma+\frac32)(z-1)^{\frac12}(z+1)^{\beta+\gamma+1}}C_{-2\gamma-2}^{\beta+2\gamma+2}\left(\sqrt{\frac{z-1}{z+1}}\right),$$

where $-2\gamma-1,\gamma+\frac32,\beta+2\gamma+2\notin-\mathbb{N}_0$, and

(2.89)
$$Q_\gamma^{(-\frac12,\beta)}(z)=\frac{2^{\beta+3\gamma+\frac12}\Gamma(-2\gamma)\Gamma(\gamma+\frac12)\Gamma(\beta+2\gamma+1)}{\Gamma(\beta+\gamma+\frac12)(z+1)^{\beta+\gamma+\frac12}}C_{-2\gamma-1}^{\beta+2\gamma+1}\left(\sqrt{\frac{z-1}{z+1}}\right),$$

where $-2\gamma,\gamma+\frac12,\beta+2\gamma+1\notin-\mathbb{N}_0$.

Proof. Inverting Theorem 2.19 completes the proof. □

Note that the above results imply the following corollary.

Corollary 2.20.2. *Let $z,\beta,\gamma\in\mathbb{C}$ such that $z\in\mathbb{C}\setminus[-1,1]$, $\gamma+\frac32,\beta+\gamma+1\notin-\mathbb{N}_0$. Then*

$$Q_\gamma^{(\frac12,\beta)}(z)=\frac{\Gamma(\gamma+\frac32)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\beta+\gamma+\frac32)}\left(\frac{2}{z-1}\right)^{\frac12}Q_{\gamma+\frac12}^{(-\frac12,\beta)}(z). \tag{2.90}$$

Proof. Equating the two right-hand sides in Theorem 2.19 completes the proof. □

Theorem 2.21. *Let $x\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$. Then the relation between the symmetric and antisymmetric Jacobi functions of the second kind on-the-cut and the Ferrers function of the second kind are*

$$\mathsf{Q}_\gamma^{(\alpha,\alpha)}(x)=\frac{2^\alpha\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(1-x^2)^{\frac12\alpha}}\mathsf{Q}_{\gamma+\alpha}^{-\alpha}(x), \tag{2.91}$$

$$\mathsf{Q}_\gamma^{(\alpha,-\alpha)}(x)=\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{1+x}{1-x}\right)^{\frac12\alpha}\mathsf{Q}_\gamma^{-\alpha}(x), \tag{2.92}$$

where $\alpha+\gamma\notin-\mathbb{N}$, and

$$\mathsf{Q}_\gamma^{(-\alpha,\alpha)}(x)=\frac{\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma+1)}\left(\frac{1-x}{1+x}\right)^{\frac12\alpha}\mathsf{Q}_\gamma^{\alpha}(x), \tag{2.93}$$

where $\gamma-\alpha\notin-\mathbb{N}$.

Proof. The results follow by taking into account (2.67). It is the case that (cf. [**10**, Theorem 3.2])

(2.94)
$$\mathsf{Q}_\nu^\mu(x)=\frac{\pi}{2\sin(\pi\mu)}\left(\cos(\pi(\nu+\mu))\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1+x}{1-x}\right)^{\frac12\mu}{}_2\boldsymbol{F}_1\left({-\nu,\nu+1\atop 1+\mu};\frac{1+x}{2}\right)\right.$$
$$\left.-\cos(\pi\nu)\left(\frac{1-x}{1+x}\right)^{\frac12\mu}{}_2\boldsymbol{F}_1\left({-\nu,\nu+1\atop 1-\mu};\frac{1+x}{2}\right)\right),$$

where $\nu\in\mathbb{C}$, $\mu\in\mathbb{C}\setminus\mathbb{Z}$, such that $\nu+\mu\notin-\mathbb{N}$. The formula (2.91) is obtained by taking $\beta=\alpha$ and then comparing (2.67) with (2.94). The other identities follow by an analogous method, taking $\beta=-\alpha$. This completes the proof. □

3. Addition theorems for the Jacobi function of the first kind

The Flensted-Jensen–Koornwinder addition theorem for Jacobi functions of the first kind is an extension of the Koornwinder addition theorem for Jacobi polynomials to the case when the degree is allowed to be a complex number. This addition theorem has two separate contexts and some interesting special cases. We will refer to the contexts as the hyperbolic and trigonometric contexts. The hyperbolic context arises when the Jacobi function is analytically continued in the complex plane from the ray $[1,\infty)$. The trigonometric context arises when the argument of the Jacobi function is analytically continued from the real segment $(-1,1)$. First we will present the addition theorem for the Jacobi function of the first kind in the hyperbolic context. As we will see, the Jacobi function in the trigonometric context can be obtained from the Jacobi function in the hyperbolic context (and vice versa). We now present the most general form of the addition theorem for Jacobi functions of the first kind in the hyperbolic and trigonometric contexts.

Remark 3.1. The addition theorem for Jacobi polynomials was originally derived using group-theoretical methods as mentioned in [**35**], using the representation theory for $\mathrm{SU}(\alpha+2)/\mathrm{U}(\alpha+1)$ and $\mathrm{U}(\alpha+2)/\mathrm{U}(\alpha+1)$ combined with standard methods discussed, for example, in [**50**]. The Lie group-theoretic setting (and in much more detail for $\mathrm{SO}(p)\times\mathrm{SO}(q)$) is carefully described in [**31**] and is as follows. Let X be a compact symmetric group of rank one which is two-point homogeneous for an isometry group U. Normalize the metric so that the lengths of the closed geodesics is 2π. The choice of a base point in X gives an identification of X with U/K. The function space on X decomposes multiplicity-free as $\oplus\mathscr{H}^n$, and if $d(x,y)$ denotes the geodesic distance between points $x,y\in X$, then the elementary spherical function on $\mathscr{H}^n$ is the Jacobi polynomial $P_n^{(\alpha,\beta)}(\cos d(x,y))$ with the parameters α,β taking suitable 'group values,' depending on the choice of X. This observation was first made by Élie Cartan in 1929 [**5**]. For an orthonormal basis $s_k(x)$ of $\mathscr{H}^n$ it is clear that the real point-pair function $(x,y)\mapsto\sum_k s_k(x)s_k(y)$ only depends on $d(x,y)$, which in turn implies that

$$\sum_k s_k(x)\overline{s_k(y)} = c_n P_n^{(\alpha,\beta)}(\cos d(x,y)), \tag{3.1}$$

for suitable constants c_n. This identity is the Lie group-theoretic meaning of the addition formula for Jacobi polynomials. Picking a suitable basis for $\mathscr{H}^n$, this leads to an explicit summation formula for the Jacobi polynomials with an argument depending on θ_1,θ_2,ϕ, as a sum. Koornwinder rederived the addition theorem analytically in [**33**] without relying on group theory, and Flensted-Jensen and Koornwinder extended it to general n (i.e., λ) and $\alpha>\beta>-1/2$ in [**17**]. In fact, it should be pointed out that there is no group-theoretic argument which will yield addition theorems such as for Jacobi functions for general values of the parameters. Group-theoretic arguments rely on the completeness and orthogonality of the (unitary) representations of the group in question. However, the case of general parameters may be obtained by appropriate analytic continuation in the group parameters.

THEOREM 3.2. *Let* $\gamma, \alpha, \beta \in \mathbb{C}$, $\gamma \notin \mathbb{Z}$, $\alpha, \beta \notin -\mathbb{N}_0$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, -1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, *and let*

$$\begin{aligned} Z^{\pm} &:= Z^{\pm}(z_1, z_2, w, x) \\ &= 2z_1^2 z_2^2 + 2w^2(z_1^2-1)(z_2^2-1) \pm 4z_1 z_2 w x (z_1^2-1)^{\frac12}(z_2^2-1)^{\frac12} - 1 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \mathsf{X}^{\pm} &:= \mathsf{X}^{\pm}(x_1, x_2, w, x) \\ &= 2x_1^2 x_2^2 + 2w^2(1-x_1^2)(1-x_2^2) \pm 4x_1 x_2 w x (1-x_1^2)^{\frac12}(1-x_2^2)^{\frac12} - 1 \end{aligned} \tag{3.3}$$

with the complex variables $\gamma, \alpha, \beta, z_1, z_2, x_1, x_2, x, w$ *required to be in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned} P_\gamma^{(\alpha,\beta)}(Z^{\pm}) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} &\sum_{k=0}^{\infty} \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k} \\ &\times \sum_{l=0}^{k} (\mp1)^{k-l} \frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l} (z_1 z_2)^{k-l} \left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}} \\ &\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1) P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1) \\ &\times w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^{\beta}(x) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^{\pm}) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} &\sum_{k=0}^{\infty} \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k} \\ &\times \sum_{l=0}^{k} (\mp1)^{k-l} \frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l} (x_1 x_2)^{k-l} \left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}} \\ &\times \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_1^2-1) \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_2^2-1) \\ &\times w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^{\beta}(x). \end{aligned} \tag{3.5}$$

PROOF. Start with the form of the Flensted-Jensen–Koornwinder addition theorem in [**17**, Theorem 2.1] (see also [**41**, (24)]). Define the Flensted-Jensen–Koornwinder–Jacobi function of the first kind [**17**, (2.1)] (Flensted-Jensen and Koornwinder refer to this function as the Jacobi function of the first kind)

$$\varphi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\begin{matrix} \frac12(\alpha+\beta+1+i\lambda), \frac12(\alpha+\beta+1-i\lambda) \\ \alpha+1 \end{matrix}; -\sinh^2 t\right), \tag{3.6}$$

and express it in terms of the Jacobi function of the first kind using

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+1)\Gamma(-\frac12(\alpha+\beta-1+i\lambda))}{\Gamma(\frac12(\alpha-\beta+1-i\lambda))} P^{(\alpha,\beta)}_{-\frac12(\alpha+\beta+1+i\lambda)}(\cosh(2t)), \tag{3.7}$$

which follows by comparing the Gauss hypergeometric representations of the functions. Replacing $\lambda = i(\alpha+\beta+2\gamma+1)$ and setting $z_1 = \cosh t_1$, $z_2 = \cosh t_2$, and $w = \cos\psi$ produces the form of the addition theorem (3.4). Then analytically continuing (3.4) to $\mathsf{X}^{\pm} \in (-1,1)$ using (2.65) produces (3.5). This completes the proof. $\square$

In the limit as $\gamma \to n \in \mathbb{N}_0$, then Koornwinder's addition theorem for the Jacobi function of the first kind becomes the addition theorem for Jacobi polynomials and the double infinite sum becomes truncated as in (1.6).

REMARK 3.3. It is worth mentioning that in the definitions of $Z^{\pm}$ (3.2) and $\mathsf{X}^{\pm}$ (3.3), the influence of the ± 1 factor on the addition theorems in Theorem 3.2 and elsewhere in this paper is simply due to the influence of the parity relation (2.12) for ultraspherical polynomials, based upon the reflection map $x \mapsto -x$.

REMARK 3.4. Note that there are various ways of expressing the variables $Z^{\pm}$ and $\mathsf{X}^{\pm}$ defined by (3.2) and (3.3), which are useful in different applications. For instance, we may also write

(3.8)

$$\begin{aligned}Z^{\pm} &= 2z_1^2z_2^2(1-x^2) - 1 + 2(z_1^2-1)(z_2^2-1)\left(w \pm \frac{xz_1z_2}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right)^2 \\ &= 2(z_1^2-1)(z_2^2-1)\left(\frac{2z_1^2z_2^2(1-x^2)-1}{2(z_1^2-1)(z_2^2-1)} + \left(w \pm \frac{xz_1z_2}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right)^2\right)\end{aligned}$$

and

(3.9)

$$\begin{aligned}\mathsf{X}^{\pm} &= 2x_1^2x_2^2(1-x^2) - 1 + 2(1-x_1^2)(1-x_2^2)\left(w \pm \frac{xx_1x_2}{\sqrt{(1-x_1^2)(1-x_2^2)}}\right)^2 \\ &= 2(1-x_1^2)(1-x_2^2)\left(\frac{2x_1^2x_2^2(1-x^2)-1}{2(1-x_1^2)(1-x_2^2)} + \left(w \pm \frac{xx_1x_2}{\sqrt{(1-x_1^2)(1-x_2^2)}}\right)^2\right).\end{aligned}$$

First we will develop some tools which will help us prove the correct form of the double summation addition theorem for the Jacobi function of the second kind. Consider the orthogonality of the ultraspherical polynomials and the Jacobi polynomials with the argument $2w^2-1$.

LEMMA 3.5. *Let $m,n,p \in \mathbb{N}_0$, $\mu \in (-\frac{1}{2},\infty)\backslash\{0\}$, $\alpha > \beta > -1$. Then the ultraspherical and Jacobi polynomials satisfy the orthogonality relations*

$$\int_0^{\pi} C_m^{\mu}(\cos\phi)C_n^{\mu}(\cos\phi)(\sin\phi)^{2\mu}\,\mathrm{d}\phi = \frac{\pi\,\Gamma(2\mu+n)}{2^{2\mu-1}(\mu+n)n!\,\Gamma(\mu)^2}\delta_{m,n}, \tag{3.10}$$

(3.11)

$$\begin{aligned}&\int_0^1 P_m^{(\alpha-\beta-1,\beta+p)}(2w^2-1)P_n^{(\alpha-\beta-1,\beta+p)}(2w^2-1)w^{2\beta+2p+1}(1-w^2)^{\alpha-\beta-1}\,\mathrm{d}w \\ &\qquad = \frac{\Gamma(\alpha-\beta+n)\Gamma(\beta+1+p+n)}{2(\alpha+p+2n)\Gamma(\alpha+p+n)n!}\delta_{m,n}.\end{aligned}$$

PROOF. These orthogonality relations follow easily from [**30**, (9.8.20), (9.8.2)] upon making the straightforward substitutions. ∎

3.1. The parabolic biangle orthogonal polynomial system. Define a system of 2-variable orthogonal polynomials, which are sometimes referred to as parabolic biangle polynomials, by [**40**]

$$\mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi) := w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)C_{k-l}^{\beta}(\cos\phi), \tag{3.12}$$

where $k,l\in\mathbb{N}_0$ such that $l\leqslant k$. These 2-variable polynomials are orthogonal over $(w,\phi)\in(0,1)\times(0,\pi)$ with orthogonality measure

$$\mathrm{d}m^{(\alpha,\beta)}(w,\phi):=(1-w^2)^{\alpha-\beta-1}w^{2\beta+1}(\sin\phi)^{2\beta}\,\mathrm{d}w\,\mathrm{d}\phi. \tag{3.13}$$

The orthogonal polynomial system $\mathcal{P}^{(\alpha,\beta)}_{k,l}$ is deeply connected to the addition theorem for Jacobi functions of the first and second kind. Using the orthogonality relations in Lemma 3.5 we can derive the orthogonality relation for the 2-variable parabolic biangle polynomials.

LEMMA 3.6. *Let $k,l,k',l'\in\mathbb{N}_0$ such that $l\leqslant k$, $l'\leqslant k'$, $\alpha>\beta>-1$. Then the 2-variable parabolic biangle polynomials satisfy the orthogonality relation*

$$\begin{aligned}\int_0^1\int_0^\pi &\mathcal{P}^{(\alpha,\beta)}_{k,l}(w,\phi)\mathcal{P}^{(\alpha,\beta)}_{k',l'}(w,\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)\\ &=\frac{\pi\,\Gamma(\beta+1+k)\Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)}{2^{2\beta}\Gamma(\beta)^2(\alpha+k+l)(\beta+k-l)\Gamma(\alpha+k)(k-l)!l!}\delta_{k,k'}\delta_{l,l'}.\end{aligned} \tag{3.14}$$

PROOF. Starting with the definition of the 2-variable parabolic biangle polynomials (3.12), integrating over $(w,\phi)\in(0,1)\times(0,\pi)$ with measure (3.13), and using the orthogonality relations in Lemma 3.5 completes the proof. □

The following result is a Jacobi function of the first kind generalization of [**33**, (4.10)] for Jacobi polynomials.

THEOREM 3.7. *Let $k,l\in\mathbb{N}_0$ with $l\leqslant k$, $\gamma,\alpha,\beta\in\mathbb{C}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,-1]$, $Z^\pm$ defined in (3.2), such that $x=\cos\phi$ and the complex variables $\gamma,\alpha,\beta,z_1,z_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned}\int_0^1\int_0^\pi &P^{(\alpha,\beta)}_\gamma(Z^\pm)\,w^{k-l}P^{(\alpha-\beta-1,\beta+k-l)}_l(2w^2-1)C^\beta_{k-l}(\cos\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)\\ &=(\mp1)^{k+l}\,\mathsf{A}^{(\alpha,\beta,\gamma)}_{k,l}(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac12(k+l)}\\ &\quad\times P^{(\alpha+k+l,\beta+k-l)}_{\gamma-k}(2z_1^2-1)P^{(\alpha+k+l,\beta+k-l)}_{\gamma-k}(2z_2^2-1),\end{aligned} \tag{3.15}$$

where

$$\mathsf{A}^{(\alpha,\beta,\gamma)}_{k,l}:=\frac{\pi\Gamma(\gamma+1)(\alpha+\beta+\gamma+1)_k\Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)(-\beta-\gamma)_l}{2^{2\beta}\Gamma(\beta)(-\gamma)_k\,(k-l)!\,l!\,\Gamma(\alpha+\gamma+1+l)}. \tag{3.16}$$

PROOF. Start with the addition theorem for the Jacobi function of the first kind (3.4) and consider the (k,l)-th term in the double series. It involves a product of two Jacobi functions of the first kind with degree $\gamma-k$ and parameters $(\alpha+k+l,\beta+k-l)$. Replace in (3.4) the summation indices k,l by k',l', multiply both sides of (3.4) by $\mathcal{P}^{(\alpha,\beta)}_{k,l}(w,\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)$, and integrate both sides over $(w,\phi)\in(0,1)\times(0,\pi)$ using (3.10), (3.11). This completes the proof. □

We will return to the parabolic biangle polynomials in Section 4.

3.2. Special cases of the addition theorem for the Jacobi function of the first kind. In the case when $z_1,z_2,x_1,x_2,w,x=\cos\phi$ are real numbers then the argument of the Jacobi function of the first kind in the addition theorem takes a simpler form convenient form and was proved in Flensted-Jensen–Koornwinder [**17**].

REMARK 3.8. In the case where the variables z_1, z_2, x_1, x_2, x, w are real then one may write $Z^\pm$ and $\mathsf{X}^\pm$ as follows:

$$Z^\pm = 2\left|z_1z_2 \pm \mathrm{e}^{i\phi}\, w\,(z_1^2-1)^{\frac12}(z_2^2-1)^{\frac12}\right|^2 - 1, \tag{3.17}$$

$$\mathsf{X}^\pm = 2\left|x_1x_2 \pm \mathrm{e}^{i\phi}\, w\,(1-x_1^2)^{\frac12}(1-x_2^2)^{\frac12}\right|^2 - 1. \tag{3.18}$$

We now give a result which appears to be identical to Theorem 3.2, but it must be emphasized that it is only in the real case that we are able to write $Z^\pm$, $\mathsf{X}^\pm$ using (3.17), (3.18). Otherwise one must use (3.2), (3.3).

THEOREM 3.9. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $\gamma \notin \mathbb{Z}$, $\alpha, \beta \notin -\mathbb{N}_0$, $z_1, z_2 \in (1, \infty)$, $x_1, x_2 \in (-1, 1)$, $w \in \mathbb{R}$, $\phi \in [0, \pi]$, and $Z^\pm$, $\mathsf{X}^\pm$ is defined as in* (3.17), (3.18) *respectively. Then*
(3.19)

$$\begin{aligned} P_\gamma^{(\alpha,\beta)}(Z^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k} \\ &\quad\times \sum_{l=0}^k (\mp1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(z_1z_2)^{k-l}\left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}} \\ &\qquad\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1)P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1) \\ &\qquad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(\cos\phi), \end{aligned}$$

and
(3.20)

$$\begin{aligned} \mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k} \\ &\quad\times \sum_{l=0}^k (\mp1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(x_1x_2)^{k-l}\left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}} \\ &\qquad\times \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_1^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_2^2-1) \\ &\qquad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(\cos\phi). \end{aligned}$$

PROOF. Starting with Theorem 3.2 and restricting such that the variables $z_1, z_2, x_1, x_2, w, x = \cos\phi$ are real completes the proof. □

As in Theorem 3.2, in the limit when $\gamma \to n \in \mathbb{N}_0$, Koornwinder's addition theorem for the Jacobi function of the first kind becomes the addition theorem for Jacobi polynomials, and the double infinite sum becomes truncated as in (1.6). Note that for all results below, the double or single infinite sums on the right-hand side will always be truncated when the left-hand side is a polynomial (Jacobi polynomial or ultraspherical polynomial). We will not mention this again. Next we have a specialization of Theorem 3.9 when $w = 1$.

COROLLARY 3.9.1. *Let $\gamma,\alpha,\beta\in\mathbb{C}$, $\gamma\not\in\mathbb{Z}$, $\alpha,\beta\not\in-\mathbb{N}_0$, $r_1,r_2\in[0,\infty)$, $\theta_1,\theta_2\in[0,\frac{\pi}{2}]$, $\phi\in[0,\pi]$, and*

$$Z^\pm := 2\left|\cosh r_1\cosh r_2 \pm \mathrm{e}^{i\phi}\sinh r_1\sinh r_2\right|^2-1 \\ = \cosh(2r_1)\cosh(2r_2)\pm\sinh(2r_1)\sinh(2r_2)\cos\phi, \tag{3.21}$$

and

$$\mathsf{X}^\pm := 2\left|\cos\theta_1\cos\theta_2 \pm \mathrm{e}^{i\phi}\sin\theta_1\sin\theta_2\right|^2-1 \\ = \cos(2\theta_1)\cos(2\theta_2)\pm\sin(2\theta_1)\sin(2\theta_2)\cos\phi. \tag{3.22}$$

Then

(3.23)

$$P_\gamma^{(\alpha,\beta)}(Z^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \\ \times\sum_{k=0}^\infty\frac{(\alpha)_k(\frac{\alpha}{2}+1)_k(-\beta-\gamma)_k(\alpha+\beta+\gamma+1)_k}{(\frac{\alpha}{2})_k(\beta+1)_k(-\gamma)_k(\alpha+\gamma+1)_k}(\sinh r_1\sinh r_2)^{2k} \\ \times\sum_{l=0}^k\frac{(\mp1)^l(\alpha-\beta)_{k-l}(-\alpha-\gamma-k)_l(-\alpha-2k+1)_l}{(k-l)!(-\alpha-2k)_l(\beta+\gamma+1)_l}(\coth r_1\coth r_2)^l \\ \times P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cosh(2r_1))P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cosh(2r_2))\frac{\beta+l}{\beta}C_l^\beta(\cos\phi)$$

and

(3.24)

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \\ \times\sum_{k=0}^\infty\frac{(\alpha)_k(\frac{\alpha}{2}+1)_k(-\beta-\gamma)_k(\alpha+\beta+\gamma+1)_k}{(\frac{\alpha}{2})_k(\beta+1)_k(-\gamma)_k(\alpha+\gamma+1)_k}(\sin\theta_1\sin\theta_2)^{2k} \\ \times\sum_{l=0}^k\frac{(\mp1)^l(\alpha-\beta)_{k-l}(-\alpha-\gamma-k)_l(-\alpha-2k+1)_l}{(k-l)!(-\alpha-2k)_l(\beta+\gamma+1)_l}(\cot\theta_1\cot\theta_2)^l \\ \times P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cos(2\theta_1))P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cos(2\theta_2))\frac{\beta+l}{\beta}C_l^\beta(\cos\phi).$$

PROOF. Start with Theorem 3.9 and let $w=1$ using (2.48); substituting $l\mapsto l'=k-l$ followed by relabeling $l'\mapsto l$ completes the proof. □

By letting $\alpha=\beta$ in Corollary 3.9.1 we can relate the above result to associated Legendre and Gegenbauer functions of the first kind. This is mentioned in [**35**], namely that Koornwinder's addition theorem for Jacobi polynomials generalizes Gegenbauer's addition theorem (1.4). Similarly, the extension to the Flensted-Jensen–Koornwinder addition theorem for Jacobi functions of the first kind generalizes the addition theorem for Gegenbauer functions of the first kind. First we define the variables

$$\mathcal{Z}^\pm := \mathcal{Z}^\pm(r_1,r_2,\phi) := \cosh r_1\cosh r_2\pm\sinh r_1\sinh r_2\cos\phi, \tag{3.25}$$

$$\mathcal{X}^\pm := \mathcal{X}^\pm(\theta_1,\theta_2,\phi) := \cos\theta_1\cos\theta_2\pm\sin\theta_1\sin\theta_2\cos\phi. \tag{3.26}$$

COROLLARY 3.9.2. *Let* $\gamma, \alpha \in \mathbb{C}$, $2\alpha \neq 1, 0, -1, \ldots$, $\gamma \notin -\mathbb{N}$, $r_1, r_2 \in [0,\infty)$, $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$, $\phi \in [0,\pi]$, *and* $\mathcal{Z}^{\pm}$, $\mathcal{X}^{\pm}$ *defined by* (3.25), (3.26), *respectively. Then*

$$C_\gamma^\alpha(\mathcal{Z}^\pm) = \frac{\Gamma(2\alpha)\Gamma(\gamma+1)}{\Gamma(2\alpha+\gamma)} \sum_{k=0}^{\infty} \frac{(\mp 1)^k\, 2^{2k} (\alpha)_k (\alpha)_k}{(-\gamma)_k (2\alpha+\gamma)_k} (\sinh(2r_1)\sinh(2r_2))^k \times C_{\gamma-k}^{\alpha+k}(\cosh(2r_1)) C_{\gamma-k}^{\alpha+k}(\cosh(2r_2)) \frac{\alpha-\frac12+k}{\alpha-\frac12} C_k^{\alpha-\frac12}(\cos\phi) \tag{3.27}$$

and

$$C_\gamma^\alpha(\mathcal{X}^\pm) = \frac{\Gamma(2\alpha)\Gamma(\gamma+1)}{\Gamma(2\alpha+\gamma)} \sum_{k=0}^{\infty} \frac{(\mp 1)^k\, 2^{2k} (\alpha)_k (\alpha)_k}{(-\gamma)_k (2\alpha+\gamma)_k} (\sin(2\theta_1)\sin(2\theta_2))^k \times C_{\gamma-k}^{\alpha+k}(\cos(2\theta_1)) C_{\gamma-k}^{\alpha+k}(\cos(2\theta_2)) \frac{\alpha-\frac12+k}{\alpha-\frac12} C_k^{\alpha-\frac12}(\cos\phi), \tag{3.28}$$

or equivalently

$$\frac{P_\gamma^{-\alpha}(\mathcal{Z}^\pm)}{(1-{\mathcal{Z}^\pm}^2)^{\frac12\alpha}} = \frac{2^\alpha\Gamma(\alpha+1)}{(\sinh(2\theta_1)\sinh(2\theta_1))^\alpha} \sum_{k=0}^{\infty} (\pm 1)^k (\alpha-\gamma)_k (\alpha+\gamma+1)_k \times P_\gamma^{-\alpha-k}(\cosh(2r_1)) P_\gamma^{-\alpha-k}(\cosh(2r_2)) \frac{\alpha+k}{\alpha} C_k^\alpha(\cos\phi) \tag{3.29}$$

and

$$\frac{\mathsf{P}_\gamma^{-\alpha}(\mathcal{X}^\pm)}{(1-{\mathcal{X}^\pm}^2)^{\frac12\alpha}} = \frac{2^\alpha\Gamma(\alpha+1)}{(\sin(2\theta_1)\sin(2\theta_1))^\alpha} \sum_{k=0}^{\infty} (\pm 1)^k (\alpha-\gamma)_k (\alpha+\gamma+1)_k \times \mathsf{P}_\gamma^{-\alpha-k}(\cos(2\theta_1)) \mathsf{P}_\gamma^{-\alpha-k}(\cos(2\theta_2)) \frac{\alpha+k}{\alpha} C_k^\alpha(\cos\phi). \tag{3.30}$$

PROOF. Start with Corollary 3.9.1 and let $\alpha = \beta$ using (2.71), (2.72) respectively for the Jacobi functions of the first kind on the left-hand side and those on the right-hand side. Then mapping $(2z_1^2-1, 2z_2^2-1) \mapsto (z_1, z_2)$, $(2x_1^2-1, 2x_2^2-1) \mapsto (x_1, x_2)$, where $z_1 = \cosh r_1$, $z_2 = \cosh r_2$, $x_1 = \cos\theta_1$, $x_2 = \cos\theta_2$, and simplifying using (2.71)–(2.73), completes the proof.

Another way to prove this result is to take $\beta = -\frac12$, $w = \cos\psi = 1$, $\gamma \to 2\gamma$ in (3.19) and use the quadratic transformation (2.76). After using (2.71), this produces the left-hand side of (3.27) with degree 4γ and order $\alpha + \frac12$. Because we set $w = 1$, the sum over l only survives for $l = 0, 1$. By taking $4\gamma \mapsto \gamma$ and expressing the contribution due to each of these terms, one can identify Gegenbauer's addition theorem through repeated application of (2.71) on the right-hand side; and using the fact that

$$\sum_{k=0}^{\infty} (f_{2k} + f_{2k+1}) = \sum_{k=0}^{\infty} f_k, \tag{3.31}$$

for any sequence $\{f\}_{k\in\mathbb{N}_0}$, one arrives at (3.27). The proof of (3.28) is similar. ☐

4. Addition theorems for the Jacobi function of the second kind

Now we present double summation addition theorems for the Jacobi functions of the second kind in the hyperbolic and trigonometric contexts.

REMARK 4.1. There is no direct group-theoretical derivation of an addition formula for the Jacobi functions of the second kind. However, those functions satisfy the same differential recurrence relations as the functions of the first kind, and the actions of those operators and the Jacobi differential equation (2.39) on functions of either kind for general parameters give realizations of the Lie algebras of the groups considered. One would therefore expect the Jacobi functions of the second kind to satisfy addition formulas with the same structure as those for the functions of the first kind. This is known to hold, for example, for the Gegenbauer functions of the first and second kind [**14**, §8].

4.1. The hyperbolic context for the addition theorem for the Jacobi function of the second kind.

Define

$$z_{\lessgtr} := \min_{\max}\{z_1, z_2\}, \tag{4.1}$$

when $z_1, z_2 \in (1,\infty)$, and in the case when $z_1, z_2 \in \mathbb{C}$, one takes without loss of generality $z_1 = z_>$ to lie on an ellipse with foci at ± 1, and $z_2 = z_<$ to be in the interior of that ellipse.

THEOREM 4.2. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C}\setminus(-\infty,-1]$, with $x, w \in \mathbb{C}$, $Z^\pm$ defined by (3.2), such that the complex variables $\gamma, \alpha, \beta, z_1, z_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned} Q_\gamma^{(\alpha,\beta)}(Z^\pm) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} &\sum_{k=0}^\infty \frac{(\alpha+1)_k(\gamma+1)_k}{(\alpha+k)(\beta+1)_k(1-\gamma)_k} \\ &\times \sum_{l=0}^k (\pm 1)^{k-l}(\alpha+k+l)(z_1z_2)^{k-l}\left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}} \\ &\qquad \times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1) \\ &\qquad \times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x). \end{aligned} \tag{4.2}$$

PROOF. Start with the addition formula (3.4) for the Jacobi functions of the first kind. The series was shown to converge by Flensted-Jensen and Koornwinder [**17**, Theorem 2.1]. Now apply the connection relation (2.55), which relates the Jacobi function of the first kind to two Jacobi functions of the second kind, to both sides of (3.4): to the function of the first kind on the left and to the function of the first kind on the right with argument $2z_2^2-1$, assuming without loss of generality that $z_2 = z_>$. This yields an equation in which the asymptotic behavior of the two terms on the left as $z_2 \to \infty$ matches the term-by-term asymptotic behavior of the two corresponding series on the right, suggesting that they should be identified and that the series converge separately. To exploit this, project the left and right-hand sides by the parabolic biangle polynomials, as in Theorem 3.7, with the same application of the connection formula to the Jacobi functions of the first and second kind. This results in the equation

(4.3)
$$\begin{aligned}
&\mathsf{B}_\gamma^{(\alpha,\beta)} \int_0^1 \int_0^\pi Q_\gamma^{(\alpha,\beta)}(Z^\pm)\, w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^\beta(\cos\phi)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi) \\
&\quad + \mathsf{C}_{\gamma,k,l}^{(\alpha,\beta)}\, (z_1 z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{1}{2}(k+l)} \\
&\qquad \times P_{\gamma-k}^{(\alpha+k+l,\alpha+k-l)}(2z_<^2-1) Q_{\gamma-k}^{(\alpha+k+l,\alpha+k-l)}(2z_>^2-1) \\
&= \mathsf{D}_\gamma^{(\alpha,\beta)} \int_0^1 \int_0^\pi Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(Z^\pm) w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^\beta(\cos\phi)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi) \\
&\quad + \mathsf{E}_{\gamma,k,l}^{(\alpha,\beta)}\, (z_1 z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{1}{2}(k+l)} \\
&\qquad \times P_{-\alpha-\beta-\gamma-k-1}^{(\alpha+k+l,\alpha+k-l)}(2z_<^2-1) Q_{-\alpha-\beta-\gamma-k-1}^{(\alpha+k+l,\alpha+k-l)}(2z_>^2-1),
\end{aligned}$$

where

$$\mathsf{B}_\gamma^{(\alpha,\beta)} := \frac{-2\sin(\pi\gamma)\sin(\pi(\beta+\gamma))}{\pi\sin(\pi(\alpha+\beta+2\gamma+1))}, \tag{4.4}$$

$$\begin{aligned}
\mathsf{C}_{\gamma,k,l}^{(\alpha,\beta)} &:= \frac{\sin(\pi\gamma)\sin(\pi(\beta+\gamma))}{2^{2\beta-1}\sin(\pi(\alpha+\beta+2\gamma+1))} \\
&\quad \times \frac{\Gamma(\gamma+1)(\alpha+\beta+\gamma+1)_k \Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)(-\beta-\gamma)_l}{\Gamma(\beta)(-\gamma)_k (k-l)!\, l!\, \Gamma(\alpha+\gamma+1+l)},
\end{aligned} \tag{4.5}$$

and

(4.6)
$$\mathsf{D}_\gamma^{(\alpha,\beta)} := \frac{-2\sin(\pi(\alpha+\gamma))\sin(\pi(\beta+\gamma))\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\pi\sin(\pi(\alpha+\beta+2\gamma+1))\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)},$$

(4.7)
$$\mathsf{E}_{\gamma,k,l}^{(\alpha,\beta)} := \frac{\sin(\pi(\alpha+\gamma))\sin(\pi(\beta+\gamma))\Gamma(\beta+\gamma+1)\Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)}{2^{2\beta-1}\sin(\pi(\alpha+\beta+2\gamma+1))\Gamma(\beta)\Gamma(\alpha+\beta+\gamma+1)(k-l)!\, l!}.$$

Now consider the asymptotics of all four terms of (4.3) as $z_2 \to \infty$. The asymptotic behavior of $Z^\pm$ as $z_2 \to \infty$ is $Z^\pm \sim z_2^2$. The behavior of the Jacobi function of the second kind as $|z| \to \infty$, by (2.64), is

$$Q_\gamma^{(\alpha,\beta)}(z) \sim \frac{1}{z^{\alpha+\beta+\gamma+1}}.$$

Therefore, one has the following asymptotic behaviors as $\zeta \to \infty$, where $\zeta = z_>$ with $z_<$ fixed:

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm) \sim (Z^\pm)^{-\gamma-\alpha-\beta-1} \sim \zeta^{-2\gamma-2\alpha-2\beta-2}, \tag{4.8}$$

$$Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(Z^\pm) \sim (Z^\pm)^\gamma \sim \zeta^{2\gamma}, \tag{4.9}$$

and

$$Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(\zeta) \sim \zeta^{-\gamma-\alpha-\beta-k-1}, \tag{4.10}$$

$$Q_{-\alpha-\beta-\gamma-1-k}^{(\alpha+k+l,\beta+k-l)}(\zeta) \sim \zeta^{\gamma-k}. \tag{4.11}$$

The equation (4.3) with leading order asymptotic contribution as $z_2 \to \infty$ taken out can be written in terms of two analytic functions

$$\mathsf{f}(z_2^{-1}) := \mathsf{f}^{(\alpha,\beta)}_{\gamma,k,l}(z_1, z_2^{-1}),$$
$$\mathsf{g}(z_2^{-1}) := \mathsf{g}^{(\alpha,\beta)}_{\gamma,k,l}(z_1, z_2^{-1}),$$

as

$$z_2^{-2(\gamma+\alpha+\beta+1)}\mathsf{f}(z_2^{-1}) = z_2^{2\gamma}\mathsf{g}(z_2^{-1}). \tag{4.12}$$

For $4\Re\gamma \notin -2(\alpha+\beta+1)+\mathbb{Z}$, the only way the equation can be true is if f and g vanish identically. The case of general γ then follows by analytic continuation in γ. Therefore we have now verified separately all terms in the double series expansion of the Jacobi function of the second kind appearing in (4.2), and the corresponding series for $Q^{(\alpha,\beta)}_{-\alpha-\beta-\gamma-1}(Z^{\pm})$. The two series do not mix, have different asymptotic behaviors for $z_2 = z_> \to \infty$, and must converge separately given the overall convergence proved by Flensted-Jensen and Koornwinder [**17**, Theorem 2.1]. This completes the proof. □

REMARK 4.3. If one applies (2.56) to the Jacobi functions of the second kind on the left-hand side and right-hand side of (4.2), then it becomes the addition theorem for Jacobi polynomials in the hyperbolic context.

COROLLARY 4.3.1. *Let $k,l \in \mathbb{N}_0$ with $l \leqslant k$, $\gamma,\alpha,\beta \in \mathbb{C}$, $z_1,z_2 \in \mathbb{C}\setminus(-\infty,1]$, with $Z^{\pm}$ defined by* (3.2)*, such that $x=\cos\phi$ and the complex variables $\gamma,\alpha,\beta,z_1,z_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned}\int_0^1\int_0^\pi Q^{(\alpha,\beta)}_{\gamma}(Z^{\pm})w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)C^{\beta}_{k-l}(\cos\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi),\\ =(\pm1)^{k+l}\,\mathsf{A}^{(\alpha,\beta,\gamma)}_{k,l}(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac12(k+l)}\\ \times P^{(\alpha+k+l,\beta+k-l)}_{\gamma-k}(2z_<^2-1)Q^{(\alpha+k+l,\beta+k-l)}_{\gamma-k}(2z_>^2-1),\end{aligned} \tag{4.13}$$

where $\mathsf{A}^{(\alpha,\beta,\gamma)}_{k,l}$ is defined in (3.16)*.*

PROOF. This follows directly from (4.12) since in the proof of Theorem 4.2, we showed that $\mathsf{f}=0$. The result $\mathsf{g}=0$ is equivalent to this result under the transformation $\gamma \mapsto -\gamma-\alpha-\beta-1$. □

REMARK 4.4. The integral representations Theorem 3.7 and Corollary 4.3.1 are equivalent to the double summation addition theorems for the Jacobi function of the first kind (3.4) and second kind, Theorem 4.2.

REMARK 4.5. One has the following well-known product representations which are the $k=l=0$ cases of the integral representations in Theorem 3.7 and Corollary 4.3.1. Let $x=\cos\phi$, $\gamma,\alpha,\beta\in\mathbb{C}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,-1]$, with $Z^{\pm}$ defined by (3.2), such that the complex variables $\gamma,\alpha,\beta,z_1,z_2$ are in some yet to be determined neighborhood of the real line. Then

$$\begin{aligned} &P_\gamma^{(\alpha,\beta)}(2z_1^2-1)P_\gamma^{(\alpha,\beta)}(2z_2^2-1) \\ &\quad = \frac{2\Gamma(\alpha+\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1)\Gamma(\beta+\frac12)\Gamma(\alpha-\beta)}\int_0^1\int_0^\pi P_\gamma^{(\alpha,\beta)}(Z^\pm)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi), \end{aligned} \tag{4.14}$$

$$\begin{aligned} &P_\gamma^{(\alpha,\beta)}(2z_<^2-1)Q_\gamma^{(\alpha,\beta)}(2z_>^2-1) \\ &\quad = \frac{2\Gamma(\alpha+\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1)\Gamma(\beta+\frac12)\Gamma(\alpha-\beta)}\int_0^1\int_0^\pi Q_\gamma^{(\alpha,\beta)}(Z^\pm)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi). \end{aligned} \tag{4.15}$$

For independent verification of these product representations, see [**16**, Theorem 4.1] for the product formula (4.14) and [**16**, p. 255] for the product formula (4.15).

Remark 4.6. As pointed out by one of the referees, our proof of the addition theorem for the Jacobi function of the second kind was chosen so that it was the most economical, following as it does from the role of Jacobi functions as spherical functions on symmetric spaces [**35**], extended to general parameters as in [**17**]. However, this does not mean that our chosen method is the most insightful method of proof overall. In fact, Flensted-Jensen and Koornwinder (1979) [**17**] derived the addition theorem for the Jacobi function of the first kind from the product formula (4.14), which the same authors had obtained earlier in [**16**, Theorem 4.1]. Inspection of the proof of the product formula (4.14) in [**16**] shows that it really comes from the product formula (4.15) for Jacobi functions of the second kind [**16**, Proof of Theorem 4.1, p. 252]. In this proof, the authors expand the Jacobi function of the first kind using [**16**, (4.6)], which is simply the connection formula (2.55) which expresses the Jacobi function of the first kind as a linear combination of two Jacobi functions of the second kind with different degrees. So, as the referee pointed out, a more informative proof of the addition theorem for the Jacobi function of the second kind, Theorem 4.2, would be to derive it from the product formula (4.15) in the same way as Theorem 3.2 is derived in [**17**] from the product formula (4.14).

In the case when $z_1, z_2, w, x=\cos\phi$ are real numbers, the argument of the Jacobi function of the second kind in the addition theorem for the Jacobi function of the second kind takes a simpler and more convenient form. This is analogous to the Flensted-Jensen–Koornwinder addition theorem of the first kind, (3.19). We present this result now.

Theorem 4.7. *Let $\gamma,\alpha,\beta,w\in\mathbb{R}$, such that $\gamma\notin\mathbb{Z}$, $\alpha\notin-\mathbb{N}$, $\beta\in(-\frac12,\infty)\setminus\{0\}$, $\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$, $\phi\in[0,\pi]$, $z_1,z_2\in(1,\infty)$, and $Z^\pm$, $z_\lessgtr$, defined by* (3.17), (4.1), *respectively. Then*

$$\begin{aligned} Q_\gamma^{(\alpha,\beta)}(Z^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k} \\ &\quad\times\sum_{l=0}^k(\pm1)^{k+l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(z_1z_2)^{k-l}\left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}} \\ &\quad\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1) \\ &\quad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(\cos\phi). \end{aligned} \tag{4.16}$$

Proof. This follows from Theorem 4.2 by setting the variables $\gamma, \alpha, \beta, z_1, z_2, w$, $x = \cos\phi$ to real numbers. $\square$

4.2. The trigonometric context of the addition theorem for the Jacobi function of the second kind. In the trigonometric context for the addition theorem for Jacobi functions of the second kind, one must use the Jacobi function of the second kind on-the-cut $\mathsf{Q}_\gamma^{(\alpha,\beta)}$ (2.66), which is defined in Section 2.2.3 and has a hypergeometric representation given by (2.67). Note that this representation is not unique and there are many other double Gauss hypergeometric representations of this function. For more about this see the discussion immediately above (2.67). Define

$$x_\lessgtr := \begin{matrix}\min\\ \max\end{matrix}\,\{x_1, x_2\}, \tag{4.17}$$

where $x_1, x_2 \in (-1, 1)$, and in the case where $x_1, x_2 \in \mathbb{C}$, then if one takes without loss of generality $x_1 = x_>$ to lie on an ellipse with foci at ± 1, then $x_2 = x_<$ must be chosen to be in the interior of that ellipse.

Theorem 4.8. *Let $k, l \in \mathbb{N}_0$, $l \leqslant k$, $\gamma, \alpha, \beta \in \mathbb{C}$, $x_1, x_2 \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $\alpha \notin \mathbb{Z}$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, with $\mathsf{X}^\pm$, $x_\lessgtr$ defined by* (3.3), (4.17) *respectively, such that the complex variables $\gamma, \alpha, \beta, x_1, x_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned}\int_0^1\int_0^\pi &\mathsf{Q}_\gamma^{(\alpha,\beta)}(X^\pm)\, w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^\beta(\cos\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)\\ &= (\mp 1)^{k+l}\mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)}(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac12(k+l)}\\ &\quad\times \mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1),\end{aligned} \tag{4.18}$$

where $\mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)}$ is defined in (3.16).

Proof. Starting with Corollary 4.3.1 and applying the definition (2.66) completes the proof. $\square$

Corollary 4.8.1. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $x_1, x_2 \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $\alpha \notin \mathbb{Z}$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, with $\mathsf{X}^\pm$, $x_\lessgtr$ defined by* (3.3), (4.17) *respectively, such that the complex variables $\gamma, \alpha, \beta, x_1, x_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned}\int_0^1\int_0^\pi &\mathsf{Q}_\gamma^{(\alpha,\beta)}(X^\pm)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)\\ &= \frac{\sqrt{\pi}\,\Gamma(\gamma+1)\Gamma(\beta+\frac12)\Gamma(\alpha-\beta)}{2\Gamma(\alpha+\gamma+1)}\mathsf{Q}_\gamma^{(\alpha,\beta)}(2x_<^2-1)\mathsf{P}_\gamma^{(\alpha,\beta)}(2x_>^2-1).\end{aligned} \tag{4.19}$$

Proof. Starting with Theorem 4.8 and setting $k = l = 0$ completes the proof. $\square$

Theorem 4.9. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $x_1, x_2 \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $\alpha \notin \mathbb{Z}$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, $x, w \in \mathbb{C}$ with $\mathsf{X}^\pm$, $x_\lessgtr$ defined by* (3.3), (4.17) *respectively, such that the complex variables $\gamma, \alpha, \beta, x_1, x_2, x, w$ are in some yet to be determined* *neighborhood of the real line. Then*

(4.20)

$$\begin{aligned}\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}\\ &\quad\times\sum_{l=0}^k(\mp 1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(x_1x_2)^{k-l}\left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}}\\ &\qquad\times \mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)\\ &\qquad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x).\end{aligned}$$

Proof. The result follows by starting with the addition theorem for Jacobi functions of the second kind (4.16) and applying the definition (2.66) completes the proof. □

In the case when $z_1, z_2, w, x = \cos\phi$ are real numbers, then the argument of the Jacobi function of the second kind on-the-cut in the addition theorem for the Jacobi function of the second kind takes a simpler and more convenient form. This is analogous to the addition theorem (3.19). We present this result now.

Corollary 4.9.1. *Let $\gamma, \alpha, \beta, w \in \mathbb{R}$, such that $\gamma \notin \mathbb{Z}$, $\alpha, \alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, $x_1, x_2 \in (-1,1)$, $\phi \in [0,\pi]$, with $\mathsf{X}^\pm$, $x_\lessgtr$ defined by* (3.18), (4.17) *respectively. Then*

(4.21)

$$\begin{aligned}\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}\\ &\quad\times\sum_{l=0}^k(\mp 1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(x_1x_2)^{k-l}\left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}}\\ &\qquad\times \mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)\\ &\qquad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(\cos\phi).\end{aligned}$$

Proof. This follows from Theorem 4.7 by setting the complex variables to be real. □

4.3. Olver-normalized Jacobi functions and their addition theorems. Koornwinder's addition theorem (1.6) for Jacobi polynomials of degree $n \in \mathbb{N}_0$ involves a terminating sum. One can attribute this to the infinite sum appearing in the generalizations (3.19), (3.20), first recognizing that Jacobi polynomials $P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}$ vanish for both $\gamma = n \in \mathbb{N}_0$ and $k \geqslant n+1$. Considering the limit as $\gamma \to n$ for all values of $k \in \mathbb{N}_0$ in Koornwinder's addition theorem, one sees that the factor $1/(-\gamma)_k$ blows up for $k \geqslant n+1$. On the other hand, this factor is multiplied in the limit by the Jacobi function of the first kind prefactor containing $1/\Gamma(\gamma-k+1)$, and considering the residues of the gamma function, one sees that the product will be finite, namely

$$\lim_{\gamma\to n}\frac{1}{(-\gamma)_k\Gamma(\gamma-k+1)} = \lim_{\gamma\to n}\frac{\Gamma(-\gamma)}{\Gamma(-\gamma+k)\Gamma(\gamma-k+1)} = \frac{(-1)^k}{n!}, \tag{4.22}$$

for $k \geqslant n+1$. When this finite factor is multiplied by the second Jacobi polynomial $P_{\gamma-k}^{(\alpha+\beta+k+l,\beta+k-l)}$, which vanishes for $k \geqslant n+1$, the resulting expression vanishes for all these k values, which yields a terminating sum over $k \in \{0,\ldots,n\}$.

Unlike Koornwinder's addition theorem for Jacobi polynomials, the addition theorem for the Jacobi functions of the second kind (see §4) involves a nonterminating sum. One can see this by examining (4.20), first recognizing that the Jacobi polynomials $P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}$ vanish for $\gamma = n \in \mathbb{N}_0$ and $k \geqslant n+1$. Considering the limit as $\gamma \to n$ for all values of $k \in \mathbb{N}_0$ in Koornwinder's addition theorem, one sees that the factor $1/(-\gamma)_k$ blows up for $k \geqslant n+1$. On the other hand, this factor in the limit is multiplied by the Jacobi function of the first kind prefactor containing $1/\Gamma(\gamma-k+1)$, and considering the residues of the gamma function, one sees that the product will be finite, as shown in (4.22), for $k \geqslant n+1$. This finite factor is then multiplied by the Jacobi function of the second kind $Q_{\gamma-k}^{(\alpha+\beta+k+l,\beta+k-l)}(2z_>^2-1)$, which does not vanish for $k \geqslant n+1$ for $\alpha,\beta \notin \mathbb{Z}$, unlike the case for Jacobi polynomials.

4.4. Olver-normalized Jacobi functions. In (2.8), we introduced Olver's normalization of the Gauss [**11**, (15.2.2)] and generalized hypergeometric function (see also [**11**, (16.2.5)]), which results in these functions being entire functions of all their parameters, including the denominator parameters. Olver applied this concept of a special normalization to the associated Legendre function of the second kind [**11**, (14.3.10)] (see also [**46**, pp. 170 and 178]). We now demonstrate how to apply it to the Jacobi functions of the first and second kind.

In the above description, instead of carefully determining the limits of the relevant functions when there are removable singularities due to the appearance of various gamma function prefactors, an alternative is to use appropriately defined Olver-normalized Jacobi functions and recast the addition theorems correspondingly. The benefit of using Olver-normalized definitions of the Jacobi functions is that one avoids complications due to gamma functions with removable singularities. Typical examples occur in often appearing examples when one has degrees γ and parameters α,β given by integers. In these cases, if the standard definitions such as those which appear in Theorems 2.5, 2.7 are used, the functions that appear are not defined and careful limits must be taken. However, if one adopts carefully chosen Olver-normalized definitions in which only Olver-normalized Gauss hypergeometric functions appear, then these functions will be entire in all their parameters. As we will see, by using these definitions, we arrive at formulas for the addition theorems which are elegant and highly useful! First we give our new choice of the Olver normalization, and then relate the Olver-normalized definitions to the usual definitions. Our definitions of Olver-normalized Jacobi functions of the first and second kind in the hyperbolic and trigonometric contexts are given by

$$\boldsymbol{P}_\gamma^{(\alpha,\beta)}(z) := {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,\alpha+\beta+\gamma+1\\ \alpha+1\end{matrix};\frac{1-z}{2}\right), \tag{4.23}$$

$$\boldsymbol{Q}_\gamma^{(\alpha,\beta)}(z) := \frac{2^{\alpha+\beta+\gamma}}{(z-1)^{\alpha+\gamma+1}(z+1)^{\beta}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\gamma+1,\alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix};\frac{2}{1-z}\right), \tag{4.24}$$

and

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(x) := {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,\alpha+\beta+\gamma+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right), \tag{4.25}$$

(4.26)
$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) := \tfrac{1}{2}\Gamma(\alpha+1)\left(\frac{1+x}{2}\right)^\gamma \times \left(\frac{\cos(\pi\alpha)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,-\beta-\gamma\\1+\alpha\end{matrix};\frac{x-1}{x+1}\right) - \frac{\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)}\left(\frac{1+x}{1-x}\right)^\alpha {}_2\boldsymbol{F}_1\left(\begin{matrix}-\alpha-\gamma,-\alpha-\beta-\gamma\\1-\alpha\end{matrix};\frac{x-1}{x+1}\right)\right).$$

Therefore one has connection relations between the Jacobi functions of the first and second kinds and their Olver-normalized counterparts, namely

$$P_\gamma^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\boldsymbol{P}_\gamma^{(\alpha,\beta)}(z), \tag{4.27}$$

$$Q_\gamma^{(\alpha,\beta)}(z) = \Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)\,\boldsymbol{Q}_\gamma^{(\alpha,\beta)}(z), \tag{4.28}$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\boldsymbol{\mathsf{P}}_\gamma^{(\alpha,\beta)}(x). \tag{4.29}$$

Note that

$$\boldsymbol{\mathsf{P}}_\gamma^{(\alpha,\beta)}(x) = \boldsymbol{P}_\gamma^{(\alpha,\beta)}(x\pm i0), \tag{4.30}$$

as in (2.65). Furthermore in the special case $\gamma=0$ one has

$$\boldsymbol{Q}_0^{(\alpha,\beta)}(z) := \frac{2^{\alpha+\beta}}{(z-1)^{\alpha+1}(z+1)^\beta}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}1,\alpha+1\\\alpha+\beta+2\end{matrix};\frac{2}{1-z}\right). \tag{4.31}$$

Remark 4.10. As of the date of publication of this manuscript, we have been unable to find an Olver-normalized version of the Jacobi function of the second kind on-the-cut $\boldsymbol{\mathsf{Q}}_\gamma^{(\alpha,\beta)}$. However we have found a special normalization of this function which works well when $\gamma=0$ and the β parameter takes an integer value, which is of particular importance because this case appears in a very important application (see Section 5 below). Let $b\in\mathbb{N}_0$. Define

$$\mathcal{Q}_{-k}^{(\alpha+k+l,b+k-l)}(x) := \lim_{\gamma\to 0,\beta\to b}(-\beta-\gamma)_l\mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(x), \tag{4.32}$$

which is a well-defined function for all α,b,x,k,l in its domain.

4.5. Addition theorems for the Olver-normalized Jacobi functions. Now that we have introduced the Olver-normalized Jacobi functions of the first and second kind in the hyperbolic and trigonometric contexts, we are in a position to perform the straightforward derivation of the corresponding addition theorems for these functions.

Theorem 4.11. *Let $\gamma,\alpha,\beta\in\mathbb{C}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,-1]$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $x,w\in\mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ defined by (3.2), (3.3), respectively, such that the complex variables $\gamma,\alpha,\beta,z_1,z_2,x_1,x_2,x,w$ are in some yet to be determined* *neighborhood of the real line. Then*

(4.33)

$$P_\gamma^{(\alpha,\beta)}(Z^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^{k-l}(\alpha+k+l)(\alpha+\gamma+1)_l(-\beta-\gamma)_l(z_1z_2)^{k-l}$$
$$\times((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}\boldsymbol{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1)$$
$$\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x),$$

(4.34)

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm)=\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^{k-l}(\alpha+k+l)(\alpha+\gamma+1)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}$$
$$\times\boldsymbol{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)$$
$$\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x),$$

and

(4.35)

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^{k-l}(\alpha+k+l)(\alpha+\gamma+1)_l(-\beta-\gamma)_l(x_1x_2)^{k-l}$$
$$\times((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_1^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_2^2-1)$$
$$\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x),$$

(4.36)

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\Gamma(\alpha+1)\sum_{k=0}^\infty\frac{(-1)^k(\alpha+\beta+\gamma+1)_k(\alpha+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^{k-l}(\alpha+k+l)(-\beta-\gamma)_l(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}$$
$$\times\mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)$$
$$\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x).$$

PROOF. Substituting (4.27)–(4.29) into (3.4), (3.5), (4.2), (4.20) as necessary completes the proof. □

There are corresponding expansions that are sometimes useful in which the l sum is reversed, obtained by making the replacements $l'=k-l$ and then $l'\mapsto l$ in Theorem 4.2. These are as follows.

COROLLARY 4.11.1. *Let* $\gamma,\alpha,\beta\in\mathbb{C}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,-1]$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $x,w\in\mathbb{C}$, *with* $Z^\pm$, $\mathsf{X}^\pm$ *defined by* (3.2), (3.3), *respectively, such* *that the complex variables $\gamma,\alpha,\beta,z_1,z_2,x_1,x_2,x,w$ are in some yet to be determined neighborhood of the real line. Then*

(4.37)

$$P_\gamma^{(\alpha,\beta)}(Z^\pm)=x\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(\alpha+\gamma+1)_{k-l}(-\beta-\gamma)_{k-l}(z_1z_2)^l((z_1^2-1)(z_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\boldsymbol{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_1^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_2^2-1)$$
$$\times w^lP_{k-l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x),$$

(4.38)

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm)=\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(\alpha+\gamma+1)_{k-l}(z_1z_2)^l((z_1^2-1)(z_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\boldsymbol{Q}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_>^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_<^2-1)$$
$$\times w^lP_l^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x),$$

and

(4.39)

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(\alpha+\gamma+1)_{k-l}(-\beta-\gamma)_{k-l}(x_1x_2)^l$$
$$\times((x_1^2-1)(x_2^2-1))^{\frac{2k-l}{2}}\mathsf{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_1^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_2^2-1)$$
$$\times w^lP_{k-l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x),$$

(4.40)

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\Gamma(\alpha+1)\sum_{k=0}^\infty(-1)^k\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(-\beta-\gamma)_{k-l}(x_1x_2)^l((x_1^2-1)(x_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\mathsf{Q}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_>^2-1)$$
$$\times w^lP_{k-l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x).$$

PROOF. Making the replacement $l\mapsto k-l$ in Theorem 4.11 completes the proof. ☐

REMARK 4.12. It should be noted that removing the factors $\Gamma(\alpha+\gamma+1)/\Gamma(\gamma+1)$ from the P series and $\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)$ from the Q series would convert the P and Q on the left-hand sides to $\boldsymbol{P}$ and $\boldsymbol{Q}$, so that all expansions would be in terms of Olver-normalized functions.

By examining the expansion of the Jacobi function of the first kind, one can see that the $(-\gamma)_k$ shifted factorial in the above alternative expansion is moved from the denominator to the numerator, which is more natural for Jacobi polynomials where the sum is terminating. One can see the benefit: all the functions involved in the expansions are well-defined for all values of the parameters, including integer values. These expansions are extremely useful for expansions of fundamental solutions on rank-one symmetric spaces, where the degrees and parameters are given by integers. One no longer has any difficulties with various functions not being defined for certain parameter values. This is completely resolved. One example is that in the integer context of the Jacobi function of the second kind, the functions appear with degree equal to $\gamma - k$ for all $k \in \mathbb{N}_0$. They quickly become undefined for negative values of the degree. However, since the Olver-normalized Jacobi functions are entire functions, there is no longer any problem here. These alternative expansions are highly desirable!

5. Eigenfunction expansions of a fundamental solution of the Laplace–Beltrami operator on non-compact and compact symmetric spaces of rank one

As an application of the addition theorem for the Jacobi function of the second kind, we now give an introduction to the motivation for the material which has been presented in the previous sections. It is a study of the global analysis of the Laplace–Beltrami operator and the solutions of inhomogeneous elliptic equations: Poisson's equation on the Riemannian symmetric spaces of rank one.

Let $d = \dim_{\mathbb{R}} \mathbb{K}$, where $\mathbb{K}$ is equal to the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the octonions $\mathbb{O}$. For $d = 1$, namely the real case, the corresponding spaces are Riemannian manifolds of constant curvature, including Euclidean space $\mathbb{R}^n$, real hyperbolic geometry $\mathbb{R}\mathbf{H}^n_R$ (noncompact), and real hyperspherical geometry $\mathbb{R}\mathbf{S}^n_R$ (compact), in various models. For $d = 2, 4, 8$, it is well known that there are corresponding isotropic Riemannian manifolds of both non-compact and compact type which are referred to as rank-one symmetric spaces (see for instance [**27**]). They include the complex hyperbolic space $\mathbb{C}\mathbf{H}^n_R$, the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n_H$, the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2_R$, the complex projective space $\mathbb{C}\mathbf{P}^n_R$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n_R$, and the octonionic projective (Cayley) plane $\mathbb{O}\mathbf{P}^2_R$. In each of the preceding, $R > 0$ is the radius of curvature. The complex, quaternionic, and octonionic rank-one symmetric spaces have respective real dimensions $2n, 4n, 16$. For a description of the Riemannian manifolds given by the rank-one symmetric spaces, see for instance [**25–27**] and the references therein.

Riemannian symmetric spaces, compact and non-compact, come in infinite series (four corresponding to complex simple groups and seven to real simple groups), together with a finite class of exceptional spaces (see [**25**, p. 516, 518]). Each symmetric space comes with a commutative algebra of invariant differential operators and correspondingly a class of eigenfunctions, its spherical functions. In the case of the rank-one symmetric spaces, these are just hypergeometric (Jacobi) functions with specified parameters. This has been used as motivation for several generalizations of the classical Gauss hypergeometric functions that are beyond the scope of this paper, for example the Heckman–Opdam functions on root systems [**22, 23, 47, 48**] and the work of Macdonald [**42, 43**].

Due to the isotropy of the symmetric spaces of rank one, a fundamental solution of the Laplace–Beltrami operator on any of these manifolds can be obtained by solving an inhomogeneous *ordinary* differential equation, given in terms of the geodesic distance. Laplace's equation is satisfied on the manifold when the Laplace–Beltrami operator acts on an unknown function and the result is zero. In geodesic polar coordinates the Laplace–Beltrami operator is given on the rank-one noncompact (hyperbolic) symmetric spaces by

$$\Delta = \frac{1}{R^2}\left\{\frac{\partial^2}{\partial r^2} + [d(n-1)\coth r + 2(d-1)\coth(2r)]\,\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{K/M}\right\} \tag{5.1}$$

$$= \frac{1}{R^2}\left\{\frac{\partial^2}{\partial r^2} + [(dn-1)\coth r + (d-1)\tanh r]\,\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{K/M}\right\} \tag{5.2}$$

$$=: \frac{1}{R^2}\left(\Delta_r + \frac{1}{\sinh^2 r}\Delta_{K/M}\right), \tag{5.3}$$

and on the rank-one compact (projective) spaces by

$$\Delta = \frac{1}{R^2}\left\{\frac{\partial^2}{\partial \theta^2} + [d(n-1)\cot\theta + 2(d-1)\cot(2\theta)]\,\frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta}\Delta_{K/M}\right\} \tag{5.4}$$

$$= \frac{1}{R^2}\left\{\frac{\partial^2}{\partial \theta^2} + [(dn-1)\cot\theta + (d-1)\tan\theta]\,\frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta}\Delta_{K/M}\right\} \tag{5.5}$$

$$=: \frac{1}{R^2}\left(\Delta_\theta + \frac{1}{\sin^2 \theta}\Delta_{K/M}\right), \tag{5.6}$$

where r and θ are the geodesic distance coordinates on the noncompact and compact spaces respectively (see [**25**, Lemma 21]). For a spherically symmetric solution such as a fundamental solution, the contribution from $\Delta_{K/M}$ vanishes and the corresponding homogeneous equation becomes relatively simple, namely the radial form of Laplace's equation, which is

$$\Delta_r\, u(r) = 0 \qquad \text{or} \qquad \Delta_\theta\, v(\theta) = 0. \tag{5.7}$$

The solutions of this second-order ordinary differential equation are given by Jacobi/hypergeometric functions (see [**27**, p. 484]). It can be easily verified that 'radial' solutions, homogeneous or fundamental, are of the form

$$u(r) = a\,P_0^{(\alpha,\beta)}(\cosh(2r)) + b\,Q_0^{(\alpha,\beta)}(\cosh(2r)), \tag{5.8}$$

$$v(\theta) = c\,\mathsf{P}_0^{(\alpha,\beta)}(\cos(2\theta)) + d\,\mathsf{Q}_0^{(\alpha,\beta)}(\cos(2\theta)), \tag{5.9}$$

where for the complex, quaternionic, and octonionic rank-one symmetric spaces one has $\alpha = n-1, 2n-1, 7$ and $\beta = 0, 1, 3$ respectively [**18**, Table 1, p. 265]. Furthermore, a fundamental solution, which by definition is not a homogeneous solution, must be singular at the origin (i.e., at $r = 0$ or $\theta = 0$) and locally match to a Euclidean fundamental solution. This implies that any fundamental solution must be irregular at the origin. Therefore fundamental solutions must correspond to Jacobi functions of the second kind. For a fundamental solution, it is the case that $a = c = 0$; and we must determine b and d, which will depend on n and R.

Remark 5.1. Note that the general solution as a function of the geodesic distance includes contributions from both the function of the first kind and the function of the second kind. However, the function of the first kind with $\gamma = 0$ simply contributes a constant a or c, since $P_0^{(\alpha,\beta)}(z) = 1$ as stated in (2.50) (and the same for the function on-the-cut). On the other hand, in the case of non-spherically symmetric solutions there will be a contribution from the function of the first kind because then the contribution of the $\Delta_{K/M}$ term in (5.3) or (5.6) will be non-zero.

Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^s$. Then a Euclidean fundamental solution of Laplace's equation is (see for instance [**21**, p. 202])

$$\mathcal{G}^s(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{\Gamma(s/2)}{2\pi^{s/2}(s-2)}\|\mathbf{x}-\mathbf{x}'\|^{2-s} & \text{if } s=1 \text{ or } s\geqslant 3, \\ \dfrac{1}{2\pi}\log\|\mathbf{x}-\mathbf{x}'\|^{-1} & \text{if } s=2. \end{cases} \tag{5.10}$$

For a description of opposite antipodal fundamental solutions on the real hypersphere, see [**7**]. The above analysis leads to the following.

THEOREM 5.2. *A fundamental solution and an opposite antipodal fundamental solution, for the Laplace–Beltrami operators on the rank-one $n \geqslant 1$ noncompact and compact symmetric spaces respectively, given in terms of the respective geodesic distance coordinates $r \in [0,\infty)$ and $\theta \in [0,\pi/2]$, are*

$$\mathcal{G}^{\mathbb{C}\mathbf{H}^n_R}(r) = \frac{(n-1)!}{2\pi^n R^{2n-2}} Q_0^{(n-1,0)}(\cosh(2r)), \tag{5.11}$$

$$\mathcal{G}^{\mathbb{H}\mathbf{H}^n_R}(r) = \frac{(2n)!}{2\pi^{2n} R^{4n-2}} Q_0^{(2n-1,1)}(\cosh(2r)), \tag{5.12}$$

$$\mathcal{G}^{\mathbb{O}\mathbf{H}^2_R}(r) = \frac{302\,400}{\pi^8 R^{14}} Q_0^{(7,3)}(\cosh(2r)), \tag{5.13}$$

and

$$\mathcal{G}^{\mathbb{C}\mathbf{P}^n_R}(\theta) = \frac{(n-1)!}{2\pi^n R^{2n-2}} \mathsf{Q}_0^{(n-1,0)}(\cos(2\theta)), \tag{5.14}$$

$$\mathcal{G}^{\mathbb{H}\mathbf{P}^n_R}(\theta) = \frac{(2n)!}{2\pi^{2n} R^{4n-2}} \mathsf{Q}_0^{(2n-1,1)}(\cos(2\theta)), \tag{5.15}$$

$$\mathcal{G}^{\mathbb{O}\mathbf{P}^2_R}(\theta) = \frac{302\,400}{\pi^8 R^{14}} \mathsf{Q}_0^{(7,3)}(\cos(2\theta)). \tag{5.16}$$

PROOF. The complex, quaternionic, and octonionic rank-one symmetric spaces all have even dimension: $s = 2n, 4n, 16$, respectively. It is easy to verify that the spherically symmetric solutions of Laplace's equation on these spaces are Jacobi functions of the first and second kind in the noncompact case, and Jacobi functions of the first and second kind on-the-cut in the compact case, having parameter $\gamma = 0$ and parameters $\alpha = n-1$, $2n-1, 7$ and $\beta = 0, 1, 3$, respectively. Furthermore, each of the corresponding fundamental solutions must match to a Euclidean fundamental solution. Using (2.63) and (2.70), assuming that $\gamma = 0$, $\alpha = a$, $\beta = b$, $a \in \mathbb{N}$, $b \in \mathbb{N}_0$, one has that near the singularity at unity, the Jacobi function of the second kind and the Jacobi function of the second kind on-the-cut have the behaviors

$$Q_0^{(a,b)}(1+\epsilon) \sim \mathsf{Q}_0^{(a,b)}(1-\epsilon) \sim \frac{2^{a-1}(a-1)!b!}{(a+b)!\,\epsilon^a} \qquad \text{as } \epsilon \to 0^+. \tag{5.17}$$

Referring to the geodesic distance coordinate of the hyperbolic manifolds as $r \in [0,\infty)$ and that of the compact manifolds as $\theta \in [0,\pi/2]$, one has

$$\cosh(2r) \sim \cosh(2\tfrac{\rho}{R}) \sim 1 + \tfrac{2\rho^2}{R^2} \qquad \text{as } r \to 0, \tag{5.18}$$

$$\cos(2\theta) \sim \cos(\tfrac{2\rho}{R}) \sim 1 - \tfrac{2\rho^2}{R^2} \qquad \text{as } \theta \to 0, \tag{5.19}$$

where ρ is the Euclidean geodesic distance. Matching locally to the Euclidean fundamental solution (5.10) in the $R\to\infty$ flat-space limit (see for instance [**9**, §2.4]), one can determine the constant of proportionality which multiplies the Jacobi functions of the second kind. This completes the proof. □

Since fundamental solutions on the rank-one symmetric spaces all have $\gamma=0$, we first present the expansions in this case. The $\gamma=0$ case of any Jacobi function of the first kind equals unity. However, for the Jacobi functions of the second kind, the $\gamma=0$ functions are quite rich, and expansions involving them are quite useful in that they allow one to produce separated eigenfunction expansions of a fundamental solution of Laplace's equation.

Remark 5.3. The reader should be aware that the addition theorems presented below for the Jacobi functions of the second kind with $\gamma=0$ are well-defined except in the case where the α and β parameters on the left-hand sides are non-negative integers. Then, special care must be taken (refer to Theorems 2.10, 2.11), though the functions at these parameter values may be obtained by taking an appropriate limit.

Corollary 5.3.1. *Let $\alpha,\beta\in\mathbb{C}$, $\beta\notin\mathbb{Z}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,1]$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $x,w\in\mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ defined by* (3.2), (3.3), *respectively, such that the complex variables $\alpha,\beta,z_1,z_2,x_1,x_2,x,w$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned} Q_0^{(\alpha,\beta)}(Z^\pm) &= \Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(\beta+1)\sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+1)_k}{(\alpha+k)(\beta+1)_k}\\ &\quad\times\sum_{l=0}^k(\mp1)^{k-l}(\alpha+k+l)(\alpha+1)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}\\ &\quad\times \boldsymbol{Q}_{-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1)\boldsymbol{P}_{-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)\\ &\quad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x), \end{aligned} \tag{5.20}$$

$$\begin{aligned} \mathsf{Q}_0^{(\alpha,\beta)}(\mathsf{X}^\pm) &= \Gamma(\alpha+1)\sum_{k=0}^\infty(-1)^k \frac{(\alpha+1)_k(\alpha+\beta+1)_k}{(\alpha+k)(\beta+1)_k}\\ &\quad\times\sum_{l=0}^k(\mp1)^{k-l}(\alpha+k+l)(-\beta)_l(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}\\ &\quad\times \mathsf{Q}_{-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathbf{P}_{-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)\\ &\quad\times w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x). \end{aligned} \tag{5.21}$$

Proof. Substituting $\gamma=0$ in Theorem 4.11 for the Jacobi functions of the second kind completes the proof. □

Next we give examples of the expansions for complex and quaternionic hyperbolic spaces, where $\beta=0,1$, respectively. First we treat the complex case, which corresponds to complex hyperbolic and projective space. In order to do this we start with Theorem 4.11 and take the limit as $\beta\to0$ using [**2**, (6.4.13)]

$$\lim_{\mu\to0}\frac{n+\mu}{\mu}C_n^\mu(x)=\epsilon_nT_n(x), \tag{5.22}$$

where $\epsilon_n := 2 - \delta_{n,0}$ is the Neumann factor commonly appearing in Fourier series.

COROLLARY 5.3.2. *Let $\alpha \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ defined by (3.2), (3.3), respectively, such that the complex variables $\alpha, z_1, z_2, x_1, x_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$
\begin{aligned}
Q_0^{(\alpha,0)}(Z^\pm) = \Gamma(\alpha+1)\Gamma(\alpha+1) \sum_{k=0}^{\infty} \frac{(\alpha+1)_k(\alpha+1)_k}{(\alpha+k)k!} \\
\times \sum_{l=0}^{k} (\mp 1)^{k-l} (\alpha+k+l)(\alpha+1)_l (z_1 z_2)^{k-l} ((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}} \\
\times \boldsymbol{Q}_{-k}^{(\alpha+k+l,k-l)}(2z_>^2-1) \boldsymbol{P}_{-k}^{(\alpha+k+l,k-l)}(2z_<^2-1) \\
\times w^{k-l} P_l^{(\alpha-1,k-l)}(2w^2-1)\epsilon_{k-l} T_{k-l}(x),
\end{aligned} \tag{5.23}
$$

$$
\begin{aligned}
\mathsf{Q}_0^{(\alpha,0)}(\mathsf{X}^\pm) = \Gamma(\alpha+1) \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha+1)_k(\alpha+1)_k}{(\alpha+k)k!} \\
\times \sum_{l=0}^{k} (\mp 1)^{k-l} (\alpha+k+l)(x_1 x_2)^{k-l} ((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}} \\
\times \mathcal{Q}_{-k}^{(\alpha+k+l,k-l)}(2x_<^2-1) \mathsf{P}_{-k}^{(\alpha+k+l,k-l)}(2x_>^2-1) \\
\times w^{k-l} P_l^{(\alpha-1,k-l)}(2w^2-1)\epsilon_{k-l} T_{k-l}(x).
\end{aligned} \tag{5.24}
$$

PROOF. Taking the limit as $\beta \to 0$ in Theorem 4.11 using (5.22) completes the proof. ☐

Now we treat the case of the quaternionic hyperbolic and projective spaces, which correspond to $\beta = 1$.

COROLLARY 5.3.3. *Let $\alpha \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ defined by (3.2), (3.3), respectively, such that the complex variables $\alpha, z_1, z_2, x_1, x_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$
\begin{aligned}
Q_0^{(\alpha,1)}(Z^\pm) = \Gamma(\alpha+1)\Gamma(\alpha+1) \sum_{k=0}^{\infty} \frac{(\alpha+1)_k(\alpha+2)_k}{(\alpha+k)(2)_k} \\
\times \sum_{l=0}^{k} (\mp 1)^{k-l} (1+k-l)(\alpha+k+l)(\alpha+1)_l (z_1 z_2)^{k-l} ((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}} \\
\times \boldsymbol{Q}_{-k}^{(\alpha+k+l,1+k-l)}(2z_>^2-1) \boldsymbol{P}_{-k}^{(\alpha+k+l,1+k-l)}(2z_<^2-1) \\
\times w^{k-l} P_l^{(\alpha-2,1+k-l)}(2w^2-1) U_{k-l}(x),
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
\mathsf{Q}_0^{(\alpha,1)}(\mathsf{X}^\pm) = \Gamma(\alpha+1) \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha+1)_k(\alpha+2)_k}{(\alpha+k)(2)_k} \\
\times \sum_{l=0}^{k} (\mp 1)^{k-l} (1+k-l)(\alpha+k+l)(x_1 x_2)^{k-l} ((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}} \\
\times \mathcal{Q}_{-k}^{(\alpha+k+l,1+k-l)}(2x_<^2-1) \mathsf{P}_{-k}^{(\alpha+k+l,1+k-l)}(2x_>^2-1) \\
\times w^{k-l} P_l^{(\alpha-2,1+k-l)}(2w^2-1) U_{k-l}(x).
\end{aligned} \tag{5.26}
$$

PROOF. Take the limit as $\beta \to 1$ in Theorem 4.11 using [**11**, (18.7.4)], which connects the Chebyshev polynomial of the second kind to the Gegenbauer polynomial with parameter equal to unity, namely $C_n^1(x) = U_n(x)$. This completes the proof. $\square$

Now we treat the case of the octonionic hyperbolic and projective spaces, which correspond to $\beta = 3$.

COROLLARY 5.3.4. *Let* $\alpha \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, *with* $Z^{\pm}$, $\mathsf{X}^{\pm}$ *defined by* (3.2), (3.3), *respectively, such that the complex variables* $\alpha, z_1, z_2, x_1, x_2, x, w$ *are in some yet to be determined neighborhood of the real line. Then*

(5.27)
$$Q_0^{(\alpha,3)}(Z^{\pm}) = 2\Gamma(\alpha+1)\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\alpha+4)_k}{(\alpha+k)(4)_k}$$
$$\times \sum_{l=0}^{k}(\mp 1)^{k-l}(3+k-l)(\alpha+k+l)(\alpha+1)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}$$
$$\times \boldsymbol{Q}_{-k}^{(\alpha+k+l,3+k-l)}(2z_>^2-1)\boldsymbol{P}_{-k}^{(\alpha+k+l,3+k-l)}(2z_<^2-1)$$
$$\times w^{k-l}P_l^{(\alpha-4,3+k-l)}(2w^2-1)C_{k-l}^3(x),$$

(5.28)
$$\mathsf{Q}_0^{(\alpha,3)}(\mathsf{X}^{\pm}) = \frac{1}{3}\Gamma(\alpha+1)\sum_{k=0}^{\infty}(-1)^k\frac{(\alpha+1)_k(\alpha+4)_k}{(\alpha+k)(4)_k}$$
$$\times \sum_{l=0}^{k}(\mp 1)^{k-l}(3+k-l)(\alpha+k+l)(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}$$
$$\times \mathcal{Q}_{-k}^{(\alpha+k+l,3+k-l)}(2x_<^2-1)\mathsf{P}_{-k}^{(\alpha+k+l,3+k-l)}(2x_>^2-1)$$
$$\times w^{k-l}P_l^{(\alpha-4,3+k-l)}(2w^2-1)C_{k-l}^3(x).$$

PROOF. Setting $\beta = 3$ in Theorem 4.11 completes the proof. $\square$

The above calculations look almost trivial in that they are simply substitutions of the values $\beta = 0, 1, 3$ and $\gamma = 0$ into the addition theorems of Theorem 4.11. However, it should be understood that ordinarily these computations would be extremely difficult, particularly if one were to use the standard normalizations of the Jacobi functions. With standard normalizations, the Jacobi functions of the second kind at these parameter values, and in fact at arbitrary integer values of α, β and the degree γ, are not even defined. It is only because of the particular normalization that we have chosen that evaluations at these parameter values become quite easy. We will further take advantage of these expansions in later publications.

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