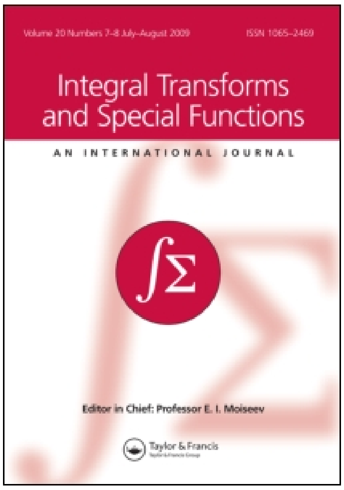

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Derivatives with respect to the degree and order of associated Legendre functions for |z|>1 using modified Bessel functions

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Derivatives with respect to the degree and order of associated Legendre functions for $|z| > 1$ using modified Bessel functions

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Expressions for the derivatives with respect to order of modified Bessel functions evaluated at integer orders and certain integral representations of associated Legendre functions with modulus argument greater than unity are used to compute derivatives of the associated Legendre functions with respect to their parameters. For the associated Legendre functions of the first and second kind, derivatives with respect to the degree are evaluated at odd-half-integer degrees, for general complex orders, and derivatives with respect to the order are evaluated at integer orders, for general complex degrees.

Keywords: Legendre functions; modified Bessel functions; derivatives

AMS Subject Classification: 31B05;31B10;33B10;33B15;33C05;33C10

1. Introduction

Here we present formulae for derivatives of associated Legendre functions (hereafter referred to as Legendre functions) of the first kind $P_\nu^\mu(z)$ and the second kind $Q_\nu^\mu(z)$, with respect to their parameters, namely the degree ν and the order μ. Some formulae relating to these derivatives have been previously noted [10] and also there has been recent work in this area [2,12–15] with Brychkov [3] giving a recent reference covering the regime for argument $z \in [-1, 1]$. In this paper, we cover parameter derivatives of Legendre functions for argument $|z| > 1$.

The strategy applied in this paper is to incorporate derivatives with respect to order evaluated at integer orders for modified Bessel functions (see [1,4,10]) to compute derivatives with respect to the degree and the order of Legendre functions. Below, we apply these results through certain integral representations of Legendre functions in terms of modified Bessel functions.

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2. A useful transformation on the complex plane

There is a transformation over an open subset of the complex plane which is particularly useful in studying Legendre functions [1,9]. This transformation, which is valid on a certain domain of the complex numbers, accomplishes the following

$$\cosh z \longleftrightarrow \coth w$$
$$\coth z \longleftrightarrow \cosh w$$
$$\sinh z \longleftrightarrow \frac{1}{\sinh w},$$

where cosh, coth, and sinh are the complex hyperbolic cosine, cotangent, and sine functions, respectively. This transformation is accomplished using the following map

$$w(z) = \log \coth \frac{z}{2}, \tag{1}$$

(where log is the complex natural logarithm) which is verified to be an involution over an open subset of the complex plane given by $-\pi < \mathrm{Im}(z) < \pi$, where one removes a branch given by $\mathrm{Re}(z) \le 0$, $\mathrm{Im}(z) = 0$. This mapping is π periodic in the imaginary direction and is locally injective over the entire complex plane when the following set of complex numbers are removed $\{z\colon z = i\pi n, n \in \mathbb{Z}\} \cup \{z\colon \mathrm{Im}(z) = 2\pi n \text{ and } \mathrm{Re}(z) \le 0,\ n \in \mathbb{Z}, z \in \mathbb{C}\}$.

This transformation is particularly useful for certain Legendre functions that have natural domain given by the real interval $(1, \infty)$, such as toroidal harmonics [5,7] (and for other Legendre functions that one might encounter in potential theory), Legendre functions of the first and second kind with odd-half-integer degree and integer order. The real argument of these Legendre functions naturally occur in $[1, \infty)$, and these are the simultaneous ranges of both the real hyperbolic cosine and cotangent functions.

One application of this map occurs with the Whipple transformation of Legendre functions [6,16] under index (degree and order) interchange. See, for instance, Equations (8.2.7) and (8.2.8) in [1], namely

$$P_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right) = \sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{1/4}\mathrm{e}^{-i\mu\pi}}{\Gamma(\nu+\mu+1)}\, Q_\nu^\mu(z), \tag{2}$$

which are valid for Re $z > 0$ and for all complex ν and μ, except where the functions are not defined. $\Gamma(z)$ is the Gamma function [1].

3. Parameter derivative formulas from $K_\nu(t)$

Starting with [8, Equation (6.628.7)], we have

$$\int_0^\infty \mathrm{e}^{-zt} K_\nu(t) t^{\alpha-1/2}\,\mathrm{d}t = \sqrt{\frac{\pi}{2}}\Gamma\left(\alpha-\nu+\frac{1}{2}\right)\Gamma\left(\alpha+\nu+\frac{1}{2}\right)(z^2-1)^{-\alpha/2} P_{\nu-1/2}^{-\alpha}(z)$$
$$= \Gamma\left(\alpha-\nu+\frac{1}{2}\right)(z^2-1)^{-\alpha/2-1/4}\mathrm{e}^{-i\pi\nu} Q_{\alpha-1/2}^{\nu}\left(\frac{z}{\sqrt{z^2-1}}\right), \tag{3}$$

where $K_\nu(t)$ is a modified Bessel function of the second kind with order ν, and the two equalities are established through the Whipple transformation (Equation (2)).

We would like to generate an analytical expression for the derivative of the Legendre function of the second kind with respect to its order, evaluated at integer orders. In order to do this, our strategy is to solve the above integral expression for the Legendre function of the second kind, differentiate with respect to the order, evaluate at integer orders, and take advantage of the corresponding formula for differentiation with respect to order for modified Bessel functions of the second kind [1,3,4,10]. Using the expression for the Legendre function of the second kind in Equation (3), we solve for $Q^{\mu}_{\nu-1/2}(z)$ and re-express using the map in Equation (1). This gives us the following expression

$$Q^{\mu}_{\nu-1/2}(z) = \frac{(z^2-1)^{-\nu/2-1/4}\mathrm{e}^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)} \int_0^{\infty} \exp\left(\frac{-zt}{z^2-1}\right) K_{\mu}(t) t^{\nu-1/2}\mathrm{d}t.$$

Differentiating with respect to the order μ and evaluating at $\mu = \pm m$, where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ yields

$$\left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = (z^2-1)^{-\nu/2-1/4} \left[\frac{\partial}{\partial\mu} \frac{\mathrm{e}^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\right]_{\mu=\pm m} \int_0^{\infty} \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) K_{\pm m}(t) t^{\nu-1/2}\mathrm{d}t$$
$$+ \frac{(z^2-1)^{-\nu/2-1/4}(-1)^m}{\Gamma\left(\nu \mp m+\frac{1}{2}\right)} \int_0^{\infty} \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} \left[\frac{\partial}{\partial\mu} K_{\mu}(t)\right]_{\mu=\pm m} \mathrm{d}t.$$

The derivative from the first term is given as

$$\left[\frac{\partial}{\partial\mu} \frac{\mathrm{e}^{i\pi\mu}}{\Gamma(\nu-\mu+1/2)}\right]_{\mu=\pm m} = \frac{(-1)^m}{\Gamma(\nu \mp m+1/2)} \left[i\pi + \psi\left(\nu \mp m + \frac{1}{2}\right)\right],$$

where the Digamma function $\psi(z)$ is defined in terms of the derivative of the Gamma function with respect to its argument z through

$$\Gamma'(z) = \Gamma(z)\psi(z).$$

The derivative in the second integral [1,3,4,10] is given by

$$\left[\frac{\partial}{\partial\mu} K_{\mu}(t)\right]_{\mu=\pm m} = \pm m! \sum_{k=0}^{m-1} \frac{1}{k!(m-k)} \frac{t^{k-m}}{2^{k-m+1}} K_k(t) \tag{4}$$

(see, for instance, [3, Equation (1.14.2.2)]). Substituting these expressions for the derivatives into the two integrals and using the map in Equation (1) to re-evaluate these integrals in terms of Legendre functions gives the following general expression for the derivative of the Legendre function of the second kind with respect to its order evaluated at integer orders as

$$\frac{\Gamma(\nu \mp m+1/2)}{\Gamma(\nu - m+1/2)} \left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = \left[i\pi + \psi\left(\nu \mp m + \frac{1}{2}\right)\right] Q^{m}_{\nu-1/2}(z)$$
$$\pm m! \sum_{k=0}^{m-1} \frac{(-1)^{k-m}(z^2-1)^{(k-m)/2}}{k!(m-k)2^{k-m+1}} Q^{k}_{\nu+k-m-1/2}(z).$$

For $\mu = 0$, there is no contribution from the sum and the result is

$$\left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=0} = \left[i\pi + \psi\left(\nu + \frac{1}{2}\right)\right] Q_{\nu-1/2}(z),$$

which agrees with that given in [10, §4.4.3]. We are now able to obtain formulas for non-zero values of μ such as for $\mu = -1$

$$\left(\nu^2 - \frac{1}{4}\right) \left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=-1} = \left[i\pi + \psi\left(\nu + \frac{3}{2}\right)\right] Q^{1}_{\nu-1/2}(z) + (z^2-1)^{-1/2} Q_{\nu-3/2}(z),$$

or for $\mu = +1$

$$\left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=1} = \left[i\pi + \psi\left(\nu - \frac{1}{2}\right)\right] Q^{1}_{\nu-1/2}(z) - (z^2-1)^{-1/2} Q_{\nu-3/2}(z),$$

or for other integer values of μ.

If we start with the expression for the Legendre function of the first kind in Equation (3) and solve for $P^{-\mu}_{\nu-1/2}(z)$ we have

$$P^{-\mu}_{\nu-1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{\mu/2}}{\Gamma(\mu-\nu+1/2)\Gamma(\mu+\nu+1/2)} \int_0^\infty \mathrm{e}^{-zt} K_\nu(t) t^{\mu-1/2} \mathrm{d}t. \qquad (5)$$

Differentiating with respect to degree ν and evaluating at $\nu = \pm n$, where $n \in \mathbb{N}_0$ yields

$$\left[\frac{\partial}{\partial\nu} P^{-\mu}_{\nu-1/2}(p)\right]_{\nu=\pm n} = \sqrt{\frac{2}{\pi}} (z^2-1)^{\mu/2} \left[\frac{\partial}{\partial\nu} \frac{1}{\Gamma(\mu-\nu+\frac{1}{2})\Gamma(\mu+\nu+\frac{1}{2})}\right]_{\nu=\pm n} \int_0^\infty \mathrm{e}^{-zt} K_{\pm n}(t) t^{\mu-1/2}\, \mathrm{d}t$$
$$+ \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{\mu/2}}{\Gamma(\mu \mp n + \frac{1}{2})\Gamma(\mu \pm n + \frac{1}{2})} \int_0^\infty \mathrm{e}^{-zt} t^{\mu-1/2} \left[\frac{\partial}{\partial\nu} K_\nu(t)\right]_{\nu=\pm n} \mathrm{d}t.$$

The derivative from the first term is given as

$$\left[\frac{\partial}{\partial\nu} \frac{1}{\Gamma(\mu-\nu+1/2)\Gamma(\mu+\nu+1/2)}\right]_{\nu=\pm n} = \frac{\psi(\mu \mp n + 1/2) - \psi(\mu \pm n + 1/2)}{\Gamma(\mu \pm n + 1/2)\Gamma(\mu \mp n + 1/2)}.$$

Substituting this expression for the derivative and that given in Equation (4) yields the following general expression for the derivative of the Legendre function of the first kind with respect to its degree evaluated at odd-half-integer degrees as

$$\pm \left[\frac{\partial}{\partial\nu} P^{-\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = \left[\psi\left(\mu - n + \frac{1}{2}\right) - \psi\left(\mu + n + \frac{1}{2}\right)\right] P^{-\mu}_{n-1/2}(z)$$
$$+ \frac{n!}{\Gamma(\mu+n+\frac{1}{2})} \sum_{k=0}^{n-1} \frac{\Gamma(\mu - n + 2k + \frac{1}{2})(z^2-1)^{(n-k)/2}}{k!(n-k)2^{k-n+1}} P^{-\mu+n-k}_{k-1/2}(z).$$

If one makes a global replacement, $-\mu \mapsto \mu$, using the properties of Gamma and Digamma functions, this result reduces to

$$\pm \left[\frac{\partial}{\partial\nu} P^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = \left[\psi\left(\mu + n + \frac{1}{2}\right) - \psi\left(\mu - n + \frac{1}{2}\right)\right] P^{\mu}_{n-1/2}(z)$$
$$+ n!\, \Gamma\left(\mu - n + \frac{1}{2}\right) \sum_{k=0}^{n-1} \frac{(z^2-1)^{(n-k)/2}}{\Gamma(\mu + n - 2k + \frac{1}{2})\, k!(n-k)2^{k-n+1}} P^{\mu+n-k}_{k-1/2}(z).$$

Note that by using the recurrence relation for Digamma functions [10, §1.2]

$$\psi(z+1) = \psi(z) + \frac{1}{z},$$

we can establish

$$\psi\left(\mu+n+\frac{1}{2}\right)-\psi\left(\mu-n+\frac{1}{2}\right)=2\mu\sum_{l=1}^{n}\left[\mu^2-\left(l-\frac{1}{2}\right)^2\right]^{-1}.$$

For $\nu=0$ there is no contribution from the sum and the result is

$$\left[\frac{\partial}{\partial\nu}P^{\mu}_{\nu-1/2}(z)\right]_{\nu=0}=0,$$

which agrees with that given in [10, §4.4.3]. We are now able to obtain formulas for non-zero values of ν such as for $\nu=\pm1$

$$\pm\left(\mu^2-\frac{1}{4}\right)\left[\frac{\partial}{\partial\nu}P^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm1}=2\mu P^{\mu}_{1/2}(z)+(z^2-1)^{1/2}P^{\mu+1}_{-1/2}(z),$$

or for other integer values of ν.

Note that this method does not seem amenable to computing derivatives with respect to the degree of Legendre functions of the form $P^{\mu}_{\nu}(z)$ evaluated at integer degrees, since shifting the degree by $+1/2$ in Equation (5) converts the modified Bessel function of the second kind to a form like $K_{\nu+1/2}(t)$ and the derivative with respect to order of this Bessel function [1,4,10] is not of a form which is easily integrated.

4. Parameter derivative formulas from $I_\nu(t)$

Starting with [8, Equation (6.624.5)] (see also [11]) we have

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-zt}I_\nu(t)t^{\alpha-1/2}\,\mathrm{d}t&=\sqrt{\frac{2}{\pi}}\mathrm{e}^{-i\pi\alpha}(z^2-1)^{-\alpha/2}Q^{\alpha}_{\nu-1/2}(z)\\&=\Gamma\left(\alpha+\nu+\frac{1}{2}\right)(z^2-1)^{-\alpha/2-1/4}P^{-\nu}_{\alpha-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),\end{aligned}\qquad(6)$$

where $I_\nu(t)$ is a modified Bessel function of the first kind with order ν, and the two equalities are established through the Whipple transformation (Equation (2)). We will use this particular integral representation of Legendre functions to compute certain derivatives of the Legendre functions with respect to the degree and order.

We start with the integral representation of the Legendre function of the second kind in Equation (6). Differentiating with respect to the degree ν and evaluating at $\nu=\pm n$, where $n\in\mathbb{N}_0$, one obtains

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n}=\sqrt{\frac{\pi}{2}}\mathrm{e}^{i\pi\mu}(z^2-1)^{\mu/2}\int_0^\infty \mathrm{e}^{-zt}t^{\mu-1/2}\left[\frac{\partial}{\partial\nu}I_\nu(t)\right]_{\nu=\pm n}\mathrm{d}t.\qquad(7)$$

The derivative of the modified Bessel function of the first kind in Equation (7) [1,3,4,10] is given by

$$\left[\frac{\partial}{\partial\nu}I_\nu(t)\right]_{\nu=\pm n}=(-1)^{n+1}K_n(t)\pm n!\sum_{k=0}^{n-1}\frac{(-1)^{k-n}}{k!(n-k)}\frac{t^{k-n}}{2^{k-n+1}}I_k(t)\qquad(8)$$

(see, for instance, [3, Equation (1.13.2.1)]). Inserting Equation (8) into Equation (7) and using Equations (3) and (6), we obtain the following general expression for the derivative of the Legendre function of the second kind with respect to its degree evaluated at odd-half-integer degrees as

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu-n+\frac{1}{2}\right)(z^2-1)^{-1/4}Q^{n}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right) \pm n!\sum_{k=0}^{n-1}\frac{(z^2-1)^{(n-k)/2}}{2^{k-n+1}k!(n-k)}Q^{\mu+k-n}_{k-1/2}(z).$$

For $\nu=0$, there is no contribution from the sum and the result is

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=0} = -\sqrt{\frac{\pi}{2}}\mathrm{e}^{i\pi\mu}\Gamma\left(\mu+\frac{1}{2}\right)(z^2-1)^{-1/4}Q_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

which agrees with that given in [10, §4.4.3]. We are now able to obtain formulas for non-zero values of ν such as for $\nu=\pm1$

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm1} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu-\frac{1}{2}\right)(z^2-1)^{-1/4}Q^{1}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right) \pm (z^2-1)^{1/2}\;Q^{\mu-1}_{-1/2}(z),$$

or for other integer values of ν.

We can see that this method does not seem amenable to computing derivatives with respect to the degree of Legendre functions of the form $Q^{\mu}_{\nu}(z)$ evaluated at integer degrees, since shifting the degree by $+1/2$ in Equation (7) converts the modified Bessel function of the first kind to a form like $I_{\nu+1/2}(t)$ and the derivative with respect to order of this Bessel function [1,4,10] is not of a form which is easily integrated.

Finally, we obtain a formula for the derivative with respect to the order for the Legendre function of the first kind evaluated at integer orders. In order to do this, we use the integral expression for the Legendre function of the first kind given in Equation (6) and the map given in Equation (1) to convert to the appropriate argument. Now use the negative order condition for Legendre functions of the first kind [6, Equation (22)] to convert to a positive order.

Differentiating both sides of the resulting expression with respect to the order μ and evaluating at $\mu=\pm m$, where $m\in\mathbb{N}_0$ yields

$$\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = 2Q^{\pm m}_{\nu-1/2}(z) + (z^2-1)^{-\nu/2-1/4}\left\{\frac{\partial}{\partial\mu}\left[\Gamma\left(\nu-\mu+\frac{1}{2}\right)\right]^{-1}\right\}_{\mu=\pm m}\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)I_{\pm m}(t)t^{\nu-1/2}\mathrm{d}t + \frac{(z^2-1)^{-\nu/2-1/4}}{\Gamma(\nu\mp m+1/2)}\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)t^{\nu-1/2}\left[\frac{\partial}{\partial\mu}I_\mu(t)\right]_{\mu=\pm m}\mathrm{d}t.$$

The derivative of the reciprocal of the Gamma function reduces to $\psi(\nu\mp m+1/2)/\Gamma(\nu\mp m+1/2)$. The derivative with respect to order for the modified Bessel function of the first kind is given in Equation (8). The integrals are easily obtained by applying the map given by Equation (1) as necessary to Equations (3) and (6). Hence, by also using standard properties of Legendre, Gamma, and Digamma functions, we obtain the following compact form

$$\frac{\Gamma(\nu \mp m + 1/2)}{\Gamma(\nu - m + 1/2)} \left[\frac{\partial}{\partial \mu} P^{\mu}_{\nu-1/2}(z) \right]_{\mu=\pm m} = Q^{m}_{\nu-1/2}(z) + \psi\left(\nu \mp m + \frac{1}{2}\right) P^{m}_{\nu-1/2}(z) \pm m! \sum_{k=0}^{m-1} \frac{(-1)^{k-m}(z^2-1)^{(k-m)/2}}{2^{k-m+1} k!(m-k)} P^{k}_{\nu+k-m-1/2}(z).$$

For $\mu = 0$, there is no contribution from the sum and the result is

$$\left[\frac{\partial}{\partial \mu} P^{\mu}_{\nu-1/2}(z) \right]_{\mu=0} = Q_{\nu-1/2}(z) + \psi\left(\nu + \frac{1}{2}\right) P_{\nu-1/2}(z),$$

which agrees with that given in [10, §4.4.3]. We are now able to obtain formulas for non-zero values of μ such as for $\mu = -1$

$$\left(\nu^2 - \frac{1}{4}\right) \left[\frac{\partial}{\partial \mu} P^{\mu}_{\nu-1/2}(z) \right]_{\mu=-1} = Q^{1}_{\nu-1/2}(z) + \psi\left(\nu + \frac{3}{2}\right) P^{1}_{\nu-1/2}(z) + (z^2-1)^{-1/2} P_{\nu-3/2}(z),$$

or for $\mu = +1$

$$\left[\frac{\partial}{\partial \mu} P^{\mu}_{\nu-1/2}(z) \right]_{\mu=1} = Q^{1}_{\nu-1/2}(z) + \psi\left(\nu - \frac{1}{2}\right) P^{1}_{\nu-1/2}(z) - (z^2-1)^{-1/2} P_{\nu-3/2}(z),$$

or for other integer values of μ.

Notes

It has recently come to our attention that the argument domain of applicability of the formulas for the derivatives of the Legendre functions that we have presented in this paper are actually valid in the complex domain $z \in \mathbb{C} \backslash [-1, 1] \supset \{z : |z| > 1, z \in \mathbb{C}\}$.

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