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Fourier and Gegenbauer expansions for a fundamental solution of the Laplacian in the hyperboloid model of hyperbolic geometry

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Abstract
Due to the isotropy of d-dimensional hyperbolic space, there exists a spherically symmetric fundamental solution for its corresponding Laplace–Beltrami operator. The R-radius hyperboloid model of hyperbolic geometry with $R > 0$ represents a Riemannian manifold with negative-constant sectional curvature. We obtain a spherically symmetric fundamental solution of Laplace's equation on this manifold in terms of its geodesic radius. We give several matching expressions for this fundamental solution including a definite integral over reciprocal powers of the hyperbolic sine, finite summation expressions over hyperbolic functions, Gauss hypergeometric functions and in terms of the associated Legendre function of the second kind with order and degree given by $d/2 - 1$ with real argument greater than unity. We also demonstrate uniqueness for a fundamental solution of Laplace's equation on this manifold in terms of a vanishing decay at infinity. In rotationally invariant coordinate systems, we compute the azimuthal Fourier coefficients for a fundamental solution of Laplace's equation on the R-radius hyperboloid. For $d \geqslant 2$, we compute the Gegenbauer polynomial expansion in geodesic polar coordinates for a fundamental solution of Laplace's equation on this negative-constant curvature Riemannian manifold. In three dimensions, an addition theorem for the azimuthal Fourier coefficients of a fundamental solution for Laplace's equation is obtained through comparison with its corresponding Gegenbauer expansion.

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1. Introduction

In this paper, we discuss eigenfunction expansions for a fundamental solution of Laplace's equation in the hyperboloid model of d-dimensional hyperbolic geometry. In particular, for a fixed $R \in (0, \infty)$ and $d \geqslant 2$, we derive and discuss Fourier cosine and Gegenbauer polynomial expansions in rotationally invariant coordinate systems, for a spherically symmetric Green's function (fundamental solution) of the Laplacian (Laplace–Beltrami operator) on a Riemannian manifold of negative-constant sectional curvature, namely the hyperboloid model of hyperbolic geometry. Useful background material relevant for this paper can be found in [42, 40, 28, 37].

This paper is organized as follows. In section 2, for the hyperboloid model of d-dimensional hyperbolic geometry, we describe some of its global properties, such as its geodesic distance function, geodesic polar coordinates, and Laplacian operator. In section 3, for the hyperboloid model we show how to compute radial harmonics in a geodesic polar coordinate system and derive several alternative expressions for a radial fundamental solution of the Laplacian on the d-dimensional R-radius hyperboloid with $R > 0$, and that our derived fundamental solution is unique in terms of a vanishing decay at infinity. In section 4, for $d \geqslant 2$, we derive and discuss Fourier cosine series for a fundamental solution of Laplace's equation on the hyperboloid about an appropriate azimuthal angle in rotationally invariant coordinate systems, and show how the resulting Fourier coefficients compare to those in Euclidean space. In section 5, for $d \geqslant 2$, we compute Gegenbauer polynomial expansions in geodesic polar coordinates and an addition theorem for the azimuthal Fourier coefficients in three dimensions for a fundamental solution of Laplace's equation on the hyperboloid. In section 6, we discuss possible directions of research in this area.

Throughout this paper, we rely on the following definitions. For $a_1, a_2, a_3, \ldots \in \mathbf{C}$, if $i, j \in \mathbf{Z}$ and $j < i$ then $\sum_{n=i}^{j} a_n = 0$ and $\prod_{n=i}^{j} a_n = 1$. The set of natural numbers is given by $\mathbf{N} := \{1, 2, 3, \ldots\}$, the set $\mathbf{N}_0 := \{0, 1, 2, \ldots\} = \mathbf{N} \cup \{0\}$, and the set $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$. The set $\mathbf{R}$ represents the real numbers and the set $\mathbf{C}$ represents the complex numbers.

2. Global analysis on the hyperboloid

2.1. *The hyperboloid model of hyperbolic geometry*

Hyperbolic space in d-dimensions is a fundamental example of a space exhibiting hyperbolic geometry. It was developed independently by Lobachevsky and Bolyai around 1830 (see [41]), and most likely by Gauss and Schweikart (although they never published this result), even earlier (see chapter 6 in [30]). It is a geometry analogous to Euclidean geometry, but such that Euclid's parallel postulate is no longer assumed to hold.

There are several models of d-dimensional hyperbolic space including the Klein (see figure 1), Poincaré (see figure 2), hyperboloid, upper-half space and hemisphere models (see [40]). The hyperboloid model for d-dimensional hyperbolic geometry is closely related to the Klein and Poincaré models: each can be obtained projectively from the others. The upper-half space and hemisphere models can be obtained from one another by inversions with the Poincaré model (see section 2.2 in [40]). The model we will be focusing on in this paper is the hyperboloid model.

The hyperboloid model, also known as the Minkowski or Lorentz models, is a model of d-dimensional hyperbolic geometry in which points are represented by the upper sheet

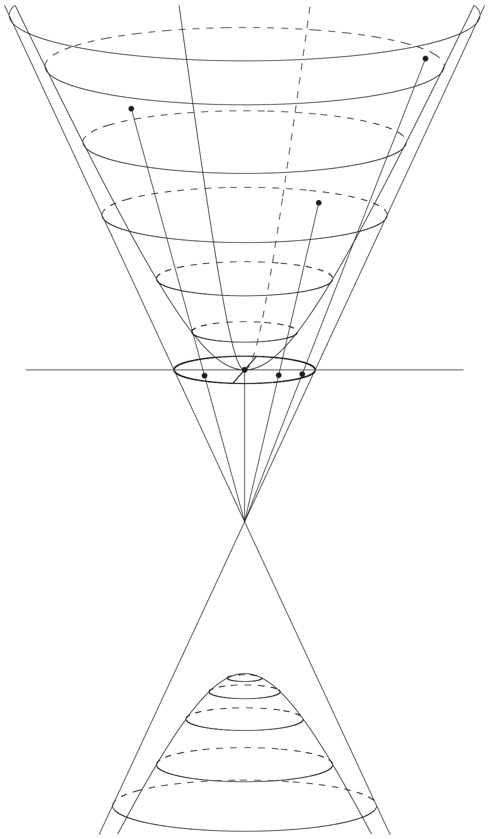

Figure 1. This figure is a graphical depiction of stereographic projection from the hyperboloid model to the Klein model of hyperbolic space.

(submanifold) S^+ of a two-sheeted hyperboloid embedded in the Minkowski space $\mathbf{R}^{d,1}$. The Minkowski space is a $(d+1)$-dimensional pseudo-Riemannian manifold which is a real finite-dimensional vector space, with coordinates given by $\mathbf{x} = (x_0, x_1, \ldots, x_d)$. It is equipped with a nondegenerate, symmetric bilinear form, the Minkowski bilinear form $[\cdot, \cdot] : \mathbf{R}^{d,1} \times \mathbf{R}^{d,1} \to \mathbf{R}$ defined such that

$$[\mathbf{x}, \mathbf{y}] := x_0y_0 - x_1y_1 - \cdots - x_dy_d.$$

The above bilinear form is symmetric, but not positive-definite, so it is not an inner product. It is defined analogously with the Euclidean inner product $(\cdot, \cdot) : \mathbf{R}^{d+1} \times \mathbf{R}^{d+1} \to \mathbf{R}$ defined such that

$$(\mathbf{x}, \mathbf{y}) := x_0y_0 + x_1y_1 + \cdots + x_dy_d.$$

The variety $[\mathbf{x}, \mathbf{x}] = x_0^2 - x_1^2 - \cdots - x_d^2 = R^2$ for $\mathbf{x} \in \mathbf{R}^{d,1}$, using the language of Beltrami [3] (see also p 504 in [42]), defines a pseudo-sphere of radius R. Points on the pseudo-sphere with zero radius coincide with the cone. Points on the pseudo-sphere with radius greater than zero lie within this cone, and points on the pseudo-sphere with purely imaginary radius lie outside the cone. The upper sheets of the positive radii pseudo-spheres are maximally symmetric, simply connected, negative-constant sectional curvature (given by $-1/R^2$; see for

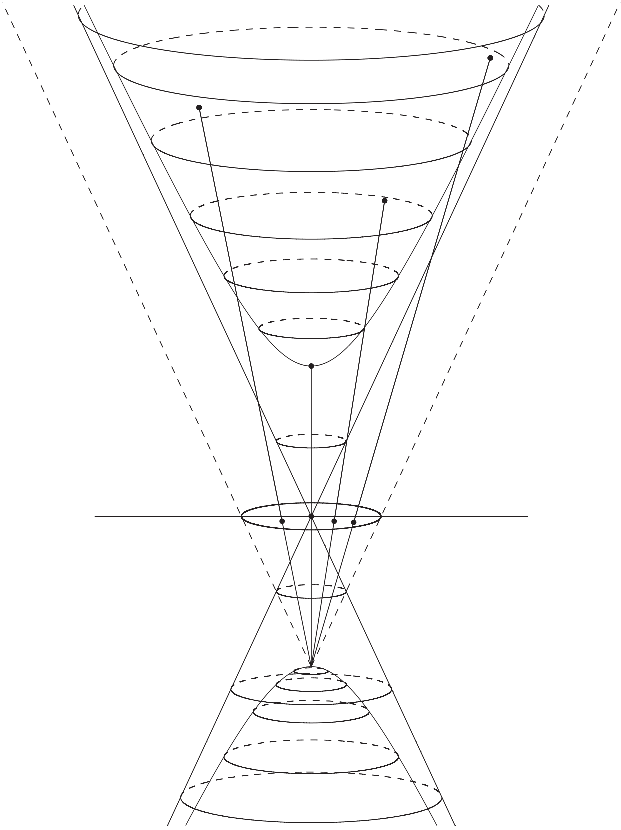

Figure 2. This figure is a graphical depiction of stereographic projection from the hyperboloid model to the Poincaré model of hyperbolic space.

instance p 148 in [28]), d-dimensional Riemannian submanifolds, embedded and with induced metric from the ambient Minkowski space $\mathbf{R}^{d,1}$. For $R \in (0, \infty)$, we refer to the upper sheet of this variety $[\mathbf{x}, \mathbf{x}] = R^2$, with $\mathbf{x} \in \mathbf{R}^{d,1}$, as the R-radius hyperboloid $\mathbf{H}_R^d$. Similarly, we refer to the variety $(\mathbf{x}, \mathbf{x}) = R^2$ for $R > 0$ and $\mathbf{x} \in \mathbf{R}^{d+1}$, as the R-radius hypersphere $\mathbf{S}_R^d$ which is a maximally symmetric, simply connected, positive-constant sectional curvature (given by $1/R^2$) d-dimensional Riemannian submanifold, embedded and with induced metric from the ambient Euclidean space. The Euclidean space $\mathbf{R}^d$ equipped with the Pythagorean norm is a space with zero sectional curvature. We denote the unit radius hyperboloid by $\mathbf{H}^d := \mathbf{H}_1^d$ and the unit radius hypersphere by $\mathbf{S}^d := \mathbf{S}_1^d$.

In our discussion of a fundamental solution for the Laplacian in the hyperboloid model of hyperbolic geometry, we focus on the positive radius pseudo-sphere which can be parametrized through *subgroup-type coordinates,* i.e. those which correspond to a maximal subgroup chain $O(d, 1) \supset \cdots$ (see, for instance, [37]). There exist separable coordinate systems which parametrize points on positive radius pseudo-spheres which cannot be constructed using maximal subgroup chains, e.g. such as those which are analogous to parabolic coordinates, etc. We will no longer discuss these.

Geodesic polar coordinates are coordinates which correspond to the maximal subgroup chain given by $O(d, 1) \supset O(d) \supset \cdots$. What we will refer to as *standard geodesic polar* *coordinates* correspond to the subgroup chain given by $O(d,1) \supset O(d) \supset O(d-1) \supset \cdots \supset O(2)$. Standard geodesic polar coordinates (see [34, 22]), similar to standard hyperspherical coordinates in Euclidean space, can be given by

$$\left.\begin{array}{l} x_0 = R\cosh r \\ x_1 = R\sinh r\cos\theta_1 \\ x_2 = R\sinh r\sin\theta_1\cos\theta_2 \\ \vdots \\ x_{d-2} = R\sinh r\sin\theta_1\cdots\cos\theta_{d-2} \\ x_{d-1} = R\sinh r\sin\theta_1\cdots\sin\theta_{d-2}\cos\phi \\ x_d = R\sinh r\sin\theta_1\cdots\sin\theta_{d-2}\sin\phi \end{array}\right\}, \tag{1}$$

where $r \in [0,\infty)$, $\phi \in [0, 2\pi)$ and $\theta_i \in [0,\pi]$ for $1 \leqslant i \leqslant d-2$.

The isometry group of the space $\mathbf{H}_R^d$ is the pseudo-orthogonal group $SO(d,1)$, the Lorentz group in $(d+1)$-dimensions. Hyperbolic space $\mathbf{H}_R^d$ can be identified with the quotient space $SO(d,1)/SO(d)$. The isometry group acts transitively on $\mathbf{H}_R^d$. That is, any point on the hyperboloid can be carried, with the help of a Euclidean rotation of $SO(d-1)$, to the point $(\cosh\alpha, \sinh\alpha, 0,\ldots,0)$, and a hyperbolic rotation

$$\left.\begin{array}{l} x'_0 = -x_1\sinh\alpha + x_0\cosh\alpha \\ x'_1 = -x_1\cosh\alpha - x_0\sinh\alpha \end{array}\right\}$$

maps that point to the origin $(1,0,\ldots,0)$ of the space.

In order to perform analysis on a fundamental solution of Laplace's equation on the hyperboloid, we need to describe how one computes distances in this space. One may naturally compare distances on the positive radius pseudo-sphere through analogy with the R-radius hypersphere. Distances on the hypersphere are simply given by arc lengths, angles between two arbitrary vectors, from the origin, in the ambient Euclidean space. We consider the d-dimensional hypersphere embedded in $\mathbf{R}^{d+1}$. Points on the hypersphere can be parametrized using hyperspherical coordinate systems. Any parametrization of the hypersphere $\mathbf{S}_R^d$ must have $(\mathbf{x},\mathbf{x}) = x_0^2 + \cdots + x_d^2 = R^2$ with $R > 0$. The geodesic distance between two points on the hypersphere $s : \mathbf{S}_R^d \times \mathbf{S}_R^d \to [0,\infty)$ is given by

$$s(\mathbf{x},\mathbf{x}') := R\gamma = R\cos^{-1}\left(\frac{(\mathbf{x},\mathbf{x}')}{(\mathbf{x},\mathbf{x})(\mathbf{x}',\mathbf{x}')}\right) = R\cos^{-1}\left(\frac{1}{R^2}(\mathbf{x},\mathbf{x}')\right). \tag{2}$$

This is evident from the fact that the geodesics on $\mathbf{S}_R^d$ are great circles (i.e. intersections of $\mathbf{S}_R^d$ with planes through the origin) with constant speed parametrizations (see p 82 in [28]).

Accordingly, we now look at the geodesic distance function on the d-dimensional positive radius pseudo-sphere $\mathbf{H}_R^d$. Distances between two points on the positive radius pseudo-sphere are given by the hyperangle between two arbitrary vectors, from the origin, in the ambient Minkowski space. Any parametrization of the hyperboloid $\mathbf{H}_R^d$ must have $[\mathbf{x},\mathbf{x}] = R^2$. The geodesic distance $d : \mathbf{H}_R^d \times \mathbf{H}_R^d \to [0,\infty)$ between any two points on the hyperboloid is given by

$$d(\mathbf{x},\mathbf{x}') := R\cosh^{-1}\left(\frac{[\mathbf{x},\mathbf{x}']}{[\mathbf{x},\mathbf{x}][\mathbf{x}',\mathbf{x}']}\right) = R\cosh^{-1}\left(\frac{1}{R^2}[\mathbf{x},\mathbf{x}']\right), \tag{3}$$

where the inverse hyperbolic cosine with argument $x \in (1,\infty)$ is given by (see (4.37.19) in [36])

$$\cosh^{-1} x = \log(x + \sqrt{x^2-1}).$$

Geodesics on $\mathbf{H}_R^d$ are great hyperbolas (i.e. intersections of $\mathbf{H}_R^d$ with planes through the origin) with constant speed parametrizations (see p 84 in [28]). We also define a global function

$\rho : \mathbf{H}^d \times \mathbf{H}^d \to [0, \infty)$ which represents the projection of the global geodesic distance function (3) on $\mathbf{H}^d_R$ onto the corresponding unit radius hyperboloid $\mathbf{H}^d$, namely

$$\rho(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') := d(\mathbf{x}, \mathbf{x}')/R, \tag{4}$$

where $\widehat{\mathbf{x}} = \mathbf{x}/R$ and $\widehat{\mathbf{x}}' = \mathbf{x}'/R$. Note that when we refer to $d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')$ below, we specifically mean that projected distance given by (4).

2.2. Laplace's equation and harmonics on the hyperboloid

Parametrizations of a submanifold embedded in either a Euclidean or Minkowski space are given in terms of coordinate systems whose coordinates are curvilinear. These are coordinates based on some transformation that converts the standard Cartesian coordinates in the ambient space to a coordinate system with the same number of coordinates as the dimension of the submanifold in which the coordinate lines are curved.

On a d-dimensional Riemannian manifold M (a manifold together with a Riemannian metric g), the Laplace–Beltrami operator (Laplacian) $\Delta : C^p(M) \to C^{p-2}(M)$, $p \geqslant 2$, in curvilinear coordinates $\xi = (\xi^1, \ldots, \xi^d)$ is given by

$$\Delta = \sum_{i,j=1}^{d} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial \xi^j} \right), \tag{5}$$

where $|g| = |\det(g_{ij})|$, the Riemannian structure is given by

$$\mathrm{d}s^2 = \sum_{i,j=1}^{d} g_{ij}\, \mathrm{d}\xi^i\, \mathrm{d}\xi^j \tag{6}$$

and

$$\sum_{i=1}^{d} g_{ki} g^{ij} = \delta_k^j,$$

where $\delta_i^j \in \{0, 1\}$ is the Kronecker delta defined for all $i, j \in \mathbf{Z}$ such that

$$\delta_i^j := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{7}$$

For a Riemannian submanifold, the relation between the metric tensor in the ambient space and g_{ij} of (5) and (6) is

$$g_{ij}(\xi) = \sum_{k,l=0}^{d} G_{kl} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^l}{\partial \xi^j}.$$

On $\mathbf{H}^d_R$, the ambient space is Minkowski, and therefore $G_{kl} = \mathrm{diag}(1, -1, \ldots, -1)$.

The set of all geodesic polar coordinate systems on the hyperboloid correspond to the many ways one can put coordinates on a hyperbolic hypersphere, i.e. the Riemannian submanifold $U \subset \mathbf{H}^d_R$ defined for a fixed $\mathbf{x}' \in \mathbf{H}^d_R$ such that $d(\mathbf{x}, \mathbf{x}') = b = const$, where $b \in (0, \infty)$. These are coordinate systems which correspond to maximal subgroup chains starting with $O(d, 1) \supset O(d) \supset \cdots$, with standard geodesic polar coordinates given by (1) being only one of them. (For a thorough description of these, see section X.5 in Vilenkin (1968) [42].) They all share the property that they are described by d-variables: $r \in [0, \infty)$ plus $(d - 1)$-angles each being given by the values $[0, 2\pi)$, $[0, \pi]$, $[-\pi/2, \pi/2]$ or $[0, \pi/2]$ (see [25, 26]).

In any of the geodesic polar coordinate systems, the global geodesic distance between any two points on the hyperboloid is given by (cf (3))

$$d(\mathbf{x}, \mathbf{x}') = R \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \cos \gamma), \tag{8}$$

where γ is the unique separation angle given in each hyperspherical coordinate system. For instance, the separation angle in standard geodesic polar coordinates (1) is given by the formula

$$\cos\gamma = \cos(\phi - \phi')\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i' + \sum_{i=1}^{d-2}\cos\theta_i\cos\theta_i'\prod_{j=1}^{i-1}\sin\theta_j\sin\theta_j'. \tag{9}$$

Corresponding separation angle formulas for any geodesic polar coordinate system can be computed using (2), (3), and the associated formulas for the appropriate inner-products. Note that by making use of the isometry group $SO(d, 1)$ to map $\mathbf{x}'$ to the origin, then $\rho = Rr$ for $\mathbf{H}_R^d$ and in particular $\rho = r$ for $\mathbf{H}^d$. Hence, for the unit radius hyperboloid, there is no distinction between the global geodesic distance and the radial parameter in a geodesic polar coordinate system.

The Riemannian structure in a geodesic polar coordinate system on this submanifold is given by

$$ds^2 = R^2(dr^2 + \sinh^2 r\, d\gamma^2), \tag{10}$$

where an appropriate expression for γ in a curvilinear coordinate system is given. If one combines (1), (5), (9) and (10), then in a particular geodesic polar coordinate system, Laplace's equation on $\mathbf{H}_R^d$ is given by

$$\Delta f = \frac{1}{R^2}\left[\frac{\partial^2 f}{\partial r^2} + (d-1)\coth r\frac{\partial f}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbf{S}^{d-1}} f\right] = 0, \tag{11}$$

where $\Delta_{\mathbf{S}^{d-1}}$ is the corresponding Laplace–Beltrami operator on the unit radius hypersphere $\mathbf{S}^{d-1}$.

3. Green's function in the hyperboloid model

3.1. Harmonics in geodesic polar coordinates

Geodesic polar coordinate systems partition $\mathbf{H}_R^d$ into a family of $(d-1)$-dimensional hyperbolic hyperspheres, each with a geodesic radius Rr with $r \in (0, \infty)$ on which all possible hyperspherical coordinate systems for $\mathbf{S}^{d-1}$ may be used (see, for instance, [42]). One then must also consider the limiting case for $r = 0$ to fill out all of $\mathbf{H}_R^d$. In subgroup-type coordinate systems, one can compute the normalized hyperspherical harmonics in that space by solving the Laplace equation using separation of variables. This results in a general procedure which is given explicitly in [25, 26]. These angular harmonics are given as general expressions involving trigonometric functions, Gegenbauer polynomials and Jacobi polynomials.

The harmonics in geodesic polar coordinate systems are given in terms of a radial solution multiplied by the angular harmonics. The angular harmonics are eigenfunctions of the Laplace–Beltrami operator on $\mathbf{S}^{d-1}$ which satisfy the following eigenvalue problem:

$$\Delta_{\mathbf{S}^{d-1}} Y_l^K(\widehat{\mathbf{x}}) = -l(l+d-2)Y_l^K(\widehat{\mathbf{x}}), \tag{12}$$

where $\widehat{\mathbf{x}} \in \mathbf{S}^{d-1}$, $Y_l^K(\widehat{\mathbf{x}})$ are normalized hyperspherical harmonics, $l \in \mathbf{N}_0$ is the angular momentum quantum number, and K stands for the set of $(d-2)$-quantum numbers identifying degenerate harmonics for each l. The degeneracy

$$(2l+d-2)\frac{(d-3+l)!}{l!(d-2)!}$$

(see (9.2.11) in [42]) tells us how many linearly independent solutions exist for a particular l value and dimension d. The hyperspherical harmonics are normalized such that

$$\int_{\mathbf{S}^{d-1}} Y_l^K(\widehat{\mathbf{x}})\overline{Y_{l'}^{K'}(\widehat{\mathbf{x}})}d\omega = \delta_l^{l'}\delta_K^{K'},$$

where $\mathrm{d}\omega$ is the Riemannian (volume) measure (see, for instance, section 3.4 in [21]) on $\mathbf{S}^{d-1}$ which is invariant under the isometry group $SO(d)$ (cf (19)), and for $x + \mathrm{i}y = z \in \mathbf{C}$, $\bar{z} = x - \mathrm{i}y$, represents complex conjugation. The generalized Kronecker delta $\delta_K^{K'} \in \{0, 1\}$ (cf (7)) is defined such that it equals 1 if all of the $(d-2)$-quantum numbers identifying degenerate harmonics for each l coincide, and equals zero otherwise.

Since the angular solutions (hyperspherical harmonics) are well known (see, for instance, chapter IX in [42] and chapter 11 in [15]), we will now focus on the radial solutions on $\mathbf{H}_R^d$ in geodesic polar coordinates, which satisfy the following ordinary differential equation (cf (11)) for all $R \in (0, \infty)$, namely

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + (d-1)\coth r \frac{\mathrm{d}u}{\mathrm{d}r} - \frac{l(l+d-2)}{\sinh^2 r} u = 0.$$

Four solutions to this ordinary differential equation $u_{1\pm}^{d,l}, u_{2\pm}^{d,l} : (1, \infty) \to \mathbf{C}$ are given by

$$u_{1\pm}^{d,l}(\cosh r) = \frac{1}{\sinh^{d/2-1} r} P_{d/2-1}^{\pm(d/2-1+l)}(\cosh r)$$

and

$$u_{2\pm}^{d,l}(\cosh r) = \frac{1}{\sinh^{d/2-1} r} Q_{d/2-1}^{\pm(d/2-1+l)}(\cosh r),$$

where $P_\nu^\mu, Q_\nu^\mu : \mathbf{C} \setminus (\infty, 1] \to \mathbf{C}$ are associated Legendre functions of the first and second kind, respectively (see, for instance, chapter 14 in [36]), namely ((8.1.2) in [1])

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1}\right]^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right), \tag{13}$$

where $|1-z| < 2$ and ((8.1.3) in [1])

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi\mu}\Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})z^{\nu+\mu+1}} {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right), \tag{14}$$

where $|z| > 1$. The Gauss hypergeometric function ${}_2F_1 : \mathbf{C}^2 \times (\mathbf{C} \setminus -\mathbf{N}_0) \times \{z \in \mathbf{C} : |z| < 1\} \to \mathbf{C}$ can be defined in terms of the following infinite series:

$${}_2F_1(a, b; c; z) := \sum_{n=0}^\infty \frac{(a)_n(b)_n}{n!(c)_n} z^n \tag{15}$$

(see (2.1.5) in [2]), and $(\cdot)_n : \mathbf{C} \to \mathbf{C}$ is the Pochhammer symbol (rising factorial) defined by

$$(z)_n := \prod_{i=1}^n (z + \mathrm{i} - 1),$$

where $n \in \mathbf{N}_0$. Note that

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$$

for all $z \in \mathbf{C} \setminus -\mathbf{N}_0$. The gamma function $\Gamma : \mathbf{C} \setminus -\mathbf{N}_0 \to \mathbf{C}$ (see chapter 5 in [36]), which is ubiquitous in special function theory, and satisfies the following recurrence formula

$$\Gamma(z+1) = z\Gamma(z), \tag{16}$$

is an important combinatoric function which generalizes the factorial function over the natural numbers (i.e. $\Gamma(n+1) = n!$ for $n \in \mathbf{N}_0$). It is naturally defined over the right-half complex plane through Euler's integral (see (5.2.1) in [36])

$$\Gamma(z) := \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t,$$

where $\operatorname{Re} z > 0$. An important formula which the gamma function satisfies is the duplication formula (i.e. (5.5.5) in [36])

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{17}$$

provided $2z \notin -\mathbf{N}_0$.

3.2. A fundamental solution of Laplace's equation on the hyperboloid[4]

Due to the fact that the space $\mathbf{H}_R^d$ is homogeneous with respect to its isometry group, the pseudo-orthogonal group $SO(d,1)$ and therefore an isotropic manifold, we expect that there exists a fundamental solution of Laplace's equation on this space with spherically symmetric (pure radial and constant angular) dependence. We specifically expect these solutions be given in terms of the associated Legendre function of the second kind with an argument given by $\cosh r$. This associated Legendre function naturally fits our requirements because it is singular at $r = 0$ and vanishes at infinity, whereas the associated Legendre function of the first kind, with the same argument, is regular at $r = 0$ and singular at infinity.

In computing a fundamental solution of the Laplacian on $\mathbf{H}_R^d$, we know that

$$-\Delta\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = \delta_g(\mathbf{x},\mathbf{x}'), \tag{18}$$

where g is the Riemannian metric on $\mathbf{H}_R^d$ and $\delta_g(\mathbf{x},\mathbf{x}')$ is the Dirac delta function on the manifold $\mathbf{H}_R^d$. The Dirac delta function is defined for an open set $U \subset \mathbf{H}_R^d$ with $\mathbf{x},\mathbf{x}' \in \mathbf{H}_R^d$ such that

$$\int_U \delta_g(\mathbf{x},\mathbf{x}')\mathrm{dvol}_g = \begin{cases} 1 & \text{if } \mathbf{x}' \in U, \\ 0 & \text{if } \mathbf{x}' \notin U, \end{cases}$$

where dvol_g is the Riemannian (volume) measure, invariant under the isometry group $SO(d,1)$ of the Riemannian manifold $\mathbf{H}_R^d$, given in standard geodesic polar coordinates by

$$\mathrm{dvol}_g = R^d \sinh^{d-1} r\,\mathrm{d}r\,\mathrm{d}\omega := R^d \sinh^{d-1} r\,\mathrm{d}r\,\sin^{d-2}\theta_{d-1}\cdots\sin\theta_2\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{d-1}. \tag{19}$$

Note that as $r \to 0^+$, dvol_g goes to the Euclidean measure, invariant under the Euclidean motion group $E(d)$, in standard hyperspherical coordinates. Therefore, in standard hyperspherical coordinates, we have the following:

$$\delta_g(\mathbf{x},\mathbf{x}') = \frac{\delta(r-r')}{R^d \sinh^{d-1} r'}\,\frac{\delta(\theta_1-\theta_1')\cdots\delta(\theta_{d-1}-\theta_{d-1}')}{\sin\theta_2'\cdots\sin^{d-2}\theta_{d-1}'}. \tag{20}$$

In general, since we can add any harmonic function to a fundamental solution of the Laplacian and still have a fundamental solution, we will use this freedom to make our fundamental solution as simple as possible. It is reasonable to expect that there exists a particular spherically symmetric fundamental solution $\mathcal{H}_R^d(\mathbf{x},\mathbf{x}')$ on the hyperboloid with pure radial $\rho(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') = d(\mathbf{x},\mathbf{x}')/R$ (cf (4)) and constant angular dependence (invariant under rotations centered about the origin), due to the influence of the point-like nature of the Dirac delta function. For a spherically symmetric solution to the Laplace equation, the corresponding $\Delta_{\mathbf{S}^{d-1}}$ term vanishes since only the $l = 0$ term survives. In other words, we expect that there exists a fundamental solution of Laplace's equation such that, aside from a multiplicative constant which depends on R and d, $\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = f(\rho)$.

We have proven that on the R-radius hyperboloid $\mathbf{H}_R^d$, a Green's function for the Laplace operator (fundamental solution for the Laplacian) can be given as follows.

[4] An interesting history for this problem in low dimensions can be found in [39].

Theorem 3.1. *Let* $d \in \{2, 3, \ldots\}$. *Define* $\mathcal{I}_d : (0, \infty) \to \mathbf{R}$ *as*

$$\mathcal{I}_d(\rho) := \int_\rho^\infty \frac{\mathrm{d}x}{\sinh^{d-1} x},$$

$\mathbf{x}, \mathbf{x}' \in \mathbf{H}_R^d$ *and* $\mathcal{H}_R^d : (\mathbf{H}_R^d \times \mathbf{H}_R^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{H}_R^d\} \to \mathbf{R}$ *defined such that*

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') := \frac{\Gamma(d/2)}{2\pi^{d/2} R^{d-2}} \mathcal{I}_d(\rho),$$

where $\rho := \cosh^{-1}\left([\widehat{\mathbf{x}}, \widehat{\mathbf{x}}']\right)$ *is the geodesic distance between* $\widehat{\mathbf{x}}$ *and* $\widehat{\mathbf{x}}'$ *on the pseudo-sphere of unit radius* $\mathbf{H}^d$, *with* $\widehat{\mathbf{x}} = \mathbf{x}/R$, $\widehat{\mathbf{x}}' = \mathbf{x}'/R$; *then,* $\mathcal{H}_R^d$ *is a fundamental solution for* $-\Delta$ *where* Δ *is the Laplace–Beltrami operator on* $\mathbf{H}_R^d$. *Moreover,*

$$\mathcal{I}_d(\rho) = \begin{cases} (-1)^{d/2-1} \dfrac{(d-3)!!}{(d-2)!!} \left[\log\coth\dfrac{\rho}{2} + \cosh\rho \displaystyle\sum_{k=1}^{d/2-1} \frac{(2k-2)!!(-1)^k}{(2k-1)!! \sinh^{2k}\rho} \right] & \text{if } d \text{ even}, \\ \left\{ \begin{array}{l} (-1)^{(d-1)/2} \left[\dfrac{(d-3)!!}{(d-2)!!} \right. \\ \qquad \left. + \left(\dfrac{d-3}{2}\right)! \displaystyle\sum_{k=1}^{(d-1)/2} \frac{(-1)^k \coth^{2k-1}\rho}{(2k-1)(k-1)!((d-2k-1)/2)!} \right] \\ \text{or} \\ (-1)^{(d-1)/2} \dfrac{(d-3)!!}{(d-2)!!} \left[1 + \cosh\rho \displaystyle\sum_{k=1}^{(d-1)/2} \frac{(2k-3)!!(-1)^k}{(2k-2)!! \sinh^{2k-1}\rho} \right] \end{array} \right\} & \text{if } d \text{ odd}, \end{cases}$$

$$= \frac{1}{(d-1)\cosh^{d-1}\rho}\, {}_2F_1\left(\frac{d-1}{2}, \frac{d}{2}; \frac{d+1}{2}; \frac{1}{\cosh^2\rho}\right),$$

$$= \frac{1}{(d-1)\cosh\rho \sinh^{d-2}\rho}\, {}_2F_1\left(\frac{1}{2}, 1; \frac{d+1}{2}; \frac{1}{\cosh^2\rho}\right),$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\pi(d/2-1)}}{2^{d/2-1}\Gamma(d/2)\sinh^{d/2-1}\rho} Q_{d/2-1}^{d/2-1}(\cosh\rho),$$

where !! *is the double factorial,* ${}_2F_1$ *is the Gauss hypergeometric function and* Q_ν^μ *is the associated Legendre function of the second kind.*

In the rest of this section, we develop the material in order to prove this theorem.

Due to the fact that the space $\mathbf{H}_R^d$ is homogeneous with respect to its isometry group $SO(d, 1)$, and therefore an isotropic manifold, without loss of generality, we are free to map the point $\mathbf{x}' \in \mathbf{H}_R^d$ to the origin. In this case, the global distance function ρ coincides with the radial parameter r in geodesic polar coordinates, and we may interchange r with ρ accordingly (cf (8) with $r' = 0$) in our representation of a fundamental solution for Laplace's equation on this manifold. Since a spherically symmetric choice for a fundamental solution of Laplace's equation is harmonic everywhere except at the origin, we may first set $g = f'$ in (11) and solve the first-order equation

$$g' + (d-1)\coth\rho\, g = 0,$$

which is integrable and clearly has the general solution

$$g(\rho) = \frac{\mathrm{d}f}{\mathrm{d}\rho} = c_0 \sinh^{1-d}\rho, \tag{21}$$

where $c_0 \in \mathbf{R}$ is a constant. Now we integrate (21) to obtain a fundamental solution for the Laplacian on $\mathbf{H}_R^d$

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') = c_0 \mathcal{I}_d(\rho) + c_1, \tag{22}$$

where

$$\mathcal{I}_d(\rho) := \int_\rho^\infty \frac{\mathrm{d}x}{\sinh^{d-1} x}, \tag{23}$$

and $c_0, c_1 \in \mathbf{R}$ are constants which depend on d and R. This definite integral result is mentioned in section II.5 of [23] as well as in [31] (see also example 4 in [13]). Note that we can add any harmonic function to (22) and still have a fundamental solution of the Laplacian since a fundamental solution of the Laplacian must satisfy

$$\int_{\mathbf{H}_R^d} (-\Delta\varphi)(\mathbf{x}')\mathcal{H}_R^d(\mathbf{x},\mathbf{x}')\,\mathrm{dvol}_g' = \varphi(\mathbf{x})$$

for all $\varphi \in \mathcal{D}(\mathbf{H}_R^d)$, where $\mathcal{D}$ is the space of test functions, and dvol_g' is the Riemannian (volume) measure on $\mathbf{H}_R^d$ in the primed coordinates. In particular, we note from our definition of $\mathcal{I}_d$ (23) that

$$\lim_{\rho\to\infty} \mathcal{I}_d(\rho) = 0.$$

Therefore, it is convenient to set $c_1 = 0$ leaving us with

$$\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = c_0\mathcal{I}_d(\rho). \tag{24}$$

On Euclidean space $\mathbf{R}^d$, a Green's function for Laplace's equation (fundamental solution for the Laplacian) is well known and is given in the following theorem (see for instance [16] p 94; [18] p 17; [4] p 211).

Theorem 3.2. *Let $d \in \mathbf{N}$. Define*

$$\mathcal{G}^d(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}\|\mathbf{x}-\mathbf{x}'\|^{2-d} & \text{if } d=1 \text{ or } d\geqslant 3,\\[2mm] \dfrac{1}{2\pi}\log\|\mathbf{x}-\mathbf{x}'\|^{-1} & \text{if } d=2; \end{cases}$$

then, $\mathcal{G}^d$ is a fundamental solution for $-\Delta$ on the Euclidean space $\mathbf{R}^d$, where Δ is the Laplace operator on $\mathbf{R}^d$.

Note most authors only present the above theorem for the case $d \geqslant 2$ but it is easily verified to also be valid for the case $d = 1$.

The hyperboloid $\mathbf{H}_R^d$, being a manifold, must behave locally like Euclidean space $\mathbf{R}^d$. Therefore, for small ρ we have $\mathrm{e}^\rho \simeq 1+\rho$ and $\mathrm{e}^{-\rho} \simeq 1-\rho$ and in that limiting regime

$$\mathcal{I}_d(\rho) \approx \int_\rho^1 \frac{\mathrm{d}x}{x^{d-1}} \simeq \begin{cases} -\log\rho & \text{if } d=2,\\[1mm] \dfrac{1}{\rho^{d-2}} & \text{if } d\geqslant 3, \end{cases}$$

which has exactly the same singularity as a Euclidean fundamental solution for Laplace's equation. Therefore, the proportionality constant c_0 is obtained by matching locally to a Euclidean fundamental solution of Laplace's equation

$$\mathcal{H}_R^d = c_0\mathcal{I}_d \simeq \mathcal{G}^d,$$

near the singularity located at $\mathbf{x} = \mathbf{x}'$.

We have shown how to compute a fundamental solution of the Laplace–Beltrami operator on the hyperboloid in terms of an improper integral (23). We would now like to express this integral in terms of well-known special functions. In low dimensions, a fundamental solution $\mathcal{I}_d$ can be straightforwardly computed using elementary methods through (23). In $d = 2$, we have

$$\mathcal{I}_2(\rho) = \int_\rho^\infty \frac{\mathrm{d}x}{\sinh x} = \frac{1}{2}\log\frac{\cosh\rho+1}{\cosh\rho-1} = \log\coth\frac{\rho}{2},$$

and in $d=3$ we have

$$\mathcal{I}_3(\rho)=\int_\rho^\infty \frac{\mathrm{d}x}{\sinh^2 x}=\frac{\mathrm{e}^{-\rho}}{\sinh\rho}=\coth\rho-1.$$

This exactly matches up to that given by (3.27) in [24]. In $d\in\{4,5,6,7\}$, we have

$$\mathcal{I}_4(\rho)=-\frac{1}{2}\log\coth\frac{\rho}{2}+\frac{\cosh\rho}{2\sinh^2\rho},$$

$$\mathcal{I}_5(\rho)=\frac{1}{3}(\coth^3\rho-1)-(\coth\rho-1),$$

$$\mathcal{I}_6(\rho)=\frac{3}{8}\log\coth\frac{\rho}{2}+\frac{\cosh\rho}{4\sinh^4\rho}-\frac{3\cosh\rho}{8\sinh^2\rho}\quad\text{and}$$

$$\mathcal{I}_7(\rho)=\frac{1}{5}(\coth^5\rho-1)-\frac{2}{3}(\coth^3\rho-1)+\coth\rho-1.$$

Now we prove several equivalent finite summation expressions for $\mathcal{I}_d(\rho)$. We wish to compute the antiderivative $\mathfrak{I}_m:(0,\infty)\to\mathbf{R}$, which is defined as

$$\mathfrak{I}_m(x):=\int\frac{\mathrm{d}x}{\sinh^m x},$$

where $m\in\mathbf{N}$. This antiderivative satisfies the following recurrence relation:

$$\mathfrak{I}_m(x)=-\frac{\cosh x}{(m-1)\sinh^{m-1}x}-\frac{(m-2)}{(m-1)}\mathfrak{I}_{m-2}(x),\tag{25}$$

which follows from the identity

$$\frac{1}{\sinh^m x}=\frac{\cosh x}{\sinh^m x}\cosh x-\frac{1}{\sinh^{m-2}x}$$

and integration by parts. The antiderivative $\mathfrak{I}_m(x)$ naturally breaks into two separate classes, namely

$$\int\frac{\mathrm{d}x}{\sinh^{2n+1}x}=(-1)^{n+1}\frac{(2n-1)!!}{(2n)!!}\left[\log\coth\frac{x}{2}+\cosh x\sum_{k=1}^{n}\frac{(2k-2)!!(-1)^k}{(2k-1)!!\sinh^{2k}x}\right]+C\tag{26}$$

and

$$\int\frac{\mathrm{d}x}{\sinh^{2n}x}=\begin{cases}(-1)^{n+1}\dfrac{(2n-2)!!}{(2n-1)!!}\cosh x\displaystyle\sum_{k=1}^{n}\frac{(2k-3)!!(-1)^k}{(2k-2)!!\sinh^{2k-1}x}+C & \text{or}\\[2ex] (-1)^{n+1}(n-1)!\displaystyle\sum_{k=1}^{n}\frac{(-1)^k\coth^{2k-1}x}{(2k-1)(k-1)!(n-k)!}+C,\end{cases}\tag{27}$$

where C is a constant and the double factorial $(\cdot)!!:\{-1,0,1,\ldots\}\to\mathbf{N}$ is defined by

$$n!!:=\begin{cases}n\cdot(n-2)\cdots 2 & \text{if } n \text{ even}\geqslant 2,\\ n\cdot(n-2)\cdots 1 & \text{if } n \text{ odd}\geqslant 1,\\ 1 & \text{if } n\in\{-1,0\}.\end{cases}$$

Note that $(2n)!!=2^n n!$ for $n\in\mathbf{N}_0$. The finite summation formulas for $\mathfrak{I}_m(x)$ all follow trivially by induction using (25) and the binomial expansion (cf (1.2.2) in [36]):

$$(1-\coth^2 x)^n=n!\sum_{k=0}^{n}\frac{(-1)^k\coth^{2k}x}{k!(n-k)!}.$$

Formulas (26) and (27) are essentially equivalent to (2.416.2–3) in [45], except (2.416.3) which is not defined for the integrand $1/\sinh x$. By applying the limits of integration from the definition of $\mathcal{I}_d(\rho)$ in (23) to (26) and (27) we obtain the following finite summation expressions for $\mathcal{I}_d(\rho)$:

$$\mathcal{I}_d(\rho) = \begin{cases} (-1)^{d/2-1}\dfrac{(d-3)!!}{(d-2)!!}\left[\log\coth\dfrac{\rho}{2} + \cosh\rho\displaystyle\sum_{k=1}^{d/2-1}\dfrac{(2k-2)!!(-1)^k}{(2k-1)!!\sinh^{2k}\rho}\right] & \text{if } d \text{ even,} \\ \left\{\begin{array}{l} (-1)^{(d-1)/2}\left[\dfrac{(d-3)!!}{(d-2)!!} \right. \\ \qquad \left. + \left(\dfrac{d-3}{2}\right)!\displaystyle\sum_{k=1}^{(d-1)/2}\dfrac{(-1)^k\coth^{2k-1}\rho}{(2k-1)(k-1)!((d-2k-1)/2)!}\right] \\ \text{or} \\ (-1)^{(d-1)/2}\dfrac{(d-3)!!}{(d-2)!!}\left[1 + \cosh\rho\displaystyle\sum_{k=1}^{(d-1)/2}\dfrac{(2k-3)!!(-1)^k}{(2k-2)!!\sinh^{2k-1}\rho}\right] \end{array}\right\} & \text{if } d \text{ odd.} \end{cases} \tag{28}$$

Moreover, the antiderivative (indefinite integral) can be given in terms of the Gauss hypergeometric function (15) as

$$\int \frac{d\rho}{\sinh^{d-1}\rho} = \frac{-1}{(d-1)\cosh^{d-1}\rho}\,{}_2F_1\left(\frac{d-1}{2},\frac{d}{2};\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right) + C, \tag{29}$$

where the constant $C \in \mathbf{R}$. The antiderivative (29) is verified as follows. By using

$$\frac{d}{dz}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z)$$

(see (15.5.1) in [36]), and the chain rule, we can show that

$$\begin{aligned}\frac{d}{d\rho}\frac{-1}{(d-1)\cosh^{d-1}\rho}\,{}_2F_1\left(\frac{d-1}{2},\frac{d}{2};\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right) &= \frac{\sinh\rho}{\cosh^d\rho}\,{}_2F_1\left(\frac{d-1}{2},\frac{d}{2};\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right) \\ &\quad + \frac{d\sinh\rho}{(d+1)\cosh^{d+2}\rho}\,{}_2F_1\left(\frac{d+1}{2},\frac{d+2}{2};\frac{d+3}{2};\frac{1}{\cosh^2\rho}\right).\end{aligned}$$

The second hypergeometric function can be simplified using Gauss' relations for contiguous hypergeometric functions, namely

$$z\,{}_2F_1(a+1,b+1;c+1;z) = \frac{c}{a-b}[{}_2F_1(a,b+1;c;z) - {}_2F_1(a+1,b;c;z)]$$

(see p 58 in [14]) and

$${}_2F_1(a,b+1;c;z) = \frac{b-a}{b}{}_2F_1(a,b;c;z) + \frac{a}{b}{}_2F_1(a+1,b;c;z)$$

(see (15.5.12) in [36]). By doing this, the term with the hypergeometric function cancels leaving only a term which is proportional to a binomial through

$${}_2F_1(a,b;b;z) = (1-z)^{-a}$$

(see (15.4.6) in [36]), which reduces to $1/\sinh^{d-1}\rho$. By applying the limits of integration from the definition of $\mathcal{I}_d(\rho)$ in (23) to (29) we obtain the following Gauss hypergeometric representation:

$$\mathcal{I}_d(\rho) = \frac{1}{(d-1)\cosh^{d-1}\rho}\,{}_2F_1\left(\frac{d-1}{2},\frac{d}{2};\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right). \tag{30}$$

Using (30), we can write another expression for $\mathcal{I}_d(\rho)$. Applying Euler's transformation

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1\,(c-a,c-b;c;z) $$

(see (2.2.7) in [2]) to (30) produces

$$ \mathcal{I}_d(\rho) = \frac{1}{(d-1)\cosh\rho\sinh^{d-2}\rho}\,{}_2F_1\left(\frac{1}{2},1;\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right). $$

Our derivation for a fundamental solution of Laplace's equation on the R-radius hyperboloid $\mathbf{H}_R^d$ in terms of the associated Legendre function of the second kind is as follows. By starting with (30) and the definition of the associated Legendre function of the second kind (14), we derive

$$ {}_2F_1\left(\frac{d-1}{2},\frac{d}{2};\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right) = \frac{2^{d/2}\Gamma\left(\frac{d+1}{2}\right)\cosh^{d-1}\rho}{\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\pi(d/2-1)}(d-2)!\sinh^{d/2-1}\rho}Q_{d/2-1}^{d/2-1}(\cosh\rho). \quad (31) $$

We have therefore verified that the harmonics computed in section 3.1, namely $u_{2+}^{d,0}$, give an alternate form of a fundamental solution for the Laplacian on the hyperboloid. Using the duplication formula for gamma functions (17), (30) and (31), we derive

$$ \mathcal{I}_d(\rho) = \frac{\mathrm{e}^{-\mathrm{i}\pi(d/2-1)}}{2^{d/2-1}\Gamma\,(d/2)\sinh^{d/2-1}\rho}Q_{d/2-1}^{d/2-1}(\cosh\rho). $$

Note that our chosen fundamental solutions of the Laplacian on the hyperboloid have the property that they tend toward zero at infinity (even for the $d=2$ case, unlike a Euclidean fundamental solution of the Laplacian). Therefore, these Green's functions are positive (see [19, 20]) and therefore $\mathbf{H}_R^d$ is not parabolic. Note that as a result of our proof, we see that the relevant associated Legendre functions of the second kind for $d\in\{2,3,4,5,6,7\}$ are (cf (28))

$$ \begin{aligned}
&Q_0(\cosh\rho) = \log\coth\frac{\rho}{2},\\
&\frac{1}{\sinh^{1/2}\rho}Q_{1/2}^{1/2}(\cosh\rho) = \mathrm{i}\sqrt{\frac{\pi}{2}}(\coth\rho-1),\\
&\frac{1}{\sinh\rho}Q_1^1(\cosh\rho) = \log\coth\frac{\rho}{2} - \frac{\cosh\rho}{\sinh^2\rho},\\
&\frac{1}{\sinh^{3/2}\rho}Q_{3/2}^{3/2}(\cosh\rho) = 3\mathrm{i}\sqrt{\frac{\pi}{2}}\left(-\frac{1}{3}\coth^3\rho+\coth\rho-\frac{2}{3}\right),\\
&\frac{1}{\sinh^2\rho}Q_2^2(\cosh\rho) = 3\log\coth\frac{\rho}{2} - 2\frac{\cosh\rho}{\sinh^4\rho} - 3\frac{\cosh\rho}{\sinh^2\rho}, \quad \text{and}\\
&\frac{1}{\sinh^{5/2}\rho}Q_{5/2}^{5/2}(\cosh\rho) = 15\mathrm{i}\sqrt{\frac{\pi}{2}}\left(\frac{1}{15}\coth^5\rho-\frac{2}{3}\coth^3\rho+\coth\rho-\frac{8}{15}\right).
\end{aligned} $$

The constant c_0 in a fundamental solution for the Laplace operator on the hyperboloid (24) is computed by locally matching up the singularity to a fundamental solution for the Laplace operator in Euclidean space, theorem 3.2. The coefficient c_0 depends on d and R. It is determined as follows. For $d\geqslant 3$ we take the asymptotic expansion for $c_0\mathcal{I}_d(\rho)$ as ρ approaches zero and match this to a fundamental solution of Laplace's equation for Euclidean space given in theorem 3.2. This yields

$$ c_0 = \frac{\Gamma\,(d/2)}{2\pi^{d/2}}. \quad (32) $$

For $d = 2$ we take the asymptotic expansion for

$$c_0\mathcal{I}_2(\rho) = c_0 \log\coth\frac{\rho}{2} \simeq c_0 \log \|\mathbf{x} - \mathbf{x}'\|^{-1},$$

where $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$, as ρ approaches zero, and match this to $\mathcal{G}^2(\mathbf{x}, \mathbf{x}') = (2\pi)^{-1} \log \|\mathbf{x} - \mathbf{x}'\|^{-1}$; therefore, $c_0 = (2\pi)^{-1}$. This exactly matches (32) for $d = 2$. The derivation that $\mathcal{I}_d(\rho)$ is a fundamental solution of the Laplace operator on the hyperboloid $\mathbf{H}^d$ and the functions for $\mathcal{I}_d(\rho)$ are computed above.

As mentioned earlier, the sectional curvature of a pseudo-sphere of radius $R > 0$ is $-1/R^2$. Hence, using results in [31], all equivalent expressions in theorem 3.1 can be used for a fundamental solution of the Laplace–Beltrami operator on the R-radius hyperboloid $\mathbf{H}_R^d$ (cf section 2.1), namely

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') := \frac{\Gamma(d/2)}{2\pi^{d/2}R^{d-2}}\mathcal{I}_d(\rho).$$

The proof of theorem 3.1 is complete.

Furthermore, due to a theorem proved in [31], all equivalent expressions for $\mathcal{I}_d(\rho)$ in theorem 3.1 represent upper bounds for a fundamental solution of the Laplace–Beltrami operator on non-compact Riemannian manifolds with negative sectional curvature not exceeding $-1/R^2$ with $R > 0$.

We would also like to mention that a similar computation for a fundamental solution of Laplace's equation on the positive-constant sectional curvature compact manifold, the R-radius hypersphere, has recently been computed in [7].

3.3. Uniqueness of fundamental solution in terms of decay at infinity

It is clear that in general a fundamental solution of Laplace's equation in the hyperboloid model of hyperbolic geometry $\mathcal{H}_R^d$ is not unique since one can add any harmonic function $h : \mathbf{H}_R^d \to \mathbf{R}$ to $\mathcal{H}_R^d$ and still obtain a solution to (18), since h is in the kernel of $-\Delta$.

Note. It has been pointed out by an anonymous referee that the following result, the existence of unique minimal Green's function on $\mathbf{H}_R^d$, is not new and has been proven in [29] using compact exhaustions and maximum principles as in this paper. See also [38].

Proposition 3.3. *There exists precisely one C^∞-function $H : (\mathbf{H}_R^d \times \mathbf{H}_R^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{H}_R^d\} \to \mathbf{R}$ such that for all $\mathbf{x}' \in \mathbf{H}_R^d$ the function $H_{\mathbf{x}'} : \mathbf{H}_R^d \setminus \{\mathbf{x}'\} \to \mathbf{R}$ defined by $H_{\mathbf{x}'}(\mathbf{x}) := H(\mathbf{x}, \mathbf{x}')$ is a distribution on $\mathbf{H}_R^d$ with*

$$-\Delta H_{\mathbf{x}'} = \delta_g(\cdot, \mathbf{x}')$$

and

$$\lim_{d(\mathbf{x},\mathbf{x}')\to\infty} H_{\mathbf{x}'}(\mathbf{x}) = 0, \tag{33}$$

where $d(\mathbf{x}, \mathbf{x}')$ is the geodesic distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbf{H}_R^d$.

Proof. Existence: clear. Uniqueness. Suppose H and $\tilde{H}$ are two such functions. Let $\mathbf{x}' \in \mathbf{H}_R^d$. Define the C^∞-function $h : \mathbf{H}_R^d \setminus \{\mathbf{x}'\} \to \mathbf{R}$ by $h = H_{\mathbf{x}'} - \tilde{H}_{\mathbf{x}'}$. Then, h is a distribution on $\mathbf{H}_R^d$ with $-\Delta h = 0$. Since $\mathbf{H}_R^d$ is locally Euclidean, one has by local elliptic regularity that h can be extended to a C^∞-function $\hat{h} : \mathbf{H}_R^d \to \mathbf{R}$. It follows from (33) for H and $\tilde{H}$ that

$$\lim_{d(\mathbf{x},\mathbf{x}')\to\infty} \hat{h}(\mathbf{x}) = 0. \tag{34}$$

The strong elliptic maximum/minimum principle on a Riemannian manifold for a bounded domain Ω states that if u is harmonic, then the supremum/infimum of u in Ω coincides with the supremum/infimum of u on the boundary $\partial\Omega$. By using a compact exhaustion sequence Ω_k in a non-compact connected Riemannian manifold and passing to a subsequence $\mathbf{x}_k \in \partial\Omega_k$ such that $\mathbf{x}_k \to \infty$, the strong elliptic maximum/minimum principle can be extended to non-compact connected Riemannian manifolds with boundary conditions at infinity (see, for instance, section 8.3.2 in [21]). Taking $\Omega_k \subset \mathbf{H}_R^d$, the strong elliptic maximum/minimum principle for non-compact connected Riemannian manifolds implies using (34) that $\hat{h} = 0$. Therefore, $h = 0$ and $H(\mathbf{x}, \mathbf{x}') = \tilde{H}(\mathbf{x}, \mathbf{x}')$ for all $\mathbf{x} \in \mathbf{H}_R^d \setminus \{\mathbf{x}'\}$. □

By proposition 3.3, for $d \geqslant 2$, the function $\mathcal{H}_R^d$ is the unique normalized fundamental solution of Laplace's equation which satisfies the vanishing decay (33).

4. Fourier expansions for Green's function on the hyperboloid

Now we compute the Fourier expansions for a fundamental solution of the Laplace–Beltrami operator on $\mathbf{H}_R^d$.

4.1. Fourier expansion for a fundamental solution of the Laplacian on $\mathbf{H}_R^2$

A generating function for Chebyshev polynomials of the first kind ([17], p 51) is given by

$$\frac{1-z^2}{1+z^2-2xz} = \sum_{n=0}^{\infty} \epsilon_n T_n(x) z^n, \tag{35}$$

where $|z| < 1$, $T_n : [-1, 1] \to \mathbf{R}$ is the Chebyshev polynomial of the first kind defined as

$$T_n(x) := {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) \tag{36}$$

(see for instance section 5.7.2 in [32]: note that $T_n(\cos\psi) = \cos(n\psi)$) and $\epsilon_n := 2 - \delta_n^0$ is the Neumann factor (see p 744 in [33]), commonly occurring in Fourier cosine series. If one substitutes $z = \mathrm{e}^{-\eta}$ with $\eta \in (0, \infty)$ in (35), then we obtain

$$\frac{\sinh\eta}{\cosh\eta - \cos\psi} = \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi)\, \mathrm{e}^{-n\eta}. \tag{37}$$

Integrating both sides of (37) with respect to η, we obtain the following generating function (cf [32], p 259):

$$\log(1 + z^2 - 2z\cos\psi) = -2\sum_{n=1}^{\infty} \frac{\cos(n\psi)}{n} z^n. \tag{38}$$

If we take $z = r_</r_>$ in (38), where $r_{\lessgtr} := {}^{\min}_{\max}\{r, r'\}$ with $r, r' \in [0, \infty)$, then using polar coordinates, we can derive the Fourier expansion for a fundamental solution of the Laplacian in Euclidean space for $d = 2$ (cf theorem 3.2), namely

$$\mathfrak{g}^2 := \log\|\mathbf{x} - \mathbf{x}'\| = \log r_> - \sum_{n=1}^{\infty} \frac{\cos(n(\phi - \phi'))}{n} \left(\frac{r_<}{r_>}\right)^n, \tag{39}$$

where $\mathfrak{g}^2 = -2\pi\mathcal{G}^2$ (cf theorem 3.2). On the hyperboloid for $d = 2$, we have a fundamental solution of Laplace's equation given by

$$\mathfrak{h}^2 := \log\coth\frac{1}{2}d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \frac{1}{2}\log\frac{\cosh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') + 1}{\cosh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') - 1},$$

where $\mathfrak{h}^2 = 2\pi\mathcal{H}_R^2$ (cf theorem 3.1 and (52) below). Note that because of the R^{d-2} dependence for a fundamental solution of Laplace's equation for $d = 2$ in theorem 3.1, there is no strict dependence on R for $\mathcal{H}_R^2$ or $\mathfrak{h}^2$, but will retain the notation nonetheless. In standard geodesic polar coordinates on $\mathbf{H}_R^2$ (cf (1)), using (8) and $\cos\gamma = \cos(\phi - \phi')$ (cf (9)) produces

$$\cosh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \cosh r \cosh r' - \sinh r \sinh r' \cos(\phi - \phi');$$

therefore,

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r\cosh r' + 1 - \sinh r\sinh r'\cos(\phi - \phi')}{\cosh r\cosh r' - 1 - \sinh r\sinh r'\cos(\phi - \phi')}.$$

Replacing $\psi = \phi - \phi'$ and rearranging the logarithms yield

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r\cosh r' + 1}{\cosh r\cosh r' - 1} + \frac{1}{2}\log\left(1 - z_+\cos\psi\right) - \frac{1}{2}\log\left(1 - z_-\cos\psi\right),$$

where

$$z_\pm := \frac{\sinh r\sinh r'}{\cosh r\cosh r' \pm 1}.$$

Note that $z_\pm \in (0, 1)$ for $r, r' \in (0, \infty)$. We have the following MacLaurin series:

$$\log(1 - x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

where $x \in [-1, 1)$. Therefore, away from the singularity at $\mathbf{x} = \mathbf{x}'$ we have

$$\lambda_\pm := \log\left(1 - z_\pm\cos\psi\right) = -\sum_{k=1}^{\infty}\frac{z_\pm^k}{k}\cos^k\psi. \tag{40}$$

We can expand powers of the cosine function using the following trigonometric identity:

$$\cos^k\psi = \frac{1}{2^k}\sum_{n=0}^{k}\binom{k}{n}\cos[(2n - k)\psi],$$

which is the standard expansion for powers using Chebyshev polynomials of the first kind (see, for instance, p 52 in [17]). Inserting this expression in (40), we obtain the following double-summation expression:

$$\lambda_\pm = -\sum_{k=1}^{\infty}\sum_{n=0}^{k}\frac{z_\pm^k}{2^k k}\binom{k}{n}\cos[(2n - k)\psi]. \tag{41}$$

Now we perform a double-index replacement in (41). We break this sum into two separate sums, one for $k \leqslant 2n$ and another for $k \geqslant 2n$. There is an overlap when both sums satisfy the equality, and in that situation, we must halve after we sum over both sums. If $k \leqslant 2n$, make the substitution $k' = k - n$ and $n' = 2n - k$. It follows that $k = 2k' + n'$ and $n = n' + k'$; therefore,

$$\binom{k}{n} = \binom{2k' + n'}{n' + k'} = \binom{2k' + n'}{n' + k'}.$$

If $k \geqslant 2n$ make the substitution $k' = n$ and $n' = k - 2n$. Then, $k = 2k' + n'$ and $n = k'$; therefore,

$$\binom{k}{n} = \binom{2k' + n'}{n} = \binom{2k' + n'}{k' + n'},$$

where the equalities of the binomial coefficients are confirmed using the following identity:

$$\binom{n}{k} = \binom{n}{n - k},$$

where $n, k \in \mathbf{Z}$, except where $k < 0$ or $n - k < 0$. To take into account the double-counting which occurs when $k = 2n$ (which occurs when $n' = 0$), we introduce a factor of $\epsilon_{n'}/2$ into the expression (and relabel $k' \mapsto k$ and $n' \mapsto n$). We are left with

$$\lambda_\pm = -\frac{1}{2}\sum_{k=1}^\infty \frac{z_\pm^{2k}}{2^k k}\binom{2k}{k} - 2\sum_{n=1}^\infty \cos(n\psi)\sum_{k=0}^\infty \frac{z_\pm^{2k+n}}{2^{2k+n}(2k+n)}\binom{2k+n}{k}. \tag{42}$$

If we substitute

$$\binom{2k}{k} = \frac{2^{2k}\left(\frac{1}{2}\right)_k}{k!},$$

into the first term of (42), then we obtain

$$I_\pm := -\frac{1}{2}\sum_{k=1}^\infty \frac{\left(\frac{1}{2}\right)_k z_\pm^{2k}}{k!k} = -\int_0^{z_\pm} \frac{dz'_\pm}{z'_\pm}\sum_{k=1}^\infty \frac{\left(\frac{1}{2}\right)_k {z'_\pm}^{2k}}{k!} = -\int_0^{z_\pm} \frac{dz'_\pm}{z'_\pm}\left[\frac{1}{\sqrt{1-{z'_\pm}^2}} - 1\right].$$

We are left with

$$I_\pm = -\log 2 + \log\left(1+\sqrt{1-z_\pm^2}\right) = -\log 2 + \log\left(\frac{(\cosh r_> \pm 1)(\cosh r_< + 1)}{\cosh r \cosh r' \pm 1}\right).$$

If we substitute

$$\binom{2k+n}{k} = \frac{2^{2k}\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k},$$

into the second term of (42), then the Fourier coefficient reduces to

$$J_\pm := \frac{1}{2^{n-1}}\sum_{k=0}^\infty \frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}\frac{z_\pm^{2k+n}}{2k+n}$$
$$= \frac{1}{2^{n-1}}\int_0^{z_\pm} dz'_\pm {z'_\pm}^{n-1}\sum_{k=0}^\infty \frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}{z'_\pm}^{2k}.$$

The series in the integrand is a Gauss hypergeometric function which can be given as

$$\sum_{k=0}^\infty \frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}z^{2k} = \frac{2^n n!}{z^n\sqrt{1-z^2}}P_0^{-n}\left(\sqrt{1-z^2}\right),$$

where P_0^{-n} is the associated Legendre function of the first kind with vanishing degree and order given by $-n$ (cf (13) and (8.3.2) in [1]). The above formula is also a consequence of

$${}_2F_1\left(a, b; a+b-\frac{1}{2}; x\right) = 2^{2+b-3/2}\Gamma\left(a+b-\frac{1}{2}\right)\frac{x^{(3-2a-2b)/4}}{\sqrt{1-x}}P_{b-a-1/2}^{3/2-a-b}\left(\sqrt{1-x}\right),$$

where $x \in (0, 1)$ (see, for instance, [32], p 53), and the Legendre function of the first kind is evaluated using (cf (8.1.2) in [1])

$$P_0^{-n}(x) = \frac{1}{n!}\left(\frac{1-x}{1+x}\right)^{n/2},$$

where $n \in \mathbf{N}_0$. Therefore, the Fourier coefficient is given by

$$J_\pm = 2\int_{\sqrt{1-z_\pm^2}}^1 \frac{dz'_\pm}{1-{z'_\pm}^2}\left(\frac{1-z'_\pm}{1+z'_\pm}\right)^{n/2} = \frac{2}{n}\left[\frac{1-\sqrt{1-z_\pm^2}}{1+\sqrt{1-z_\pm^2}}\right]^{n/2}.$$

Finally we have

$$\lambda_\pm = -\log 2 + \log\left(\frac{(\cosh r_> \pm 1)(\cosh r_< + 1)}{\cosh r \cosh r' \pm 1}\right)$$
$$-2\sum_{n=1}^{\infty}\frac{\cos(n\psi)}{n}\left[\frac{(\cosh r_> \mp 1)(\cosh r_< - 1)}{(\cosh r_> \pm 1)(\cosh r_< + 1)}\right]^{n/2},$$

and the Fourier expansion for a fundamental solution of Laplace's equation for the $d = 2$ hyperboloid is given by

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r_> + 1}{\cosh r_> - 1} + \sum_{n=1}^{\infty}\frac{\cos(n(\phi-\phi'))}{n}\left[\frac{\cosh r_< - 1}{\cosh r_< + 1}\right]^{n/2}$$
$$\times\left\{\left[\frac{\cosh r_> + 1}{\cosh r_> - 1}\right]^{n/2} - \left[\frac{\cosh r_> - 1}{\cosh r_> + 1}\right]^{n/2}\right\}. \qquad (43)$$

This exactly matches up to the Euclidean Fourier expansion $\mathfrak{g}^2$ (39) as $r, r' \to 0^+$.

4.2. Fourier expansion for a fundamental solution of the Laplacian on $\mathbf{H}_R^3$

The Fourier expansion for a fundamental solution of the Laplacian in three-dimensional Euclidean space (here given in standard spherical coordinates $\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$) is given by (cf theorem 3.2, and see (1.3) in [9])

$$\mathfrak{g}^3 := \frac{1}{\|\mathbf{x}-\mathbf{x}'\|}$$
$$= \frac{1}{\pi\sqrt{rr'\sin\theta\sin\theta'}}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} Q_{m-1/2}\left(\frac{r^2 + r'^2 - 2rr'\cos\theta\cos\theta'}{2rr'\sin\theta\sin\theta'}\right),$$

where $\mathfrak{g}^3 = 4\pi\mathcal{G}^3$. These associated Legendre functions, toroidal harmonics, are given in terms of complete elliptic integrals of the first and second kind (cf (22–26) in [10]). Since $Q_{-1/2}(z)$ is given through (cf (8.13.3) in [1])

$$Q_{-1/2}(z) = \sqrt{\frac{2}{z+1}}K\left(\sqrt{\frac{2}{z+1}}\right),$$

the $m = 0$ component for $\mathfrak{g}^3$ is given by

$$\mathfrak{g}^3\big|_{m=0} = \frac{2}{\pi\sqrt{r^2 + r'^2 - 2rr'\cos(\theta+\theta')}}K\left(\sqrt{\frac{4rr'\sin\theta\sin\theta'}{r^2 + r'^2 - 2rr'\cos(\theta+\theta')}}\right) \qquad (44)$$

(see (51) for the definition of the complete elliptic integral of the first kind K). A fundamental solution of the Laplacian in standard geodesic polar coordinates on $\mathbf{H}_R^3$ is given by (cf theorem 3.1 and (52) below)

$$\mathfrak{h}^3(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') := \coth d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') - 1 = \frac{\cosh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')}{\sqrt{\cosh^2 d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') - 1}} - 1$$
$$= \frac{\cosh r\cosh r' - \sinh r\sinh r'\cos\gamma}{\sqrt{(\cosh r\cosh r' - \sinh r\sinh r'\cos\gamma)^2 - 1}} - 1,$$

where $\mathfrak{h}^3 = 4\pi R\mathcal{H}_R^3$ and $\mathbf{x}, \mathbf{x}' \in \mathbf{H}_R^3$, such that $\widehat{\mathbf{x}} = \mathbf{x}/R$ and $\widehat{\mathbf{x}}' = \mathbf{x}'/R$. In standard geodesic polar coordinates (cf (9)), we have

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \qquad (45)$$

Replacing $\psi = \phi - \phi'$ and defining

$$A := \cosh r \cosh r' - \sinh r \sinh r' \cos\theta \cos\theta'$$

and

$$B := \sinh r \sinh r' \sin\theta \sin\theta',$$

we have in the standard manner, the Fourier coefficients $\mathsf{H}_m^{1/2} : [0,\infty)^2 \times [0,\pi]^2 \to \mathbf{R}$ of the expansion (cf (53))

$$\mathfrak{h}^3(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \sum_{m=0}^{\infty} \cos(m(\phi-\phi'))\mathsf{H}_m^{1/2}(r, r', \theta, \theta'), \tag{46}$$

defined by

$$\mathsf{H}_m^{1/2}(r, r', \theta, \theta') := -\delta_n^0 + \frac{\epsilon_m}{\pi}\int_0^{\pi} \frac{(A/B - \cos\psi)\cos(m\psi)\,\mathrm{d}\psi}{\sqrt{\left(\cos\psi - \frac{A+1}{B}\right)\left(\cos\psi - \frac{A-1}{B}\right)}}. \tag{47}$$

If we make the substitution $x = \cos\psi$, this integral can be converted into

$$\mathsf{H}_m^{1/2}(r, r', \theta, \theta') = -\delta_n^0 + \frac{\epsilon_m}{\pi}\int_{-1}^{1} \frac{(A/B - x)\,T_m(x)\,\mathrm{d}x}{\sqrt{(1-x)(1+x)\left(x - \frac{A+1}{B}\right)\left(x - \frac{A-1}{B}\right)}}, \tag{48}$$

where $T_m(x)$ is the Chebyshev polynomial of the first kind (cf (36)). Since $T_m(x)$ is expressible as a finite sum over powers of x, (48) involves the square root of a quartic multiplied by a rational function of x, which by definition is an elliptic integral (see, for instance, [5]). We can directly compute (48) using [5] (253.11). If we define

$$d := -1, \quad y := -1, \quad c := 1, \quad b := \frac{A-1}{B}, \quad a := \frac{A+1}{B}, \tag{49}$$

(clearly $d \leqslant y < c < b < a$), then we can express the Fourier coefficient (48), as a linear combination of integrals, each of the form (see [5], (253.11))

$$\int_y^c \frac{x^p\,\mathrm{d}x}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = c^p g \int_0^{u_1} \left[\frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\right]^p \mathrm{d}u, \tag{50}$$

where $p \in \{0, \ldots, m+1\}$. In this expression, sn is a Jacobi elliptic function (see for instance, chapter 22 in [36]).

Byrd and Friedman [5] gave a procedure for computing (50) for all $m \in \mathbf{N}_0$. These integrals will be given in terms of complete elliptic integrals of the first three kinds (see the discussion in [5], p 201, 204 and 205). To this effect, we have the following definitions from (253.11) in [5]:

$$\alpha^2 = \frac{c-d}{b-d} < 1,$$

$$\alpha_1^2 = \frac{b(c-d)}{c(b-d)},$$

$$g = \frac{2}{\sqrt{(a-c)(b-d)}},$$

$$\varphi = \sin^{-1}\sqrt{\frac{(b-d)(c-y)}{(c-d)(b-y)}},$$

$$u_1 = F(\varphi, k),$$

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)},$$

where $k^2 < \alpha^2$, and $F : [0, \pi/2] \times [0, 1) \to \mathbf{R}$ is Legendre's incomplete elliptic integral of the first kind which can be defined through the following definite integral (see for instance section 19.2(ii) in [36]):

$$F(\varphi, k) := \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

For our specific choices in (49), these reduce to

$$\alpha^2 = \frac{2B}{A + B - 1},$$
$$\alpha_1^2 = \frac{2(A - 1)}{A + B - 1},$$
$$g = \frac{2B}{\sqrt{(A + B - 1)(A - B + 1)}},$$
$$k^2 = \frac{4B}{(A + B - 1)(A - B + 1)},$$
$$\varphi = \frac{\pi}{2}$$

and

$$u_1 = K(k),$$

where $K : [0, 1) \to [1, \infty)$ is Legendre's complete elliptic integral of the first kind which is given by

$$K(k) := F\left(\frac{\pi}{2}, k\right) \tag{51}$$

(see, for instance, section 19.2(ii) in [36]). Specific cases include

$$\int_y^c \frac{dx}{\sqrt{(a - x)(b - x)(c - x)(x - d)}} = gK(k)$$

([5], (340.00)) and

$$\int_y^c \frac{x\, dx}{\sqrt{(a - x)(b - x)(c - x)(x - d)}} = \frac{cg}{\alpha^2}\left[\alpha_1^2 K(k) + (\alpha^2 - \alpha_1^2)\Pi(\alpha^2, k)\right]$$

([5], (340.01)), where $\Pi : [0, \infty)\setminus\{1\} \times [0, 1) \to \mathbf{R}$ is Legendre's complete elliptic integral of the third kind which can be defined by the following definite integral (see for instance section 19.2(ii) in [36]):

$$\Pi(\alpha^2, k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}\,(1 - \alpha^2 \sin^2\theta)}.$$

In general, we have

$$\int_y^c \frac{x^p dx}{\sqrt{(a - x)(b - x)(c - x)(x - d)}} = \frac{c^p g \alpha_1^{2p} p!}{\alpha^{2p}} \sum_{j=0}^{p} \frac{(\alpha^2 - \alpha_1^2)^j}{\alpha_1^{2j} j!(p - j)!} V_j$$

([5], (340.04)), where

$$V_0 = K(k),$$

$$V_1 = \Pi(\alpha^2, k),$$

$$V_2 = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\left[(k^2 - \alpha^2)K(k) + \alpha^2 E(k) + (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\Pi(\alpha^2, k)\right],$$

and larger values of V_j can be computed using the following recurrence relation:

$$V_{m+3} = \frac{1}{2(m+2)(1-\alpha^2)(k^2-\alpha^2)}\big[(2m+1)k^2V_m + 2(m+1)(\alpha^2k^2+\alpha^2-3k^2)V_{m+1} + (2m+3)(\alpha^4-2\alpha^2k^2-2\alpha^2+3k^2)V_{m+2}\big]$$

(see [5], (336.00–03)), where $E : [0, 1] \to [1, \frac{\pi}{2}]$ is Legendre's complete elliptic of the second kind which can be defined by the following definite integral (see for instance section 19.2(ii) in [36]):

$$E(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta.$$

For one particular example ($p = 2$), we have

$$\int_y^c \frac{x^2\,\mathrm{d}x}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{c^2g}{\alpha^4}\big[\alpha_1^4K(k) + 2\alpha_1^2(\alpha^2-\alpha_1^2)\Pi(\alpha^2,k) + (\alpha^2-\alpha_1^2)^2V_2\big]$$

(see [5], (340.02)).

In general, the Fourier coefficients for $\mathfrak{h}^3$ will be given in terms of complete elliptic integrals of the first three kinds. Let us directly compute the $m = 0$ component, in which (48) reduces to

$$\mathsf{H}_0^{1/2}(r,r',\theta,\theta') = -1 + \frac{1}{\pi}\int_{-1}^{1} \frac{(A/B-x)\,\mathrm{d}x}{\sqrt{(1-x)(1+x)\left(x-\frac{A+1}{B}\right)\left(x-\frac{A-1}{B}\right)}}.$$

Therefore, using the above formulas, we have

$$\begin{aligned}\mathfrak{h}_3|_{m=0} &= \mathsf{H}_0^{1/2}(r,r',\theta,\theta')\\ &= -1 + \frac{2K(k)}{\pi\sqrt{(A-B+1)(A+B-1)}} + \frac{2(A-B-1)\Pi(\alpha^2,k)}{\pi\sqrt{(A-B+1)(A+B-1)}}\\ &= -1 + \frac{2}{\pi}\left\{K(k) + \big[\cosh r\cosh r' - \sinh r\sinh r'\cos(\theta-\theta') - 1\big]\Pi(\alpha^2,k)\right\}\\ &\quad\times[\cosh r\cosh r' - \sinh r\sinh r'\cos(\theta-\theta')+1]^{-1/2}\\ &\quad\times[\cosh r\cosh r' - \sinh r\sinh r'\cos(\theta+\theta')-1]^{-1/2}.\end{aligned}$$

Note that the Fourier coefficients

$$\mathfrak{h}^3|_{m=0} \to \mathfrak{g}^3|_{m=0},$$

in the limit as $r, r' \to 0^+$, where $\mathfrak{g}^3|_{m=0}$ is given in (44). This is expected since $\mathbf{H}_R^3$ is a manifold.

4.3. Fourier expansion for a fundamental solution of the Laplacian on $\mathbf{H}_R^d$

For the d-dimensional Riemannian manifold $\mathbf{H}_R^d$, with $d \geqslant 2$, one can expand a fundamental solution of the Laplace–Beltrami operator in an azimuthal Fourier series. One may Fourier expand, in terms of the azimuthal coordinate, a fundamental solution of the Laplace–Beltrami operator in any rotationally invariant coordinate system which admits solutions via separation of variables. All separable coordinate systems for Laplace's equation on d-dimensional Euclidean space $\mathbf{R}^d$ are known. In fact, this is also true for separable coordinate systems on $\mathbf{H}_R^d$ (see [27]). There has been considerable work in two and three dimensions; however, there still remains a lot of work to be done for a detailed analysis of fundamental solutions.

We define an unnormalized fundamental solution of Laplace's equation on the unit radius hyperboloid $\mathfrak{h}^d : (\mathbf{H}^d \times \mathbf{H}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{H}^d\} \to \mathbf{R}$ such that

$$\mathfrak{h}^d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') := \mathcal{I}_d(\rho(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')) = \frac{2\pi^{d/2}R^{d-2}}{\Gamma(d/2)}\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}'). \tag{52}$$

In our current azimuthal Fourier analysis, we therefore will focus on the relatively easier case of separable subgroup-type coordinate systems on $\mathbf{H}_R^d$, and specifically for geodesic polar coordinates. In these coordinates, the Riemannian metric is given by (10) and we further restrict our attention by adopting standard geodesic polar coordinates (1).

In these coordinates, one would like to expand a fundamental solution of Laplace's equation on the hyperboloid in an azimuthal Fourier series, namely

$$\mathfrak{h}^d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \sum_{m=0}^{\infty} \cos(m(\phi - \phi'))\mathsf{H}_m^{d/2-1}(r, r', \theta_1, \ldots, \theta_{d-2}, \theta_1', \ldots, \theta_{d-2}'), \tag{53}$$

where $\mathsf{H}_m^{d/2-1} : [0, \infty)^2 \times [0, \pi]^{2d-4} \to \mathbf{R}$ is defined such that

$$\mathsf{H}_m^{d/2-1}(r, r', \theta_1, \ldots, \theta_{d-2}, \theta_1', \ldots, \theta_{d-2}') := \frac{\epsilon_m}{\pi}\int_0^{\pi} \mathfrak{h}^d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')\cos(m(\phi - \phi'))d(\phi - \phi') \tag{54}$$

(see, for instance, [10]). According to theorem 3.1 and (52), we may write $\mathfrak{h}^d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')$ in terms of the associated Legendre function of the second kind (14) as follows:

$$\mathfrak{h}^d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \frac{\mathrm{e}^{-\mathrm{i}\pi(d/2-1)}}{2^{d/2-1}\Gamma(d/2)\,(\sinh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}'))^{d/2-1}} Q_{d/2-1}^{d/2-1}\left(\cosh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')\right). \tag{55}$$

By (3) we know that in any geodesic polar coordinate system

$$\cosh d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') = \cosh r \cosh r' - \sinh r \sinh r' \cos\gamma, \tag{56}$$

and therefore through (54), (55) and (56), in standard geodesic polar coordinates, the azimuthal Fourier coefficient can be given by

$$\begin{aligned}\mathsf{H}_m^{d/2-1}&(r, r', \theta_1, \ldots, \theta_{d-2}, \theta_1', \ldots, \theta_{d-2}') \\ &= \frac{\epsilon_m \mathrm{e}^{-\mathrm{i}\pi(d/2-1)}}{2^{d/2-1}\pi\Gamma(d/2)}\int_0^{\pi} \frac{Q_{d/2-1}^{d/2-1}(A - B\cos\psi)\cos(m\psi)}{\left[(A - B\cos\psi)^2 - 1\right]^{(d-2)/4}}\,\mathrm{d}\psi,\end{aligned} \tag{57}$$

where $\psi = \phi - \phi'$, $A, B : [0, \infty)^2 \times [0, \pi]^{2d-4} \to \mathbf{R}$ are defined through (9) and (56) as

$$A(r, r', \theta_1, \ldots, \theta_{d-2}, \theta_1', \ldots, \theta_{d-2}') := \cosh r \cosh r' \sum_{i=1}^{d-2} \cos\theta_i \cos\theta_i' \prod_{j=1}^{i-1} \sin\theta_j \sin\theta_j'$$

and

$$B(r, r', \theta_1, \ldots, \theta_{d-2}, \theta_1', \ldots, \theta_{d-2}') := \sinh r \sinh r' \prod_{i=1}^{d-2} \sin\theta_i \sin\theta_i'.$$

Even though (56) is a compact expression for the Fourier coefficient of a fundamental solution of Laplace's equation on $\mathbf{H}_R^d$ for $d \in \{2, 3, 4, \ldots\}$, it may be informative to use any of the representations of a fundamental solution of the Laplacian on $\mathbf{H}_R^d$ from theorem 3.1 to express the Fourier coefficients. For instance, if one uses the finite-summation expression in the odd dimensions, one can write the Fourier coefficients as a linear combination of integrals of the form

$$\int_{-1}^{1} \frac{[(a+b)/2 - x]^{2k-1}\, x^p \,\mathrm{d}x}{(a-x)^{k-1}(b-x)^{k-1}\sqrt{(a-x)(b-x)(c-x)(x-d)}},$$

where $x = \cos\psi$, $k \in \{1, \ldots, (d-1)/2\}$, $p \in \{0, \ldots, m\}$, and we have used the nomenclature of section 4.2. This integral is a rational function of x multiplied by a reciprocal square root of a quartic in x. Because of this and due to the limits of integration, we see that by definition, these are all given in terms of complete elliptic integrals. The special functions which represent the azimuthal Fourier coefficients on $\mathbf{H}_R^d$ are unlike the odd-half-integer degree, integer-order, associated Legendre functions of the second kind (toroidal harmonics) which appear on Euclidean space $\mathbf{R}^d$ for d odd (see [6, 8]) in that they include complete elliptic integrals of the third kind (in addition to complete elliptic integrals of the first and second kind) (cf section 4.2) in their basis functions. For $d \geqslant 2$, through (4.1) in [8] and that $\mathbf{H}_R^d$ is a manifold (and therefore must locally represent Euclidean space), the functions $\mathsf{H}_m^{d/2-1}$ are hyperbolic generalizations of associated Legendre functions of the second kind with odd-half-integer degree and order given by either an odd-half-integer or an integer.

5. Gegenbauer expansion in geodesic polar coordinates

We begin this section by deriving a Gegenbauer polynomial expansion for a fundamental solution of Laplace's equation on the hyperboloid in geodesic polar coordinates for $d \geqslant 3$. Since the spherical harmonics for $d = 2$ are just trigonometric functions with argument given in terms of the azimuthal angle, this case has already been covered in section 4.1. Through the limiting process from zero order Gegenbauer polynomials to Chebyshev polynomials of the first kind, we will show that our Gegenbauer polynomial expansion for $d \geqslant 3$ is consistent with our Fourier expansion for $d = 2$. As an interesting consequence of this connection, we then show how our Gegenbauer polynomial result is just a special case of a complex-valued addition formula given in [12] for Gegenbauer functions of the second kind (which are intimately related to associated Legendre functions of the second kind, cf (76) below).

In geodesic polar coordinates, Laplace's equation is given by (cf (11))

$$\Delta f = \frac{1}{R^2}\left[\frac{\partial^2 f}{\partial r^2} + (d-1)\coth r\frac{\partial f}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbf{S}^{d-1}}\right] f = 0, \tag{58}$$

where $f : \mathbf{H}_R^d \to \mathbf{R}$ and $\Delta_{\mathbf{S}^{d-1}}$ is the corresponding Laplace–Beltrami operator on the $(d-1)$-dimensional unit radius hypersphere $\mathbf{S}^{d-1}$. Eigenfunctions $Y_l^K : \mathbf{S}^{d-1} \to \mathbf{C}$ of the Laplace–Beltrami operator $\Delta_{\mathbf{S}^{d-1}}$, where $l \in \mathbf{N}_0$ and K is a set of quantum numbers which label representations for l in separable subgroup-type coordinate systems on $\mathbf{S}^{d-1}$ (i.e. angular momentum-type quantum numbers; see [26]), are given by solutions to the eigenvalue problem (12).

In standard geodesic polar coordinates (1), $K = (k_1, \ldots, k_{d-3}, k_{d-2}) \in \mathbf{N}_0^{d-3} \times \mathbf{Z}$ with $k_0 = l \geqslant k_1 \geqslant \cdots \geqslant k_{d-3} \geqslant |k_{d-2}| \geqslant 0$, and in particular $-k_{d-3} \leqslant k_{d-2} \leqslant k_{d-3}$. A positive fundamental solution $\mathcal{H}_R^d : (\mathbf{H}_R^d \times \mathbf{H}_R^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{H}_R^d\} \to \mathbf{R}$ on the R-radius hyperboloid satisfies (18). The completeness relation for hyperspherical harmonics in standard hyperspherical coordinates is given by

$$\sum_{l=0}^{\infty}\sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1})\overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')} = \frac{\delta(\theta_1 - \theta_1')\ldots\delta(\theta_{d-1} - \theta_{d-1}')}{\sin^{d-2}\theta_{d-1}'\ldots\sin\theta_2'}.$$

Therefore, through (20), we can write

$$\delta_g(\mathbf{x}, \mathbf{x}') = \frac{\delta(r - r')}{R^d \sinh^{d-1} r'}\sum_{l=0}^{\infty}\sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1})\overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')}. \tag{59}$$

For fixed $r, r' \in [0, \infty)$ and $\theta_1', \ldots, \theta_{d-1}' \in [0, \pi]$, since $\mathcal{H}_R^d$ is harmonic on its domain, its restriction is in $C^2(\mathbf{S}^{d-1})$, and therefore has a unique expansion in hyperspherical harmonics, namely

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} \sum_K u_l^K(r, r', \theta_1', \ldots, \theta_{d-1}') Y_l^K(\theta_1, \ldots, \theta_{d-1}), \tag{60}$$

where $u_l^K : [0, \infty)^2 \times [0, \pi]^{d-1} \to \mathbf{C}$. If we substitute (59) and (60) into (18) and use (12) and (58), we obtain

$$\sum_{l=0}^{\infty} \sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1}) \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (d-1)\coth r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+d-2)}{\sinh^2 r} \right] u_l^K(r, r', \theta_1', \ldots, \theta_{d-1}')$$
$$= \sum_{l=0}^{\infty} \sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1}) \overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')} \frac{\delta(r - r')}{R^{d-2} \sinh^{d-1} r'}. \tag{61}$$

This indicates that for $u_l : [0, \infty)^2 \to \mathbf{R}$,

$$u_l^K(r, r', \theta_1', \ldots, \theta_{d-1}') = u_l(r, r') \overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')}, \tag{62}$$

and from (60), the expression for a fundamental of the Laplace–Beltrami operator in standard hyperspherical coordinates on the hyperboloid is given by

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} u_l(r, r') \sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1}) \overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')}. \tag{63}$$

The above expression can be simplified using the addition theorem for hyperspherical harmonics (see, for instance, [43], section 10.2.1 in [44], chapter 9 in [2] and especially chapter XI in [15]), which is given by

$$\sum_K Y_l^K(\widehat{\mathbf{x}}) \overline{Y_l^K(\widehat{\mathbf{x}}')} = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)} (2l + d - 2) C_l^{d/2-1}(\cos\gamma), \tag{64}$$

where γ is the angle between two arbitrary vectors $\widehat{\mathbf{x}}, \widehat{\mathbf{x}}' \in \mathbf{S}^{d-1}$ given in terms of (2). The Gegenbauer polynomial $C_l^{\mu} : [-1, 1] \to \mathbf{C}$, $l \in \mathbf{N}_0$, $\mathrm{Re}\,\mu > -1/2$, can be defined in terms of the Gauss hypergeometric function as

$$C_l^{\mu}(x) := \frac{(2\mu)_l}{l!} {}_2F_1\left(-l, 2\mu + l; \mu + \frac{1}{2}; \frac{1-x}{2}\right).$$

The above expression (63) can be simplified using (64); therefore,

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)} \sum_{l=0}^{\infty} u_l(r, r')(2l + d - 2) C_l^{d/2-1}(\cos\gamma). \tag{65}$$

Now we compute the exact expression for $u_l(r, r')$. By separating the angular dependence in (61) and using (62), we obtain the differential equation

$$\frac{\mathrm{d}^2 u_l}{\mathrm{d}r^2} + (d-1)\coth r \frac{\mathrm{d}u_l}{\mathrm{d}r} - \frac{l(l+d-2)u_l}{\sinh^2 r} = -\frac{\delta(r - r')}{R^{d-2}\sinh^{d-1} r'}. \tag{66}$$

Away from $r = r'$, solutions to the differential equation (66) must be given by solutions to the homogeneous equation, which are given in section 2.2. Therefore, the solution to (66) is given by

$$u_l(r, r') = \frac{A}{(\sinh r \sinh r')^{d/2-1}} P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>), \tag{67}$$

such that $u_l(r, r')$ is continuous at $r = r'$, and $A \in \mathbf{R}$.

In order to determine the constant A, we first make the substitution

$$v_l(r, r') = (\sinh r \sinh r')^{(d-1)/2} u_l(r, r'). \tag{68}$$

This converts (66) into the following differential equation:

$$\frac{\partial^2 v_l(r, r')}{\partial r^2} - \frac{1}{4}\left[\frac{(d-1+2l)(d-3+2l)}{\sinh^2 r} + (d-1)^2\right] v_l(r, r') = -\frac{\delta(r-r')}{R^{d-2}},$$

which we then integrate over r from $r' - \epsilon$ to $r' + \epsilon$, and take the limit as $\epsilon \to 0^+$. We are left with a discontinuity condition for the derivative of $v_l(r, r')$ with respect to r evaluated at $r = r'$, namely

$$\lim_{\epsilon \to 0^+} \frac{\mathrm{d}v_l(r, r')}{\mathrm{d}r}\bigg|_{r'-\epsilon}^{r'+\epsilon} = \frac{-1}{R^{d-2}}. \tag{69}$$

After inserting (67) with (68) into (69), substituting $z = \cosh r'$, evaluating at $r = r'$, and making use of the Wronskian relation (e.g. p 165 in [32])

$$W\left\{P_\nu^{-\mu}(z), Q_\nu^\mu(z)\right\} = -\frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{z^2 - 1},$$

which is equivalent to

$$W\left\{P_\nu^{-\mu}(\cosh r'), Q_\nu^\mu(\cosh r')\right\} = -\frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{\sinh^2 r'},$$

we obtain

$$A = \frac{\mathrm{e}^{-\mathrm{i}\pi(d/2-1+l)}}{R^{d-2}},$$

and therefore

$$u_l(r, r') = \frac{\mathrm{e}^{-\mathrm{i}\pi(d/2-1+l)}}{R^{d-2}(\sinh r \sinh r')^{d/2-1}} P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>).$$

Finally, through (65) we have

$$\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}') = \frac{\Gamma(d/2)}{2\pi^{d/2} R^{d-2}(d-2)} \frac{\mathrm{e}^{-\mathrm{i}\pi(d/2-1)}}{(\sinh r \sinh r')^{d/2-1}} \sum_{l=0}^{\infty} (-1)^l (2l + d - 2) \times P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>) C_l^{d/2-1}(\cos\gamma). \tag{70}$$

As an alternative check of our derivation, we can do the asymptotics for the product of associated Legendre functions $P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>)$ in (70) as $r, r' \to 0^+$. The appropriate asymptotic expressions for P and Q respectively can be found on p 171 and p 173 in [35]. For the associated Legendre function of the first kind there is

$$P_\nu^{-\mu}(z) \sim \frac{[(z-1)/2]^{\mu/2}}{\Gamma(\mu+1)},$$

as $z \to 1$, $\mu \neq -1, -2, \ldots$, and for the associated Legendre function of the second kind there is

$$Q_\nu^\mu(z) \sim \frac{\mathrm{e}^{\mathrm{i}\pi\mu}\Gamma(\mu)}{2\,[(z-1)/2]^{\mu/2}},$$

as $z \to 1^+$, $\mathrm{Re}\ \mu > 0$, and $\nu + \mu \neq -1, -2, -3, \ldots$. To second order, the hyperbolic cosine is given by $\cosh r \simeq 1 + r^2/2$. Therefore, to lowest order we can insert $\cosh r_< \simeq 1 + r_<^2/2$ and $\cosh r_> \simeq 1 + r_>^2/2$ into the above expressions yielding

$$P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) \sim \frac{(r_</2)^{d/2-1+l}}{\Gamma(d/2+l)}$$

and

$$Q_{d/2-1}^{d/2-1+l}(\cosh r_>) \sim \frac{\mathrm{e}^{\mathrm{i}\pi(d/2-1+l)}\Gamma(d/2-1+l)}{2(r_>/2)^{d/2-1+l}},$$

as $r, r' \to 0^+$. Therefore, the asymptotics for the product of associated Legendre functions in (70) is given by

$$P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<)Q_{d/2-1}^{d/2-1+l}(\cosh r_>) \sim \frac{\mathrm{e}^{\mathrm{i}\pi(d/2-1+l)}}{2l+d-2}\left(\frac{r_<}{r_>}\right)^{l+d/2-1} \tag{71}$$

(the factor $2l+d-2$ is a term which one encounters regularly with hyperspherical harmonics). Gegenbauer polynomials obey the following generating function:

$$\frac{1}{\left(1+z^2-2zx\right)^{\mu}} = \sum_{l=0}^{\infty} C_l^{\mu}(x)z^l, \tag{72}$$

where $x \in [-1, 1]$ and $|z| < 1$ (see, for instance, p 222 in [32]). The generating function for Gegenbauer polynomials (72) can be used to expand a fundamental solution of Laplace's equation on Euclidean space $\mathbf{R}^d$ (for $d \geqslant 3$, cf theorem 3.2) in geodesic polar coordinates, namely

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|^{d-2}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+d-2}} C_l^{d/2-1}(\cos\gamma), \tag{73}$$

where γ was defined in (64). Using (73) and theorem 3.2, since $\mathcal{H}_R^d \to \mathcal{G}^d$, $\sinh r, \sinh r' \to r, r'$ and (71) is satisfied to lowest order as $r, r' \to 0^+$, we see that (70) obeys the correct asymptotics and our fundamental solution expansion locally reduces to the appropriate expansion for Euclidean space, as it should since $\mathbf{H}_R^d$ is a manifold.

Note that (70) can be further expanded over the remaining $(d-2)$-quantum numbers in K in terms of a simply separable product of normalized hyperspherical harmonics $Y_l^K(\widehat{\mathbf{x}})\overline{Y_l^K(\widehat{\mathbf{x}}')}$, where $\widehat{\mathbf{x}}, \widehat{\mathbf{x}}' \in \mathbf{S}^{d-1}$, using the addition theorem for hyperspherical harmonics (64) (see [6] for several examples).

It is intriguing to observe how one might obtain the Fourier expansion for $d = 2$ (43) from the expansion (70), which is strictly valid for $d \geqslant 3$. If one makes the substitution $\mu = d/2-1$ in (70), then we obtain the following proposition (which matches up to the generating function for Gegenbauer polynomials in the Euclidean limit $r, r' \to 0^+$).

Proposition 5.1. *For all* $\mu \in \mathbf{C}$ *such that* $\mathrm{Re}\,\mu > -1/2$, *one has*

$$\frac{1}{\sinh^{\mu}\rho} Q_{\mu}^{\mu}(\cosh\rho) = \frac{2^{\mu}\Gamma(\mu+1)}{(\sinh r \sinh r')^{\mu}} \times \sum_{n=0}^{\infty}(-1)^n \frac{n+\mu}{\mu} P_{\mu}^{-(\mu+n)}(\cosh r_<)Q_{\mu}^{\mu+n}(\cosh r_>)C_n^{\mu}(\cos\gamma), \tag{74}$$

where $\cosh\rho = \cosh r\cosh r' - \sinh r \sinh r' \cos\gamma$.

Proof. If we start with (8.6) in [12], namely

$$D_{\lambda}^{\alpha}(\zeta) = \frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\sum_{n=0}^{\infty}(-1)^n 2^{2n}(2\alpha+2n-1)\frac{\Gamma(\lambda-n+1)[\Gamma(\alpha+n)]^2}{\Gamma(\lambda+2\alpha+n)} \times (x_1^2-1)^{n/2}(x_2^2-1)^{n/2}D_{\lambda-n}^{\alpha+n}(x_1)C_{\lambda-n}^{\alpha+n}(x_2)C_n^{\alpha-1/2}(z), \tag{75}$$

where

$$\zeta = x_1x_2 - z\sqrt{x_1^2-1}\sqrt{x_2^2-1},$$

where $C_{\lambda-n}^{\alpha+n}(x_2)$ vanishes when $\lambda-n \in -\mathbf{N}$, so that the apparent singularities due to $\Gamma(\lambda-n+1)$ are not present. This series converges such that $\zeta \in \mathbf{C}$ lies in the region interior to the two ellipses with foci at ± 1 which pass through x_1 and x_2, respectively. In this formula $x_1, x_2 \in \mathbf{C} \setminus (-\infty, 1]$ and $z \in \mathbf{C}$. The Gegenbauer function of the first kind $C_\lambda^\alpha : \mathbf{C} \to \mathbf{C}$ with general degree λ and order α is defined as (see (2.5) in [12])

$$C_\lambda^\alpha(z) := \frac{\Gamma(2\alpha+\lambda)}{\Gamma(\lambda+1)\Gamma(2\alpha)}\,{}_2F_1\left(-\lambda, 2\alpha+\lambda; \alpha+\frac{1}{2}; \frac{1-z}{2}\right),$$

where $|1-z| < 2$ and elsewhere by analytic continuation. The Gegenbauer function of the second kind $D_\lambda^\alpha : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$ is defined as (see (2.3) in [12])

$$D_\lambda^\alpha(z) := \frac{e^{i\pi\alpha}\Gamma(2\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\lambda+\alpha+1)(2z)^{2\alpha+\lambda}}\,{}_2F_1\left(\frac{\lambda+2\alpha}{2}, \frac{\lambda+2\alpha+1}{2}; \lambda+\alpha+1; \frac{1}{z^2}\right)$$

for $|z| > 1$ and elsewhere by analytic continuation. The Gegenbauer functions of the first and second kind are linearly independent solutions to the Gegenbauer differential equation (see, for instance, section 4.10 in [32]) and can be expressed in terms of associated Legendre functions of the first and second kind ((13) and (14)), respectively, by ((2.8) in [12])

$$C_\lambda^\alpha(z) = \frac{\sqrt{\pi}\,\Gamma(\lambda+2\alpha)}{2^{\alpha-1/2}\Gamma(\alpha)\Gamma(\lambda+1)(z^2-1)^{\alpha/2-1/4}}P_{\lambda+\alpha-1/2}^{-\alpha+1/2}(z), \tag{76}$$

and ((2.4) in [12])

$$D_\lambda^\alpha(z) = \frac{e^{2\pi i(\alpha-1/4)}\Gamma(\lambda+2\alpha)}{\sqrt{\pi}\,2^{\alpha-1/2}\Gamma(\alpha)\Gamma(\lambda+1)(z^2-1)^{\alpha/2-1/4}}Q_{\lambda+\alpha-1/2}^{-\alpha+1/2}(z). \tag{77}$$

If we make the substitutions $\lambda = 2\mu$ and $\alpha = 1/2-\mu$, use (77), and the negative-order condition for associated Legendre functions of the second kind, cf (8.2.6) in [1],

$$Q_\nu^{-\mu}(z) = e^{-2i\mu\pi}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}Q_\nu^\mu(z), \tag{78}$$

this converts the Gegenbauer function of the second kind on the left-hand side of (75) to the associated Legendre function of the second kind with degree and order equal to μ, namely

$$\frac{1}{(\zeta^2-1)^{\mu/2}}Q_\mu^\mu(\zeta) = \frac{\sqrt{\pi}\,2^{\mu+1}\Gamma(2\mu)}{i\Gamma(\mu+\frac{1}{2})}\sum_{n=0}^\infty(-1)^n 2^{2n}(n+\mu)\frac{\Gamma(1-n)[\Gamma(\mu+\frac{1}{2}+n)]^2}{\Gamma(2\mu+1+n)}$$
$$\times(x_1^2-1)^{n/2}(x_2^2-1)^{n/2}D_{-n}^{\mu+1/2+n}(x_1)C_{-n}^{\mu+1/2+n}(x_2)C_n^\mu(z).$$

We can now express the first two Gegenbauer functions on the right-hand side in terms of the associated Legendre functions. Using (76) and (77), with (16) and the duplication formula for gamma functions (17), we obtain

$$\frac{1}{(\zeta^2-1)^{\mu/2}}Q_\mu^\mu(\zeta) = \frac{2^\mu\Gamma(\mu+1)}{[(x_1^2-1)(x_2^2-1)]^{\mu/2}}\sum_{n=0}^\infty(-1)^n\frac{n+\mu}{\mu}P_\mu^{-(\mu+n)}(x_2)Q_\mu^{\mu+n}(x_1)C_n^\mu(z).$$

By taking $\zeta = \cosh\rho$, $x_1 = \cosh r_>$, $x_2 = \cosh r_<$ and $z = \cos\gamma$, by (4) and (8) we have the desired result. □

If we take the limit as $\mu \to 0$ in (74) and use

$$\lim_{\mu\to 0}\frac{n+\mu}{\mu}C_n^\mu(x) = \epsilon_n T_n(x)$$

(see, for instance, (6.4.13) in [2]), where T_n is the Chebyshev polynomial of the first kind defined by (36), then we obtain the following formula:

$$\frac{1}{2}\log\frac{\cosh\rho+1}{\cosh\rho-1} = \sum_{n=0}^\infty\epsilon_n(-1)^n P_0^{-n}(\cosh r_<)Q_0^n(\cosh r_>)\cos(n(\phi-\phi')).$$

By taking advantage of the following formulas:

$$P_0^{-n}(z) = \frac{1}{n!}\left[\frac{z-1}{z+1}\right]^{n/2} \tag{79}$$

for $n \geqslant 0$,

$$Q_0(z) = \frac{1}{2}\log\frac{z+1}{z-1}$$

(see (8.4.2) in [1]), and

$$Q_0^n(z) = \frac{1}{2}(-1)^n(n-1)!\left\{\left[\frac{z+1}{z-1}\right]^{n/2} - \left[\frac{z-1}{z+1}\right]^{n/2}\right\} \tag{80}$$

for $n \geqslant 1$, then (43) is reproduced. The representation (79) follows easily from the Gauss hypergeometric representation of the associated Legendre function of the first kind (13). One way to derive the representation of the associated Legendre function of the second kind (80) is to use Whipple's formula for associated Legendre functions (cf (8.2.7) in [1])

$$Q_\nu^\mu(z) = \sqrt{\pi}2\Gamma(\nu+\mu+1)(z^2-1)^{-1/4}\mathrm{e}^{\mathrm{i}\pi\mu}P_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

where $\operatorname{Re} z > 0$ and (8.6.9) in [1], namely

$$P_\nu^{-1/2}(z) = \sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{-1/4}}{2\nu+1}\left\{\left[z+\sqrt{z^2-1}\right]^{\nu+1/2} - \left[z+\sqrt{z^2-1}\right]^{-\nu-1/2}\right\}$$

for $\nu \neq -1/2$.

5.1. Addition theorem for the azimuthal Fourier coefficient on $\mathbf{H}_R^3$

One can compute addition theorems for the azimuthal Fourier coefficients of a fundamental solution for Laplace's equation on $\mathbf{H}_R^d$ for $d \geqslant 3$ by relating directly obtained Fourier coefficients to the expansion over hyperspherical harmonics for the same fundamental solution. By using the expansion of $\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}')$ in terms of Gegenbauer polynomials (70) in combination with the addition theorem for hyperspherical harmonics (64) expressed in, for instance, one of Vilenkin's polyspherical coordinates (see section IX.5.2 in [42], [25, 26]), one can obtain through series rearrangement a multi-summation expression for the azimuthal Fourier coefficients. Vilenkin's polyspherical coordinates are simply subgroup-type coordinate systems which parametrize points on $\mathbf{S}^{d-1}$ (for a detailed discussion of these coordinate systems, see chapter 4 in [6]). In this section, we will give an explicit example of just such an addition theorem on $\mathbf{H}_R^3$.

The azimuthal Fourier coefficients on $\mathbf{H}_R^3$ expressed in standard hyperspherical coordinates (1) are given by the function $\mathsf{H}_m^{1/2} : [0,\infty)^2 \times [0,\pi]^2 \to \mathbf{R}$ which is defined by (47). By expressing (70) in $d = 3$, we obtain

$$\mathcal{H}_R^3(\mathbf{x}, \mathbf{x}') = \frac{-\mathrm{i}}{4\pi R\sqrt{\sinh r \sinh r'}} \times \sum_{l=0}^{\infty}(-1)^l(2l+1)P_{1/2}^{-(1/2+l)}(\cosh r_<)Q_{1/2}^{1/2+l}(\cosh r_>)P_l(\cos\gamma), \tag{81}$$

where $P_l : [-1, 1] \to \mathbf{R}$ is the Legendre polynomial defined by $P_l(x) = C_l^{1/2}(x)$, or through (13) with $\mu = 0$ and $\nu \in \mathbf{N}_0$. By using the addition theorem for hyperspherical harmonics (64) with $d = 3$ using standard spherical coordinates $(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$ to parametrize points on $\mathbf{S}^2$, we have, since the normalized spherical harmonics are

$$Y_{l,m}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi},$$

the addition theorem for spherical harmonics ((63) with $d = 3$), namely

$$P_l(\cos\gamma) = \sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')\mathrm{e}^{\mathrm{i}m(\phi-\phi')}, \tag{82}$$

where $\cos\gamma$ is given by (45). By combining (81) and (82), reversing the order of the two summation symbols and comparing the result with (46), we obtain the following single summation addition theorem for the azimuthal Fourier coefficients of a fundamental solution of Laplace's equation on $\mathbf{H}_R^3$, namely since $\mathfrak{h}^3 = 4\pi R\mathcal{H}_R^3$,

$$\mathsf{H}_m^{1/2}(r,r',\theta,\theta') = \frac{-\mathrm{i}\epsilon_m}{\sqrt{\sinh r\sinh r'}}\sum_{l=|m|}^{\infty}(-1)^l(2l+1)\frac{(l-m)!}{(l+m)!}$$
$$\times P_l^m(\cos\theta)P_l^m(\cos\theta')P_{1/2}^{-(1/2+l)}(\cosh r_<)Q_{1/2}^{1/2+l}(\cosh r_>).$$

This addition theorem reduces to the corresponding result ((2.4) in [9]) in the Euclidean $\mathbf{R}^3$ limit as $r, r' \to 0^+$.

6. Discussion

Re-arrangement of the multi-summation expressions in section 5 is possible through modification of the order in which the countably infinite space of quantum numbers is summed over in a standard hyperspherical coordinate system, namely

$$\sum_{l=0}^{\infty}\sum_{K} = \sum_{l=0}^{\infty}\sum_{k_1=0}^{l}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{d-4}=0}^{k_{d-5}}\sum_{k_{d-3}=0}^{k_{d-4}}\sum_{k_{d-2}=-k_{d-3}}^{k_{d-3}}$$
$$= \sum_{k_{d-2}=-\infty}^{\infty}\sum_{k_{d-3}=|k_{d-2}|}^{\infty}\sum_{k_{d-4}=k_{d-2}}^{\infty}\cdots\sum_{k_2=k_3}^{\infty}\sum_{k_1=k_2}^{\infty}\sum_{k_0=k_1}^{\infty}.$$

Similar multi-summation re-arrangements have been accomplished previously for azimuthal Fourier coefficients of fundamental solutions for the Laplacian in Euclidean space (see, for instance, [11, 9]). A comparison of the azimuthal Fourier expansions in section 4 (and in particular (56)) with re-arranged Gegenbauer expansions in section 5 (and in particular (70)) will yield new addition theorems for the special functions representing the azimuthal Fourier coefficients of a fundamental solution for the Laplacian on the hyperboloid. These implied addition theorems will provide new special function identities for the azimuthal Fourier coefficients, which are hyperbolic generalizations of particular associated Legendre functions of the second kind.

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