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Generalized Heine's identity for complex Fourier series of binomials

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In his treatise, Heine (Heine 1881 In *Theorie und Anwendungen*) gave an identity for the Fourier series of the function $(z-\cos\psi)^{-1/2}$, with $z,\psi\in\mathbb{R}$, and $z>1$, in terms of associated Legendre functions of the second kind $Q^0_{n-1/2}(z)$. In this paper, we generalize Heine's identity for the function $(z-\cos\psi)^{-\mu}$, with $\mu\in\mathbb{C}$, $\psi\in\mathbb{R}$ and $z\in\mathbb{C}\setminus(-\infty,1]$, in terms of $Q^{\mu-1/2}_{n-1/2}(z)$. We also compute closed-form expressions for some $Q^\mu_\nu(z)$.

Keywords: Heine's identity; Legendre functions; Fourier series

1. Introduction

Algebraic functions of the form $(z-\cos\psi)^{-\mu}$ arise naturally in classical physics through the theory of fundamental solutions of Laplace's equation, and they represent powers of distance between two points in a Euclidean space. Their expansions in Fourier series have a rich history, appearing in the theory of arbitrarily shaped charge distributions in electrostatics (Barlow 2003; Popsueva *et al.* 2007; Waltersson & Lindroth 2007; Pustovitov 2008*a*,*b*; Verdú *et al.* 2008), magnetostatics (Selvaggi *et al.* 2008*b*; Beleggia *et al.* 2009), quantum direct and exchange Coulomb interactions (Cohl *et al.* 2001; Enriquez & Blum 2005; Gattobigio *et al.* 2005; Poddar & Deb 2007; Bagheri & Ebrahimi 2008), Newtonian gravity (Fromang 2005; Huré 2005; Huré & Pierens 2005; Chan *et al.* 2006; Ou 2006; Saha & Jog 2006; Boley & Durisen 2008; Mellon & Li 2008; Selvaggi *et al.* 2008*a*; Even & Tohline 2009; Schachar *et al.* 2009), the Laplace coefficients of the planetary disturbing function (D'Eliseo 1989, 2007), and potential fluid flow around actuator discs (Hough & Ordway 1965; Breslin & Andersen 1994), just to name a few physical applications. A precise Fourier analysis is extremely useful to fully describe the general non-axisymmetric nature of these problems.

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This classical theory is very important in mathematical physics for several reasons. First, fundamental solutions of Laplace's equation are ubiquitous in pure and applied mathematics, physics and engineering. Second, Fourier expansions encapsulate rotationally invariant symmetries of geometrical shapes, making them ideal for studying symmetric configurations. They also provide a powerful tool, when implemented numerically, in a variety of important problems. Third, Fourier expansions of fundamental solutions arise in many branches of the theory of special functions, and allow one to further explore the properties of higher transcendental functions.

2. Generalized Heine's identity

Gauss (1812) was able to write closed-form expressions for the Fourier series of the function $(r_1^2 + r_2^2 - 2r_1r_2\cos\psi)^{-\mu}$. Gauss recognized that the coefficients of the expansion were given in terms of his ${}_2F_1$ hypergeometric function, and was able to write a closed-form solution given by (we have used modern notations)

$$(r_1^2 + r_2^2 - 2r_1r_2\cos\psi)^{-\mu} = \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) \frac{(\mu)_n}{n!} \frac{r_2^n}{r_1^{2\mu+n}} \, {}_2F_1\left(n+\mu, \mu; n+1; \frac{r_2^2}{r_1^2}\right), \tag{2.1}$$

where $r_1, r_2 \in \mathbb{R}$, $r_2 < r_1$, ϵ_n is the Neumann factor (Morse & Feshbach 1953),

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n \neq 0, \end{cases}$$

commonly occurring in Fourier expansions, $(\mu)_n$ is the Pochhammer symbol for a rising factorial defined by

$$(z)_0 = 1, \quad (z)_n = \prod_{j=0}^{n-1}(z+j), \quad n \geq 1, \quad z \in \mathbb{C}, \tag{2.2}$$

and the Gauss hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1, \tag{2.3}$$

with $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$. The function ${}_2F_1(a, b; c; z)$ can be analytically continued to the complex plane cut along $[1, \infty)$ (Abramowitz & Stegun 1964, ch. 15). Choosing $c = b$ in equation (2.3), we obtain the binomial expansion

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad a \in \mathbb{C}, \quad |z| < 1. \tag{2.4}$$

Carl Neumann (Neumann 1864, p. 33–34) was one of the first to study separable solutions of Laplace's equation in toroidal coordinates. He obtained a Fourier expansion in terms of hypergeometric functions. It was not until Heine's book (Heine 1881, p. 286), that these hypergeometric functions were recognized as associated Legendre functions of the second kind (hereafter referred to as

Legendre functions) with zero order and odd-half-integer degree. These Legendre functions, and in particular those with integer order, are toroidal harmonics and separate Laplace's equation in toroidal coordinates.

3. Derivation of the identity

We are interested in computing the following Fourier expansion:

$$(z-\cos\psi)^{-\mu}=\sum_{n=0}^{\infty}\cos(n\psi)A_{\mu,n}(z), \tag{3.1}$$

for $\mu\in\mathbb{C}$, $z\in\mathbb{C}\setminus(-\infty,1]$ and $\psi\in\mathbb{R}$. In the sequel, we assume

$$(z^2-w^2)^{\alpha}=(z-w)^{\alpha}(z+w)^{\alpha},\quad |\arg(z\pm w)|<\pi,$$

for any fixed $\alpha, w\in\mathbb{C}$. With these assumptions, we can rewrite the left-hand side of equation (3.1) as

$$(z-\cos\psi)^{-\mu}=z^{-\mu}\left(1-\frac{\cos\psi}{z}\right)^{-\mu},$$

where $z\in\mathbb{C}\setminus(-\infty,1]$. Choosing $|z|>1$, we have $|z^{-1}\cos\psi|<1$, and we can employ the binomial series (2.4) and obtain

$$(z-\cos\psi)^{-\mu}=z^{-\mu}\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}z^{-k}(\cos\psi)^k,\quad |z|>1. \tag{3.2}$$

We can expand the powers of $\cos\psi$ using the following trigonometric identity:

$$(\cos\psi)^k=\frac{1}{2^k}\sum_{n=0}^{k}\binom{k}{n}\cos[(2n-k)\psi],$$

which can be obtained by expanding

$$\left(\frac{e^{i\psi}+e^{-i\psi}}{2}\right)^k,$$

or using Chebyshev polynomials (Fox & Parker 1968, p. 52). Inserting this expression in equation (3.2), we obtain the following double summation expression:

$$(z-\cos\psi)^{-\mu}=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(\mu)_k}{k!}\frac{1}{2^k z^{\mu+k}}\binom{k}{n}\cos[(2n-k)\psi],\quad |z|>1. \tag{3.3}$$

Now, we perform a double-index replacement in equation (3.3). There are two separate cases: $k\le 2n$ and $k\ge 2n$. There is an overlap if both have an equality and in that case, we must multiply by 1/2 after we sum over both cases. For $k\le 2n$, we make the substitution $k' = k - n$ and $n' = 2n - k$. It follows that $k = 2k' + n'$, $n = n' + k'$, and therefore

$$\binom{k}{n} = \binom{2k'+n'}{n'+k'} = \binom{2k'+n'}{k'}.$$

For $k \geq 2n$, we make the substitution $k' = n$ and $n' = k - 2n$. Then, $k = 2k' + n'$, $n = k'$, and therefore

$$\binom{k}{n} = \binom{2k'+n'}{k'} = \binom{2k'+n'}{k'+n'},$$

where the equalities of the binomial coefficients follow from the identity

$$\binom{l}{m} = \binom{l}{l-m}. \tag{3.4}$$

To take into account the double counting, which occurs when $k = 2n$ $(n' = 0)$, we introduce a factor of $(1/2)\epsilon_{n'}$ into the expression (and relabel $k' \mapsto k$ and $n' \mapsto n$). We are left with

$$(z - \cos\psi)^{-\mu} = \frac{1}{2z^{\mu}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) \sum_{k=0}^{\infty} \frac{(\mu)_{2k+n}}{(2k+n)!} \frac{1}{(2z)^{2k+n}} \times \left[\binom{2k+n}{k} + \binom{2k+n}{k+n}\right], \quad |z| > 1,$$

which is simplified using the definition of the binomial coefficients and equation (3.4),

$$(z - \cos\psi)^{-\mu} = \frac{1}{z^{\mu}} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\psi)}{2^n z^{\mu+n}} \sum_{k=0}^{\infty} \frac{(\mu)_{2k+n}}{k!(k+n)!} \frac{1}{4^k} \left(\frac{1}{z^2}\right)^k, \quad |z| > 1.$$

The second sum is given in terms of a hypergeometric series

$$(z - \cos\psi)^{-\mu} = \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\psi)(\mu)_n}{n! 2^n z^{\mu+n}} \times {}_2F_1\left(\frac{\mu+n}{2}, \frac{\mu+n+1}{2}; n+1; \frac{1}{z^2}\right), \quad |z| > 1. \tag{3.5}$$

Setting

$$z = \frac{r_1^2 + r_2^2}{2r_1 r_2},$$

we recover Gauss' original formula (2.1), only now $\mu \in \mathbb{C}$.

Using the quadratic transformation (Abramowitz & Stegun 1964, eqn 15.3.19)

$$ {}_2F_1\left(a, a+\frac{1}{2}; c; z\right) = 2^{2a}(1+\sqrt{1-z})^{-2a} \times {}_2F_1\left(2a, 2a-c+1; c; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right), \quad |\arg(1-z)| < \pi, $$

in equation (3.5), we obtain

$$ \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\psi)(\mu)_n}{n!2^n z^{\mu+n}} {}_2F_1\left(\frac{\mu+n}{2}, \frac{\mu+n+1}{2}; n+1; \frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{\epsilon_n 2^{\mu} \cos(n\psi)(\mu)_n}{n!\phi^{\mu+n}(z)} {}_2F_1\left(\mu+n, \mu; n+1; \frac{1}{\phi^2(z)}\right), \quad |\arg(z-1)| < \pi, \tag{3.6} $$

with $\phi(z) = z + \sqrt{z^2-1}$. The function $\phi(z)$ is uniquely defined on the entire complex plane in which a cut is made from $-\infty$ to $+1$. Here, we are considering that branch of $\sqrt{z^2-1}$, which is positive when $z \in (1, \infty)$. Hence, equation (3.6) provides the analytic continuation of equation (3.5) to $z \in \mathbb{C} \setminus (-\infty, 1]$.

The hypergeometric function in equation (3.5) is expressible in terms of Legendre functions (Abramowitz & Stegun 1964, p. 332) by

$$ {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; \frac{1}{z^2}\right) = \sqrt{\frac{2}{\pi}} \frac{2^{\nu}\Gamma(\nu+1) z^{\nu+\mu}(z^2-1)^{-\mu/2+1/4} e^{-i\pi(\mu-1/2)}}{\Gamma(\nu+\mu)} Q_{\nu-1/2}^{\mu-1/2}(z), \tag{3.7} $$

where $\nu, \mu \in \mathbb{C}$ and $|z| > 1$, and by (Gradshteyn & Ryzhik 2007, eqn 8.777)

$$ {}_2F_1\left(\alpha+\nu+1, \alpha+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{\phi^2(z)}\right) = \frac{2^{-\alpha}}{\sqrt{\pi}} e^{-i\pi\alpha} \frac{\Gamma(\nu+3/2)}{\Gamma(\nu+\alpha+1)} \frac{\phi^{\alpha+\nu+1}(z)}{(z^2-1)^{\alpha/2}} Q_{\nu}^{\alpha}(z), \tag{3.8} $$

where $\alpha, \nu \in \mathbb{C}$ and $|\arg(z-1)| < \pi$. Using equation (3.8) in (3.6), with $\alpha = \mu - 1/2$ and $\nu = n - 1/2$, we conclude that

$$ (z-\cos\psi)^{-\mu} = \frac{1}{\Gamma(\mu)} \sqrt{\frac{2}{\pi}} \frac{e^{-i\pi(\mu-1/2)}}{(\sqrt{z^2-1})^{\mu-1/2}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) Q_{n-1/2}^{\mu-1/2}(z), \tag{3.9} $$

for $\mu \in \mathbb{C}$ and $|\arg(z-1)| < \pi$.

If we consider $n \in \mathbb{Z}$ in equation (3.9) and take advantage of the following property (Cohl *et al.* 2000):

$$ Q_{-n-1/2}^{\mu}(z) = Q_{n-1/2}^{\mu}(z), \quad n \in \mathbb{Z}, \quad \mu \in \mathbb{C}, $$

we obtain a complex generalization of Heine's reciprocal square-root identity given as

$$(z-\cos\psi)^{-\mu}=\sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\mathrm{i}\pi(\mu-1/2)}(z^2-1)^{-\mu/2+1/4}}{\Gamma(\mu)}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\psi}Q_{n-1/2}^{\mu-1/2}(z), \tag{3.10}$$

with $\mu\in\mathbb{C}$, $\psi\in\mathbb{R}$ and $|\arg(z-1)|<\pi$, where Heine's original identity (Heine 1881, p. 286) is the case $\mu=1/2$ given by

$$\frac{1}{\sqrt{z-\cos\psi}}=\frac{\sqrt{2}}{\pi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\psi}Q_{m-1/2}(z). \tag{3.11}$$

See Cohl & Tohline (1999) for exact forms of these toroidal harmonics and their recurrence relations.

The expansion (3.10) appeared in Magnus *et al.* (1966, p. 182) and more recently in Conway (2007). The results in these references contain the restriction that $\mathrm{Re}(\mu)>0$. Recently, Selvaggi *et al.* (2008*b*) generalized equation (3.11) for μ given by odd-half-integers, even those less than or equal to $-1/2$ (this was also suggested in Cohl 2003). Our result confirms the fact suggested by Selvaggi *et al.* (2008*b*) that equation (3.10) is not only valid for $\mathrm{Re}(\mu)>0$, but over the entire complex μ-plane.

The generalization given by equation (3.10) can also be expressed in terms of associated Legendre functions of the first kind through Whipple's transformation (Abramowitz & Stegun 1964; Cohl *et al.* 2000), as

$$(z-\cos\psi)^{-\mu}=\frac{(z^2-1)^{-\mu/2}}{\Gamma(\mu)}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\psi}\Gamma(\mu-n)P_{\mu-1}^{n}\left(\frac{z}{\sqrt{z^2-1}}\right).$$

In the following section, we describe some specific examples of the generalized identity and present some interesting implications.

4. Examples and implications

In the previous section, we derived the value of the integral

$$\int_{-\pi}^{\pi}\frac{\cos(nt)}{(z-\cos t)^{\mu}}\,\mathrm{d}t=2^{3/2}\sqrt{\pi}\frac{\mathrm{e}^{-\mathrm{i}\pi(\mu-1/2)}(z^2-1)^{-\mu/2+1/4}}{\Gamma(\mu)}Q_{n-1/2}^{\mu-1/2}(z), \tag{4.1}$$

for $z\in\mathbb{C}\setminus(-\infty,1]$, $\mu\in\mathbb{C}$. We may compare this definite integral with the integral representation of $Q_{\nu}^{\mu}(z)$ given in Gradshteyn & Ryzhik (2007, eqn 8.713.1). In that integral representation, if we make a replacement $\mu\mapsto\mu-(1/2)$ and set $\nu=n-1/2$, where $n\in\mathbb{Z}$, we can obtain the same form as equation (4.1). However, in that reference, the restriction that $\mathrm{Re}(\mu)>0$ is given, while we have shown equation (4.1) to hold for all $\mu\in\mathbb{C}$.

The principle example for the generalized Heine identity when $\mu=q+1/2$, $q\in\mathbb{Z}$ is given by

$$\frac{1}{(z-\cos\psi)^{q+1/2}}=\frac{2^{q+1/2}(-1)^q}{\pi(2q-1)!!(z^2-1)^{q/2}}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)\,Q_{n-1/2}^{q}(z), \tag{4.2}$$

where $z!!$ is the double factorial (Spanier & Oldham 1987, §2:13), defined by $(-1)!! = (0)!! = 1$ and

$$(n)!! = \prod_{j=0}^{\lfloor n/2 \rfloor - 1} (n-2j), \quad n \in \mathbb{N}.$$

The expansion (4.2) was first proved by Selvaggi *et al.* (2008*b*) (see also Cohl (2003) and Magnus *et al.* (1966, p. 182)). Notice that the double factorial is well-defined for negative odd integers (Arfken & Weber 1995),

$$(-2n-1)!! = \frac{(-1)^n 2^n \; n!}{(2n)!}.$$

For instance, for $q = -1$, we have

$$\sqrt{z - \cos\psi} = \frac{\sqrt{z^2-1}}{\sqrt{2}\pi} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\psi)}{n^2 - 1/4} Q^1_{n-1/2}(z),$$

and for $q = 1$,

$$(z - \cos\psi)^{-3/2} = -\frac{2^{3/2}}{\pi\sqrt{z^2-1}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi)\, Q^1_{n-1/2}(z). \tag{4.3}$$

The minus sign in the expansion (4.3) might seem troublesome, but note that the unit-order Legendre functions are all negative, as can be easily seen from the hypergeometric function representation (3.7).

We also have

$$Q^1_{-1/2}(z) = -\frac{1}{\sqrt{2(z-1)}} E\left(\sqrt{\frac{2}{z+1}}\right)$$

and

$$Q^1_{1/2}(z) = -\frac{z}{\sqrt{2(z-1)}} E\left(\sqrt{\frac{2}{z+1}}\right) + \sqrt{\frac{z-1}{2}} K\left(\sqrt{\frac{2}{z+1}}\right),$$

where K and E are the complete elliptic integrals of the first and second kind, respectively (Abramowitz & Stegun 1964, ch. 17). The rest of the unit-order, odd-half-integer-degree Legendre functions can be computed using the following recurrence relation (Abramowitz & Stegun 1964, eqn 8.5.3):

$$(\nu - \mu + 1)\, Q^{\mu}_{\nu+1}(z) = (2\nu + 1) z Q^{\mu}_{\nu}(z) - (\nu + \mu)\, Q^{\mu}_{\nu-1}(z).$$

Thus, all odd-half-integer-degree, integer-order Legendre functions can be written in terms of elliptic integrals of the first and second kind. The analogous formulas for $q = 0$ were given by Cohl & Tohline (1999).

When μ is a negative integer, the binomial expansion reduces to a polynomial in z. Using the definition of the Pochhammer symbol $(-q)_n$, equation (2.2) and the negative-order condition for the Legendre functions (Cohl *et al.* 2000, p. 367), we see that

$$\frac{1}{\Gamma(-q)}\,Q_{n-1/2}^{-q-1/2}(z) = -\frac{(-q)_n}{\Gamma(n+q+1)}\,Q_{n-1/2}^{q+1/2}(z).$$

Using this property and setting $\mu = -q$ in equation (3.10), we obtain

$$(z-\cos\psi)^q = -\mathrm{i}\sqrt{\frac{2}{\pi}}(-1)^q(z^2-1)^{q/2+1/4} \times \sum_{n=0}^{q} \epsilon_n \cos(n\psi)\frac{(-q)_n}{(q+n)!}\,Q_{n-1/2}^{q+1/2}(z), \tag{4.4}$$

where $q \in \mathbb{N}_0$.

Now we examine the case when μ is a natural number. Substituting $\mu = q \in \mathbb{N}$ in equation (3.10), yields

$$(z-\cos\psi)^{-q} = \mathrm{i}\sqrt{\frac{2}{\pi}}\frac{(-1)^q(z^2-1)^{-q/2+1/4}}{(q-1)!}\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)\,Q_{n-1/2}^{q-1/2}(z), \tag{4.5}$$

where the right-hand side is positive, as can be seen using equation (3.7). For $q = 1$, the coefficients are Legendre functions with odd-half-integer degree and order half. These can be evaluated using (Abramowitz & Stegun 1964, eqns 8.6.10–8.6.11)

$$Q_\nu^{1/2}(z) = -\left(\nu+\frac{1}{2}\right)Q_\nu^{-1/2}(z) = \mathrm{i}\sqrt{2\pi}(z^2-1)^{-1/4}\left[z+\sqrt{z^2-1}\right]^{-\nu-1/2}. \tag{4.6}$$

If we take $z = \cosh\eta$ and $\nu = n - 1/2$, where $n \in \mathbb{Z}$, and insert the resulting expression in equation (4.5), we obtain

$$\frac{1}{\cosh\eta-\cos\psi} = \frac{1}{\sinh\eta}\sum_{n=0}^{\infty}\epsilon_n \mathrm{e}^{-n\eta}\cos(n\psi). \tag{4.7}$$

Similarly, if we use equation (4.6) and the order recurrence relation for Legendre functions (Abramowitz & Stegun 1964, eqn 8.5.1),

$$Q_\nu^{\mu+2}(z) + 2(\mu+1)\frac{z}{\sqrt{z^2-1}}\,Q_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)\,Q_\nu^{\mu}(z), \tag{4.8}$$

we are able to compute all the odd-half-integer-order Legendre functions appearing in equation (4.5).

Expansions for $(\cosh\eta - \cos\psi)^{-q}$ can be obtained from equation (4.7) by repeated differentiation with respect to η. In the next section, we offer an alternative approach.

5. Closed-form expressions for certain Legendre functions

From equations (3.1) and (3.5), we have

$$(z-\cos\psi)^{-\mu}=\sum_{n=0}^{\infty}\cos(n\psi)A_{\mu,n}(z),$$

with $\mu\in\mathbb{C}$, $z\in\mathbb{C}\setminus(-\infty,1]$ and $\psi\in\mathbb{R}$, where

$$A_{\mu,n}(z)=\frac{\epsilon_n(\mu)_n}{2^n n!z^{\mu+n}}{}_2F_1\left(\frac{\mu+n}{2},\frac{\mu+n+1}{2};n+1;\frac{1}{z^2}\right).$$

We let

$$z=\cosh\eta=\frac{1}{2}(x+x^{-1}),\qquad x=\mathrm{e}^{-\eta}.$$

Taking $\mathrm{Re}(\eta)>0$, we get $|x|<1$ and

$$A_{\mu,n}(\cosh\eta)=\epsilon_n\frac{2^{\mu}(\mu)_n}{n!}\left(\frac{x}{x^2+1}\right)^{\mu+n}\times{}_2F_1\left(\frac{\mu+n}{2},\frac{\mu+n+1}{2};n+1;\frac{4x^2}{(1+x^2)^2}\right).\tag{5.1}$$

Using (Andrews *et al.* 1999, eqn 3.1.9)

$${}_2F_1(a,b;a-b+1;w)=(1+w)^{-a}{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4w}{(1+w)^2}\right),$$

($|w|<1$), in equation (5.1), with $a=\mu+n$ and $b=\mu$, we get

$$A_{\mu,n}(\cosh\eta)=\epsilon_n\frac{(\mu)_n}{n!}2^{\mu}x^{\mu+n}\ {}_2F_1(\mu+n,\mu;n+1;x^2).\tag{5.2}$$

Using Pfaff's transformation (Andrews *et al.* 1999, eqn 2.2.6),

$${}_2F_1(a,b;c;w)=(1-w)^{-a}\ {}_2F_1\left(a,c-b;b;\frac{w}{w-1}\right),\qquad|\arg(1-w)|<\pi,$$

in equation (5.2), we obtain

$$A_{\mu,n}(\cosh\eta)=\epsilon_n\frac{(\mu)_n}{n!}2^{\mu}x^{\mu+n}(1-x^2)^{-\mu}\ {}_2F_1\left(\mu,1-\mu;n+1;\frac{x^2}{x^2-1}\right),$$

or

$$A_{\mu,n}(\cosh\eta)=\epsilon_n\frac{(\mu)_n}{n!}\frac{\mathrm{e}^{-n\eta}}{(\sinh\eta)^{\mu}}{}_2F_1\left(\mu,1-\mu;n+1;-\frac{\mathrm{e}^{-\eta}}{2\sinh\eta}\right).$$

Hence, we conclude that

$$\frac{(\sinh\eta)^{\mu}}{(\cosh\eta-\cos\psi)^{\mu}}=\sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)\frac{(\mu)_n}{n!}\mathrm{e}^{-n\eta}\ {}_2F_1\left(1-\mu,\mu;n+1;-\frac{\mathrm{e}^{-\eta}}{2\sinh\eta}\right),\tag{5.3}$$

for $\mu\in\mathbb{C}$, $\psi\in\mathbb{R}$ and $\mathrm{Re}(\eta)>0$.

Taking $\mu = 1$ in equation (5.3), we get

$$\frac{\sinh\eta}{\cosh\eta - \cos\psi} = \sum_{n=0}^{\infty} \epsilon_n \cos(n\psi) \frac{(\mu)_n}{n!} \mathrm{e}^{-n\eta},$$

since ${}_2F_1(0,1;c;w) = 1$, and we recover equation (4.7). In general, for $\mu = q \in \mathbb{N}$, we can write

$$A_{q,n}(\cosh\eta) = \frac{\mathrm{e}^{-n\eta}}{(\sinh\eta)^q} \sum_{k=0}^{q-1} \binom{q+n-1}{n+k} \binom{q+k-1}{k} \frac{\mathrm{e}^{-k\eta}}{2^k (\sinh\eta)^k},$$

or using equation (3.4)

$$A_{q,n}(\cosh\eta) = \frac{\mathrm{e}^{-n\eta}}{(\sinh\eta)^q} \sum_{k=0}^{q-1} \binom{n+q-1}{q-k-1} \binom{q+k-1}{q-1} \frac{\mathrm{e}^{-k\eta}}{2^k (\sinh\eta)^k}.$$

Using this formula, we are able to write a formula for odd-half-integer-degree, odd-half-integer-order Legendre functions

$$Q_{n-1/2}^{q-1/2}(z) = \sqrt{\frac{\pi}{2}} \frac{\mathrm{i}(-1)^{q+1}(z - \sqrt{z^2-1})^n}{(q-1)!(z^2-1)^{1/4}} \times \sum_{k=0}^{q-1} \binom{n+q-1}{q-k-1} \binom{q+k-1}{q-1} \left[\frac{z - \sqrt{z^2-1}}{2(z^2-1)^{1/2}}\right]^k,$$

or in terms of Pochhammer symbols

$$Q_{n-1/2}^{q-1/2}(z) = \mathrm{i}(-1)^{q+1} \sqrt{\frac{\pi}{2}} \frac{\Gamma(q+n)}{n!} \frac{(z - \sqrt{z^2-1})^n}{(z^2-1)^{1/4}} \times \sum_{k=0}^{q-1} \frac{(q)_k (1-q)_k}{(n+1)_k k!} \left[\frac{-z + \sqrt{z^2-1}}{2(z^2-1)^{1/2}}\right]^k,$$

since

$$(-1)^k \frac{(q)_n (q)_k (1-q)_k}{(n+1)_k k! n!} = \binom{q+k-1}{k} \binom{q+n-1}{n+k}.$$

This leads us to the following general formula for $q \in \mathbb{Z}$ (Magnus *et al.* 1966, p. 162):

$$Q_{\nu}^{q-1/2}(z) = \mathrm{i}(-1)^{q+1} \sqrt{\frac{\pi}{2}} \frac{\Gamma(q+\nu+1/2)}{\Gamma(\nu+3/2)} \frac{(z + \sqrt{z^2-1})^{-\nu-1/2}}{(z^2-1)^{1/4}} \times \sum_{k=0}^{|q-1/2|-1/2} \frac{(q)_k (1-q)_k}{(\nu+3/2)_k k!} \left[\frac{-z + \sqrt{z^2-1}}{2(z^2-1)^{1/2}}\right]^k. \qquad (5.4)$$

This is a generalization of the very important formulae in Abramowitz & Stegun (1964, eqns 8.6.10–8.6.11). Taking $z = \cosh\eta$, we have

$$\begin{aligned} Q_\nu^{q-1/2}(\cosh\eta) &= \mathrm{i}(-1)^{q+1}\sqrt{\frac{\pi}{2}}\frac{\Gamma(q+\nu+1/2)}{\Gamma(\nu+3/2)}\frac{\mathrm{e}^{-\eta(\nu+1/2)}}{\sqrt{\sinh\eta}} \\ &\quad\times\sum_{k=0}^{|q-1/2|-1/2}\frac{(q)_k(1-q)_k}{(\nu+3/2)_k\ k!}\frac{(-1)^k\mathrm{e}^{-k\eta}}{2^k(\sinh\eta)^k}. \end{aligned} \tag{5.5}$$

An alternative procedure for computing these Legendre functions is to start with equation (4.6) and use the order recurrence relation (4.8). On the other hand, the expressions (5.4) and (5.5) give closed-form expressions for the Legendre functions by evaluating a finite sum.

6. Conclusions

The generalized Heine identity is useful for studying both Poisson's equation in three dimensions and fundamental solutions of Δ^k in $\mathbb{R}^n$, where Δ denotes the Laplacian operator. These solutions are given in terms of a functional form, which matches the generalized Heine identity when n is odd and for $k \le n/2 - 1$ when n is even. The generalized Heine identity can also be used as a powerful tool for expressing geometric properties (multi-summation addition theorems) of fundamental solutions in rotationally invariant coordinate systems.

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