



Regular Articles

Gegenbauer expansions and addition theorems for a binomial and logarithmic fundamental solution of the even-dimensional Euclidean polyharmonic equation



Howard S. Cohl^{a,*}, Jessie E. Hirtenstein^b, Jim Lawrence^{c,a}, Lisa Ritter^a

^a Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, MD 20899-8910, USA

^b Department of Physics, University of California, Davis, CA 95616, USA

^c Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA

ARTICLE INFO

Article history:

Received 17 December 2021

Available online 4 August 2022

Submitted by S. Cooper

Keywords:

Logarithmic fundamental solutions

Even dimensional Cartesian space

Gegenbauer polynomial expansions

ABSTRACT

On even-dimensional Euclidean space for integer powers of the positive Laplace operator greater than or equal to half the dimension, a fundamental solution of the polyharmonic equation has binomial and logarithmic behavior. Gegenbauer polynomial expansions of these fundamental solutions are obtained through a limit applied to Gegenbauer expansions of a power-law fundamental solution of the polyharmonic equation. This limit is accomplished through parameter differentiation. By combining these results with previously derived azimuthal Fourier series expansions for these binomial and logarithmic fundamental solutions, we are able to obtain addition theorems for the azimuthal Fourier coefficients. These logarithmic and binomial addition theorems are expressed in Vilenkin polyspherical geodesic polar coordinate systems and as well in generalized Hopf coordinates on hyperspheres in arbitrary even dimensions.

Published by Elsevier Inc.

1. Introduction

Analysis of polyharmonic operators (natural powers of the Laplace operator) are ubiquitous in many areas of pure and applied mathematics and as well in physics and engineering problems. Here we concern ourselves with a fundamental solution of the polyharmonic equation (Laplace, biharmonic, etc.), which is connected to solutions of the inhomogeneous polyharmonic equation. Solutions to inhomogeneous polyharmonic equations are useful in many physical applications including those areas related to Poisson's equation such as Newtonian gravity, electrostatics, magnetostatics, quantum direct and exchange interactions [11, §1],

* Corresponding author.

E-mail addresses: howard.cohl@nist.gov (H.S. Cohl), jhirtenstein@ucdavis.edu (J.E. Hirtenstein), lawrence@gmu.edu (J. Lawrence), lisa.ritter@nist.gov (L. Ritter).

URL: <http://www.nist.gov/itl/math/msg/howard-s-cohl.cfm> (H.S. Cohl).

etc. Furthermore, applications of higher-powers of the Laplace operator include such varied areas as minimal surfaces [27], continuum mechanics [23], mesh deformation [21], elasticity [24], Stokes flow [22], geometric design [34], cubature formulae [31], mean value theorems (cf. Pizzetti's formula) [28], and Hartree-Fock calculations of nuclei [35].

Closed-form expressions for the Fourier expansions of a logarithmic fundamental solution for the polyharmonic equation are extremely useful when solving inhomogeneous polyharmonic problems on even-dimensional Euclidean space, especially when a degree of rotational symmetry is involved. A fundamental solution of the polyharmonic equation on d -dimensional Euclidean space \mathbb{R}^d has two arguments and therefore maps from a $2d$ -dimensional space to the reals. Solutions to the inhomogeneous polyharmonic equation can be obtained by convolution of a fundamental solution with an integrable source distribution. Eigenfunction decompositions of a fundamental solution reduces the dimension of the resulting convolution integral to obtain Dirichlet boundary values in order to solve the resulting elliptic system, replacing it instead by a sum or an integral over some parameter space. By taking advantage of rotational or nearly rotational symmetry in the azimuthal Fourier decomposition of the source distribution, one reduces the dimensionality of the resulting convolution integral and obtains a rapidly convergent Fourier cosine expansion. In the case of an axisymmetric (constant angular dependence) source distribution, the entire contribution to the boundary values are determined by a single term in the azimuthal Fourier series. These kinds of expansions have been previously shown to be extremely effective in solving inhomogeneous problems (see for instance the discussion [14] and those papers which cite it).

It is well-known (see for [29, p. 45], [20, p. 202]) that a fundamental solution of the polyharmonic equation on d -dimensional Euclidean space \mathbb{R}^d is given by combinations of power-law and logarithmic functions of the global distance between two points. In [11], the Fourier coefficients of a power-law fundamental solution of the polyharmonic equation were obtained. Gegenbauer and Jacobi polynomial expansions of a power-law fundamental solution of the polyharmonic equation (which generalize the Fourier expansions presented in [11]) were obtained in [9]. The coefficients of these expansions are seen to be given in terms of associated Legendre functions. Fourier expansions of a logarithmic and binomial fundamental solution of the polyharmonic equation were obtained in [8].

The work presented in this manuscript is concerned with computing the Gegenbauer coefficients of binomial and logarithmic fundamental solutions of the polyharmonic equation. One obtains a logarithmic fundamental solution for the polyharmonic equation only on even-dimensional Euclidean space and only when the power of the Laplace operator is greater than or equal to the dimension divided by two. The most familiar example of a logarithmic fundamental solution of the polyharmonic equation occurs in two-dimensions, for a single-power of the Laplace operator, i.e., Laplace's equation. In this manuscript we present an approach for obtaining the Gegenbauer expansion of binomial and logarithmic fundamental solutions of the polyharmonic equation by parameter differentiation of a power-law fundamental solution of the polyharmonic equation.

This manuscript is organized as follows. In Section 2 we introduce the fundamental mathematical sets, sequences, functions and orthogonal polynomials, which are necessary to understand the mathematical details of this manuscript. In Section 3 we describe the properties of binomial and logarithmic fundamental solutions of the polyharmonic equation in even-dimensional Euclidean space \mathbb{R}^d —and in particular in rotationally-invariant coordinate systems and in Vilenkin polyspherical coordinates. In Section 4, we derive using a limit-derivative approach, the Fourier cosine and Gegenbauer polynomial expansions for the kernels which occur in binomial and logarithmic fundamental solutions of the polyharmonic equation in Euclidean space. These kernels are $(z-x)^p$ and $(z-x)^p \log(z-x)$, where $z > 1$, $x \in (-1, 1)$ and $p \in \mathbb{N}_0$. In Section 5 we use the results presented in the previous sections to obtain the azimuthal Fourier and Gegenbauer expansions for binomial and logarithmic fundamental solution of the polyharmonic equation in rotationally-invariant coordinate systems in even-dimensional Euclidean space. In Section 7 we derive addition theorems for the

azimuthal Fourier coefficients for the binomial and logarithmic fundamental solutions of the polyharmonic equation in of even-dimensional Euclidean space.

2. Preliminaries

Here we introduce some nomenclature which we will rely upon in the text below. Throughout this paper we adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \dots\}$, and we use the sets $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ to represent the integers, real and complex numbers respectively. As is the common convention for associated Legendre functions [1, (8.1.1)], for any expression of the form $(z^2 - 1)^\alpha$, read this as $(z^2 - 1)^\alpha := (z + 1)^\alpha(z - 1)^\alpha$, for any fixed $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 1]$. Over the set of complex numbers, we assume that empty sum vanishes and the empty product is unity. Given two numbers: $r, r' \in \mathbb{R}$, define

$$r_{\leq} := \min_{\max}\{r, r'\}. \tag{2.1}$$

The harmonic number $H_n \in \mathbb{Q}, n \in \mathbb{Z}$, is given by

$$H_n := \begin{cases} 0, & \text{if } n \leq 0, \\ \sum_{k=1}^n \frac{1}{k}, & \text{if } n \geq 1. \end{cases} \tag{2.2}$$

The harmonic number is related to the polygamma function [18, (5.2.2)] with $n \in \mathbb{N}_0$ since [25, p. 14]

$$\psi(n + 1) = -\gamma + H_n,$$

where $\gamma \approx 0.57721\dots$ is the Euler-Mascheroni constant [18, (5.2.3)].

The Pochhammer symbol (shifted factorial) is defined for $a \in \mathbb{C}, n \in \mathbb{N}$, such that

$$(a)_0 := 1, \quad (a)_n := (a)(a + 1) \cdots (a + n - 1). \tag{2.3}$$

One also has

$$(-p)_n = \begin{cases} \frac{(-1)^p p!}{(p - n)!} & \text{if } 0 \leq n \leq p, \\ 0 & \text{if } n \geq p + 1. \end{cases} \tag{2.4}$$

Note for $z \in \mathbb{C}, k \in \mathbb{N}_0$, the binomial coefficient can be given in terms of the Pochhammer symbol as follows [18, (1.2.6)]

$$\binom{z}{k} = \frac{(-1)^k (-z)_k}{k!}. \tag{2.5}$$

The Gauss hypergeometric function ${}_2F_1 : \mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus -\mathbb{N}_0) \times \mathbb{C} \setminus [1, \infty)$ [18, Chapter 15] is defined as

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

for $|z| < 1$ and through analytical continuation for the rest of its domain. If one takes $b = c$ in the Gauss hypergeometric function, one produces the binomial theorem [18, (15.4.6)]

$${}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right) = (1-z)^{-a}. \quad (2.6)$$

Associated Legendre functions of the first and second kind $P_\nu^\mu, Q_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \rightarrow \mathbb{C}$ [18, (14.3.6-7) and Section 14.21] are defined as

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2F_1\left(\begin{matrix} -\nu, \nu+1 \\ 1-\mu \end{matrix}; \frac{1-z}{2}\right), \quad (2.7)$$

for $|1-z| < 2$,

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi} e^{i\mu\pi} \Gamma(\nu+\mu+1) (z^2-1)^{\frac{\mu}{2}}}{2^{\nu+1} \Gamma(\nu+\frac{3}{2}) z^{\nu+\mu+1}} {}_2F_1\left(\begin{matrix} \frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2} \\ \nu+\frac{3}{2} \end{matrix}; \frac{1}{z^2}\right), \quad (2.8)$$

for $|z| > 1$, and elsewhere in z by analytic continuation of the Gauss hypergeometric function. A property of the associated Legendre functions that we will take advantage of are the Whipple formulae [18, (14.9.16-17)] (these are equivalent to each other)

$$Q_\nu^\mu(z) = \sqrt{\frac{\pi}{2}} e^{i\mu\pi} \Gamma(\nu+\mu+1) (z^2-1)^{-\frac{1}{4}} P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right), \quad (2.9)$$

$$P_\nu^\mu(z) = i \sqrt{\frac{2}{\pi}} \frac{e^{i\nu\pi} (z^2-1)^{-\frac{1}{4}}}{\Gamma(-\nu-\mu)} Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right), \quad (2.10)$$

for $\Re z > 0$. These allow one to convert between the associated Legendre functions of the first and second kind. We also take advantage of [18, (14.9.14)]

$$Q_\nu^{-\mu}(z) = e^{-2i\pi\mu} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} Q_\nu^\mu(z). \quad (2.11)$$

Some useful special cases are (cf. [18, (14.5.17)])

$$Q_{\frac{1}{2}}^{\frac{1}{2}}(z) = \frac{i\sqrt{\pi/2}}{(z^2-1)^{\frac{1}{4}}(z+\sqrt{z^2-1})}, \quad (2.12)$$

and

$$Q_{n+\frac{1}{2}}^{\frac{3}{2}}(z) = \frac{-i\sqrt{\pi/2}(z+(n+1)\sqrt{z^2-1})}{(z^2-1)^{\frac{3}{4}}(z+\sqrt{z^2-1})^{n+1}}, \quad (2.13)$$

which follows from [18, (14.10.6)] and [18, (14.5.17)]. One also has

$$Q_{\mu-\frac{1}{2}}^{\mu+\frac{1}{2}}(z) = \frac{e^{i\pi(\mu+\frac{1}{2})} 2^{\mu-\frac{1}{2}} \Gamma(\mu+\frac{1}{2})}{(z^2-1)^{\frac{\mu}{2}+\frac{1}{4}}}, \quad (2.14)$$

where in this formula $\mu \in \mathbb{C} \setminus \{-\frac{1}{2}, -1, -\frac{3}{2}, \dots\}$, which follows using (2.8) and the binomial theorem (2.6).

Let $p, m \in \mathbb{N}_0$. In regard to parameter differentiation of associated Legendre functions of the first kind, then Szmytkowski derived [33, cf. (5.12)]

$$\left[\frac{\partial}{\partial \nu} P_\nu^m(z) \right]_{\nu=p} = \begin{cases} \left(\log \frac{z+1}{2} + 2H_{2p} - H_p - H_{p-m} \right) P_p^m(z) \\ + (-1)^{p+m} \sum_{k=0}^{p-m-1} (-1)^k \frac{(2k+2m+1) \left[1 + \frac{k!(p+m)!}{(k+2m)!(p-m)!} \right]}{(p-m-k)(p+m+k+1)} P_{k+m}^m(z) \\ + (-1)^p \frac{(p+m)!}{(p-m)!} \sum_{k=0}^{m-1} \frac{(-1)^k (2k+1)}{(p-k)(p+k+1)} P_k^{-m}(z), & \text{if } 0 \leq m \leq p, \\ (-1)^{p+m+1} (p+m)!(m-p-1)! P_p^{-m}(z), & \text{if } m \geq p+1. \end{cases} \tag{2.15}$$

One interesting consequence of Szmytkowski’s formula is the following corollary which does not seem to have appeared elsewhere in the literature.

Corollary 2.1. *Let $p, m \in \mathbb{N}_0$, $0 \leq m \leq p$, $z \in \mathbb{C} \setminus (-\infty, 1]$. Then*

$$\begin{aligned} \left[\frac{\partial}{\partial \nu} Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}}(z) \right]_{\nu=p} &= \left[2H_{2p} - H_p - \gamma + i\pi + \log \frac{z + \sqrt{z^2 - 1}}{2\sqrt{z^2 - 1}} \right] Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}}(z) \\ &+ (p-m)! \sum_{k=0}^{p-m-1} \frac{(2k+2m+1) \left[1 + \frac{k!(p+m)!}{(k+2m)!(p-m)!} \right]}{k!(p-m-k)(p+m+k+1)} Q_{m-\frac{1}{2}}^{k+m+\frac{1}{2}}(z) \\ &+ (p+m)! \sum_{k=0}^{m-1} \frac{2k+1}{(k+m)!(p-k)(p+k+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(z). \end{aligned} \tag{2.16}$$

Proof. Starting with (2.10) for $\mu = m \in \mathbb{N}_0$, and the reflection formula for the gamma function [18, (5.5.3)], we have

$$P_\nu^m(z) = \frac{-i}{(z^2 - 1)^{\frac{1}{4}}} \sqrt{\frac{2}{\pi}} \frac{e^{-i\pi\nu}}{\Gamma(\nu - m + 1)} Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}} \left(\frac{z}{\sqrt{z^2 - 1}} \right).$$

Then replace

$$\begin{aligned} \left[\frac{\partial}{\partial \nu} P_\nu^m(z) \right]_{\nu=p} &= \frac{-i}{(z^2 - 1)^{\frac{1}{4}}} \sqrt{\frac{2}{\pi}} \left[\frac{\partial}{\partial \nu} \frac{e^{-i\pi\nu}}{\Gamma(\nu - m + 1)} Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}} \left(\frac{z}{\sqrt{z^2 - 1}} \right) \right]_{\nu=p} \\ &= \frac{-i}{(z^2 - 1)^{\frac{1}{4}}} \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\partial}{\partial \nu} \frac{e^{-i\pi\nu}}{\Gamma(\nu - m + 1)} \right]_{\nu=p} Q_{m-\frac{1}{2}}^{p+\frac{1}{2}} \left(\frac{z}{\sqrt{z^2 - 1}} \right) + \frac{(-1)^p}{(p-m)!} \left[\frac{\partial}{\partial \nu} Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}} \left(\frac{z}{\sqrt{z^2 - 1}} \right) \right]_{\nu=p} \right\}, \end{aligned}$$

with

$$\left[\frac{\partial}{\partial \nu} \frac{e^{-i\pi\nu}}{\Gamma(\nu - m + 1)} \right]_{\nu=p} = \frac{(-1)^{p+1}}{(p-m)!} (i\pi + \psi(p - m + 1)),$$

in (2.15) for $0 \leq m \leq p$, together with (2.10), [18, (14.9.14)], and applying the map $z \mapsto \frac{z}{\sqrt{z^2 - 1}}$ [6, Appendix A], completes the proof. Unfortunately, applying this same method for $m \geq p + 1$ does not seem to produce an analogous result. \square

On the other hand, using the map $z \mapsto \frac{z}{\sqrt{z^2-1}}$, (2.15), and (2.9), produces the following useful relation

$$\left[\frac{\partial}{\partial \nu} P_\nu^m \left(\frac{z}{\sqrt{z^2-1}} \right) \right]_{\nu=p} = i \sqrt{\frac{2}{\pi}} (z^2-1)^{\frac{1}{4}} \times \begin{cases} \frac{(-1)^{p+1}}{(p-m)!} \left(\log \frac{z + \sqrt{z^2-1}}{2\sqrt{z^2-1}} + 2H_{2p} - H_p - H_{p-m} \right) Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(z) \\ + (-1)^{p+1} \sum_{k=0}^{p-m-1} \frac{(2k+2m+1)}{k!(p-m-k)(p+m+k+1)} \left[1 + \frac{k!(p+m)!}{(k+2m)!(p-m)!} \right] Q_{m-\frac{1}{2}}^{k+m+\frac{1}{2}}(z) \\ + \frac{(-1)^{p+1}(p+m)!}{(p-m)!} \sum_{k=0}^{m-1} \frac{2k+1}{(k+m)!(p-k)(p+k+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(z), & \text{if } 0 \leq m \leq p, \\ (-1)^m (m-p-1)! Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(z), & \text{if } m \geq p+1. \end{cases} \quad (2.17)$$

One main orthogonal polynomial that we use in this manuscript is the Gegenbauer polynomial. The Gegenbauer polynomial $C_n^\nu : \mathbb{C} \rightarrow \mathbb{C}$ can be defined by

$$C_n^\nu(x) := \frac{(2\nu)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+2\nu \\ \nu + \frac{1}{2} \end{matrix}; \frac{1-x}{2} \right), \quad (2.18)$$

where $n \in \mathbb{N}_0$, $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$. The Gegenbauer polynomial can also be written in terms of the Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut) [18, (18.11.1)]

$$C_{n-m}^{m+\frac{1}{2}}(x) = \frac{1}{(\frac{1}{2})_m (-2)^m (1-x^2)^{\frac{m}{2}}} P_n^m(x), \quad (2.19)$$

where $x \in (-1, 1)$, $n, m \in \mathbb{N}_0$ such that $0 \leq m \leq n$. The Ferrers function of the first kind is defined as [18, (14.3.1)]

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}\mu} {}_2F_1 \left(\begin{matrix} -\nu, \nu+1 \\ 1-\mu \end{matrix}; \frac{1-x}{2} \right), \quad (2.20)$$

where $\nu, \mu \in \mathbb{C}$, $x \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. One may consider the limit as $\mu \rightarrow 0$ of the Gegenbauer polynomial [2, (6.4.13)]

$$\lim_{\mu \rightarrow 0} \frac{n+\mu}{\mu} C_n^\mu(x) = \epsilon_n T_n(x) \quad (2.21)$$

where $\epsilon_n = 2 - \delta_{n,0}$ is the Neumann factor, commonly appearing in Fourier cosine series. The Chebyshev polynomial of the first kind $T_n : \mathbb{C} \rightarrow \mathbb{C}$, is defined by

$$T_n(x) := {}_2F_1 \left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix}; \frac{1-x}{2} \right),$$

with the useful Fourier cosine representation $T_n(\cos \theta) = \cos(n\theta)$. The Chebyshev polynomial of the second kind $U_n : \mathbb{C} \rightarrow \mathbb{C}$, is defined by [18, (18.7.4)] $U_n(x) := C_n^1(x)$.

For generalized Hopf coordinates (see Section 6.2 below) we use Jacobi polynomials which are defined as [18, (18.5.7)]

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, \alpha + \beta + n + 1 \\ \alpha + 1 \end{matrix}; \frac{1-z}{2} \right).$$

Theorem 2.2. Let $\alpha, \beta, n \in \mathbb{N}_0, z \in \mathbb{C}$. Then the Jacobi polynomials have the following symmetry in parameter relations

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha + n)!(\beta + n)!}{n!(\alpha + \beta + n)!} \left(\frac{z - 1}{2}\right)^{-\alpha} P_{\alpha+n}^{(-\alpha, \beta)}(z), \tag{2.22}$$

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha + n)!(\beta + n)!}{(\alpha + \beta + n)!n!} \left(\frac{z + 1}{2}\right)^{-\beta} P_{\beta+n}^{(\alpha, -\beta)}(z). \tag{2.23}$$

Proof. For (2.22), start with [25, p. 212]

$$P_n^{(\alpha, \beta)}(z) = \binom{n + \beta}{n} \left(\frac{z - 1}{2}\right)^n {}_2F_1\left(\begin{matrix} -n, -n - \alpha \\ \beta + 1 \end{matrix}; \frac{z + 1}{z - 1}\right),$$

replace $\alpha \mapsto -\alpha$, then $n \mapsto n + \alpha$. Then using (2.5), and the reflection formula [18, (5.5.3)], the final result follows since for $x, y \in \mathbb{Z}$, one has

$$\lim_{\epsilon \rightarrow 0} \frac{\sin(\pi(x + y + 2\epsilon))}{\sin(\pi(x + \epsilon))} = \cos(\pi y) = (-1)^y.$$

For (2.23) use [18, (18.5.8)]

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha + 1)_n}{n!} \left(\frac{z + 1}{2}\right)^n {}_2F_1\left(\begin{matrix} -n, -n - \beta \\ \alpha + 1 \end{matrix}; \frac{z - 1}{z + 1}\right),$$

and replace first $\beta \mapsto -\beta$, and then $n \mapsto n + \beta$. \square

Corollary 2.3. Let $\alpha, \beta \in \mathbb{Z}, n \in \mathbb{N}_0, z \in \mathbb{C}$. Then the Jacobi polynomials have the following symmetry in parameter relation

$$P_n^{(\alpha, \beta)}(z) = \left(\frac{z - 1}{2}\right)^{-\alpha} \left(\frac{z + 1}{2}\right)^{-\beta} P_{\alpha+\beta+n}^{(-\alpha, -\beta)}(z). \tag{2.24}$$

Proof. For (2.24) use (2.22) and then substitute (2.23) to replace the Jacobi polynomial which appears on the right-hand side. Notice that the restriction $\alpha, \beta \in \mathbb{N}_0$ is relaxed to $\alpha, \beta \in \mathbb{Z}$ in the resulting expression. \square

3. Binomial and logarithmic kernels for the even dimensional fundamental solution of the polyharmonic equation in Euclidean space

If $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ then the Euclidean inner product $(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$(\mathbf{x}, \mathbf{x}') := x_1 x'_1 + \cdots + x_d x'_d, \tag{3.1}$$

induces a norm (the Euclidean norm) $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$, on \mathbb{R}^d , given by $\|\mathbf{x}\| := \sqrt{(\mathbf{x}, \mathbf{x})}$. If $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the polyharmonic equation given by

$$(-\Delta)^k \Phi(\mathbf{x}) = 0, \tag{3.2}$$

where $k \in \mathbb{N}$ and $\Phi \in C^{2k}(\mathbb{R}^d)$, $\mathbf{x} \in \mathbb{R}^d$, and $\Delta : C^p(\mathbb{R}^d) \rightarrow C^{p-2}(\mathbb{R}^d)$ for $p \geq 2$, is the Laplace operator on \mathbb{R}^d defined by $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$. Then Φ is referred to as polyharmonic, and $(-\Delta)^k$ is referred to as the

polyharmonic operator. If the power k of the Laplace operator equals two, then (3.2) is referred to as the biharmonic equation and Φ is called biharmonic. The inhomogeneous polyharmonic equation is given by

$$(-\Delta)^k \Phi(\mathbf{x}) = \rho(\mathbf{x}), \quad (3.3)$$

where we take ρ to be an integrable function so that a solution to (3.3) exists. A fundamental solution for the polyharmonic equation on \mathbb{R}^d is a function $\mathcal{G}_k^d : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\} \rightarrow \mathbb{R}$ which satisfies the distributional equation

$$(-\Delta)^k \mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \quad (3.4)$$

where δ is the Dirac delta distribution and $\mathbf{x}' \in \mathbb{R}^d$. One might want to obtain solutions $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ to the inhomogeneous polyharmonic equation (3.3) given an integrable function $\rho \in C_c^2(\mathbb{R}^d)$. An integral solution of the polyharmonic equation is given by

$$\Phi(\mathbf{x}) = \int_{\mathbb{R}^d} \mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') \, d\mathbf{x}' \quad (3.5)$$

(see [5, (3.13)]). In order to solve (3.3) on some simply-connected domain $\Omega \subseteq \mathbb{R}^d$, one may require Dirichlet boundary values on the boundary $\partial\Omega$. By using (3.5), one may obtain the Dirichlet boundary values in order to solve (3.3), given a fundamental solution of the polyharmonic equation, by convolution of a fundamental solution of that partial differential operator with the source distribution over Ω .

Let $n \in \mathbb{N}$ and recall the definition of the harmonic number $H_n := \sum_{k=1}^n \frac{1}{k}$. Then a fundamental solution of the polyharmonic equation (3.2) on Euclidean space \mathbb{R}^d is given by (see for instance [4, (2.1)], [8, Theorem 1], [31, Section II.2])

$$\mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') := \begin{cases} \frac{(-1)^{k+\frac{d}{2}+1} \|\mathbf{x} - \mathbf{x}'\|^{2k-d}}{(k-1)! (k-\frac{d}{2})! 2^{2k-1} \pi^{\frac{d}{2}}} \left(\log \|\mathbf{x} - \mathbf{x}'\| - \beta_{k-\frac{d}{2}, d} \right), & \text{if } d \text{ even, } k \geq \frac{d}{2}, \\ \frac{\Gamma(\frac{d}{2} - k) \|\mathbf{x} - \mathbf{x}'\|^{2k-d}}{(k-1)! 2^{2k} \pi^{\frac{d}{2}}}, & \text{otherwise,} \end{cases} \quad (3.6)$$

where $\beta_{p,d} \in \mathbb{Q}$ is defined as $\beta_{p,d} := \frac{1}{2}(H_p + H_{\frac{d}{2}+p-1} - H_{\frac{d}{2}-1})$.

Remark 3.1. In regard to a logarithmic fundamental solution of the polyharmonic equation (d even, $k \geq \frac{d}{2}$), note that [31, Section II.2] is missing the term proportional to $\|\mathbf{x} - \mathbf{x}'\|^{2k-d}$. This term is in the kernel of the polyharmonic operator $(-\Delta)^k$, so for any constant multiple of this term $\beta_{p,d}$ may be added to a fundamental solution of the polyharmonic equation. Our choice for this constant is given so that

$$-\Delta \mathcal{G}_k^d = \mathcal{G}_{k-1}^d, \quad (3.7)$$

is satisfied for all $k > \frac{d}{2}$, and that for $k = \frac{d}{2}$, the constant vanishes. Boyling's fundamental solution satisfies (3.7) for all $k > \frac{d}{2}$, but is missing the term proportional to $H_{\frac{d}{2}-1}$, and therefore only vanishes when $k = \frac{d}{2}$ for $d = 2$. Sobolev does not include this constant term, so for him \mathcal{G}_k^d is purely logarithmic for all $k \geq \frac{d}{2}$, $d \geq 2$ even. However in that case (3.7) is not satisfied for $k > \frac{d}{2}$.

We consider parametrizations of Euclidean space \mathbb{R}^d which are given in terms of coordinate systems whose coordinates are curvilinear, i.e., based on some transformation which converts the Cartesian coordinates to a coordinate system with the same number of coordinates in which the coordinate lines are curved.

We consider solutions of the polyharmonic equation (3.2) in a curvilinear coordinate system, which arises through the theory of separation of variables. We refer to coordinate systems which yield solutions through the separation of variables method as separable. In this manuscript, we restrict our attention to separable rotationally-invariant coordinate systems for the polyharmonic equation on \mathbb{R}^d which are given by

$$\left. \begin{aligned} x_1 &= x_1(\xi_1, \dots, \xi_{d-1}) \\ &\vdots \\ x_{d-2} &= x_{d-2}(\xi_1, \dots, \xi_{d-1}) \\ x_{d-1} &= R(\xi_1, \dots, \xi_{d-1}) \cos \phi \\ x_d &= R(\xi_1, \dots, \xi_{d-1}) \sin \phi \end{aligned} \right\}. \tag{3.8}$$

These coordinate systems are described by d coordinates: an angle $\phi \in \mathbb{R}$ plus $(d-1)$ -curvilinear coordinates $(\xi_1, \dots, \xi_{d-1})$. Rotationally-invariant coordinate systems parametrize points on the $(d-1)$ -dimensional half-hyperplane given by $\phi = \text{const}$ and $R \geq 0$ using the curvilinear coordinates $(\xi_1, \dots, \xi_{d-1})$. A separable rotationally-invariant coordinate system transforms the polyharmonic equation into a set of d -uncoupled ordinary differential equations with separation constants $m \in \mathbb{Z}$ and k_j for $1 \leq j \leq d-2$. For a separable rotationally-invariant coordinate system, this uncoupling is accomplished, in general, by assuming a solution to (3.2) of the form

$$\Phi(\mathbf{x}) = e^{im\phi} \mathcal{R}(\xi_1, \dots, \xi_{d-1}) \prod_{i=1}^{d-1} A_i(\xi_i, m, k_1, \dots, k_{d-2}),$$

where the properties of the functions \mathcal{R} and A_i , for $1 \leq i \leq d-1$, and the constants k_j for $1 \leq j \leq d-2$, depend on the specific separable rotationally-invariant coordinate system in question. Separable coordinate systems are divided into two separate classes, those which are simply separable ($\mathcal{R} = \text{const}$), and those which are \mathcal{R} -separable. For an extensive description of the theory of separation of variables see [26].

The Euclidean distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$, expressed in a rotationally-invariant coordinate system, is given by

$$\|\mathbf{x} - \mathbf{x}'\| = \sqrt{2RR'} [\chi - \cos(\phi - \phi')]^{\frac{1}{2}}, \tag{3.9}$$

where the hypertoroidal parameter $\chi > 1$, is given by

$$\chi := \chi(R, R', x_1, \dots, x_{d-2}, x'_1, \dots, x'_{d-2}) := \frac{R^2 + R'^2 + \sum_{k=1}^{d-2} (x_k - x'_k)^2}{2RR'} \tag{3.10}$$

and $R, R' \in (0, \infty)$ are defined in (3.8). The hypersurfaces given by $\chi = \text{const}$ are independent of coordinate system and represent hypertori of revolution.

One type of coordinate system which parametrizes points in d -dimensional Euclidean space which has a high degree of symmetry are Vilenkin’s polyspherical coordinates (for a detailed description of Vilenkin’s polyspherical coordinates, see [9, Appendix B and references therein]). These curvilinear orthogonal coordinate systems are composed of a radius $r \in [0, \infty)$ and $(d-1)$ angles which must have domains given in $\{[0, \frac{1}{2}\pi], [0, \pi], [-\pi, \pi]\}$. Using these coordinate systems, we can also express the distance between two points as

$$\|\mathbf{x} - \mathbf{x}'\| = \sqrt{2rr'} (\zeta - \cos \gamma)^{\frac{1}{2}}, \tag{3.11}$$

where $\zeta : [0, \infty)^2 \rightarrow [1, \infty)$ is defined by

$$\zeta = \zeta(r, r') := \frac{r^2 + r'^2}{2rr'}, \tag{3.12}$$

and the separation angle $\gamma \in [0, \pi]$ is defined through the relation

$$\cos \gamma := \frac{(\mathbf{x}, \mathbf{x}')}{\|\mathbf{x}\| \|\mathbf{x}'\|} = \frac{r r'}{r r'}, \tag{3.13}$$

using the Euclidean inner product and norm. Note that

$$\sqrt{\zeta^2 - 1} = \frac{r^2_{>} - r^2_{\leq}}{2rr'}. \tag{3.14}$$

From (3.6) we see that apart from a multiplicative constant, the algebraic expression of a fundamental solution for the polyharmonic equation on Euclidean space \mathbb{R}^d for d even, $k \geq \frac{d}{2}$, is given by $\mathfrak{I}_k^d : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\} \rightarrow \mathbb{R}$ defined by

$$\mathfrak{I}_k^d(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2k-d} \left(\log \|\mathbf{x} - \mathbf{x}'\| - \beta_{k-\frac{d}{2},d} \right). \tag{3.15}$$

By expressing \mathfrak{I}_k^d in a rotationally-invariant coordinate system (3.8) one has the following result. Let $p = k - d/2 \in \mathbb{N}_0$. Then

$$\begin{aligned} \mathfrak{I}_k^d(\mathbf{x}, \mathbf{x}') &= (2RR')^p \left(\frac{1}{2} \log(2RR') - \beta_{p,d} \right) (\chi - \cos(\phi - \phi'))^p \\ &\quad + \frac{1}{2} (2RR')^p (\chi - \cos(\phi - \phi'))^p \log(\chi - \cos(\phi - \phi')). \end{aligned} \tag{3.16}$$

Also, in a Vilenkin polyspherical coordinate system, one has

$$\mathfrak{I}_k^d(\mathbf{x}, \mathbf{x}') = (2rr')^p \left(\frac{1}{2} \log(2rr') - \beta_{p,d} \right) (\zeta - \cos \gamma)^p + \frac{1}{2} (2rr')^p (\zeta - \cos \gamma)^p \log(\zeta - \cos \gamma). \tag{3.17}$$

For the polyharmonic equation on even-dimensional Euclidean space \mathbb{R}^d , if $1 \leq k \leq \frac{d}{2} - 1$ then a fundamental solution is given by $\mathfrak{h}_k^d : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\} \rightarrow \mathbb{R}$ which is a power-law given by $\mathfrak{h}_k^d(\mathbf{x}, \mathbf{x}') = \|\mathbf{x} - \mathbf{x}'\|^{2k-d}$, where $2k - d \in -2\mathbb{N}$. For this range of k values then $2k - d$ is a negative even integer and this case and all its implications are fully covered by the material presented in [9].

For the case in which a logarithmic fundamental solution exists, namely $k \geq \frac{d}{2}$, then $2k - d \in 2\mathbb{N}$ is a positive even integer and the kernel $\mathfrak{j}_k^d : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$\mathfrak{j}_k^d(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2k-d}, \tag{3.18}$$

corresponds to a binomial expression $(z - x)^p$, where $p = k - d/2 \in \mathbb{N}_0$ and its series expansion either in terms of Chebyshev polynomials of the first kind or in terms of Gegenbauer polynomials truncates in a finite number of terms. By expressing \mathfrak{j}_k^d in a rotationally-invariant coordinate system there is

$$\mathfrak{j}_k^d(\mathbf{x}, \mathbf{x}') = (2RR')^p [\chi - \cos(\phi - \phi')]^p, \tag{3.19}$$

and expressing j_k^d in a Vilenkin polyspherical coordinate system one has

$$j_k^d(\mathbf{x}, \mathbf{x}') = (2rr')^p [\zeta - \cos \gamma]^p, \tag{3.20}$$

where $p = k - d/2 \in \mathbb{N}_0$.

4. Fourier and Gegenbauer expansions of binomial and logarithmic kernels

We require Gegenbauer (polynomial) expansions for kernels which naturally arise in a logarithmic fundamental solution of the polyharmonic equation in Vilenkin’s polyspherical coordinates in even-dimensional Euclidean space. Since these coordinates are rotationally invariant, in this study of addition theorems in these coordinates, we require Fourier cosine (Chebyshev polynomials of the first kind) and Gegenbauer polynomial expansions (see [9]). By necessity, here we treat the binomial $(z - x)^p$ and logarithmic $(z - x)^p \log(z - x)$ kernels, where $x, z, \nu \in \mathbb{C}$ and $p \in \mathbb{N}_0$. We have previously derived Fourier expansions of the binomial [11, (3.10)] and the logarithmic [8, (20), (26)] kernels. We now treat those corresponding Gegenbauer polynomial expansions. By examining (3.16), (3.17), (3.19), we see that for the computation of Fourier and Gegenbauer expansions, we are interested in the Fourier and Gegenbauer expansions of the Euler and logarithmic kernels.

The formulas which are presented below all rely in one way or another on the following important Gegenbauer polynomial expansion which can be found in [9, (3.4)], namely

$$(z - x)^{-\nu} = \frac{2^{\mu+\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}\Gamma(\nu)} e^{i\pi(\mu-\nu+\frac{1}{2})} (z^2 - 1)^{\frac{\mu-\nu}{2}+\frac{1}{4}} \sum_{n=0}^{\infty} (n + \mu) C_n^\mu(x) Q_{n+\mu-\frac{1}{2}}^{\nu-\mu-\frac{1}{2}}(z), \tag{4.1}$$

where $z \in \mathbb{C} \setminus (-\infty, 1]$, and $x \in \mathbb{C}$ lies inside the ellipse with foci at ± 1 that passes through z . This result and the following results have the curious property of z, x lying on ellipses with foci at ± 1 and this is due to the important theorem of Szegő, namely [32, Theorem 12.7.3, Expansion of an analytic function in terms of orthogonal polynomials].

Lemma 4.1. *Let $p \in \mathbb{N}_0$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $x \in \mathbb{C}$. Then the expansion of Euler kernel in Chebyshev polynomials of the first kind is given by the following binomial expansion (see [11, (4.4)])*

$$(z - x)^p = i(-1)^{p+1} \sqrt{\frac{2}{\pi}} p! (z^2 - 1)^{\frac{p}{2}+\frac{1}{4}} \sum_{n=0}^p \frac{(-1)^n \epsilon_n T_n(x)}{(p - n)!(p + n)!} Q_{n-\frac{1}{2}}^{p+\frac{1}{2}}(z). \tag{4.2}$$

Proof. This truncated series result follows from (4.1) with $-\nu = p \in \mathbb{N}_0$ and then taking $\mu \rightarrow 0$ with (2.21). \square

The Fourier cosine expansion of the important logarithmic kernel is given in the following lemma.

Lemma 4.2. *Let $p \in \mathbb{N}_0$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $z - x \in \mathbb{C} \setminus (-\infty, 0]$, x lies inside the ellipse with foci at ± 1 that passes through z . Then*

$$(z - x)^p \log(z - x) = (z - x)^p \left(\log \frac{z + \sqrt{z^2 - 1}}{2} + 2H_{2p} \right) + i\sqrt{\frac{2}{\pi}} (-1)^p p! (z^2 - 1)^{\frac{p}{2}+\frac{1}{4}} \sum_{n=0}^p \frac{\epsilon_n (-1)^n T_n(x)}{(p - n)!(p + n)!} (H_{p+n} + H_{p-n}) Q_{n-\frac{1}{2}}^{p+\frac{1}{2}}(z)$$

$$\begin{aligned}
& + i\sqrt{\frac{2}{\pi}}(-1)^{p+1}p!(z^2-1)^{\frac{p}{2}+\frac{1}{4}}\sum_{n=0}^{p-1}\frac{\epsilon_n(-1)^nT_n(x)}{(p+n)!}\sum_{k=0}^{p-n-1}\frac{(2n+2k+1)\left[1+\frac{k!(p+n)!}{(2n+k)!(p-n)!}\right]}{k!(p-n-k)(p+n+k+1)}Q_{n-k+\frac{1}{2}}^{n+k+\frac{1}{2}}(z) \\
& + i\frac{2^{\frac{3}{2}}}{\sqrt{\pi}}(-1)^{p+1}p!(z^2-1)^{\frac{p}{2}+\frac{1}{4}}\sum_{n=1}^p\frac{(-1)^nT_n(x)}{(p-n)!}\sum_{k=0}^{n-1}\frac{2k+1}{(n+k)!(p-k)(p+k+1)}Q_{n-k-\frac{1}{2}}^{k+\frac{1}{2}}(z) \\
& + i\frac{2^{\frac{3}{2}}}{\sqrt{\pi}}p!(z^2-1)^{\frac{p}{2}+\frac{1}{4}}\sum_{n=p+1}^{\infty}\frac{(n-p-1)!T_n(x)}{(p+n)!}Q_{n-\frac{1}{2}}^{p+\frac{1}{2}}(z). \tag{4.3}
\end{aligned}$$

Proof. Use [8, (26)], with (2.9), (2.10), (2.11). \square

The following consequence of (4.3) is also given in [25, p. 259].

Corollary 4.3. Let $z \in \mathbb{C} \setminus (-\infty, 1]$, $z-x \in \mathbb{C} \setminus (-\infty, 0]$, x lies inside the ellipse with foci at ± 1 that passes through z . Then

$$\log(z-x) = \log\frac{z+\sqrt{z^2-1}}{2} - 2\sum_{n=1}^{\infty}\frac{T_n(x)}{n}\frac{1}{(z+\sqrt{z^2-1})^n}.$$

Proof. Let $p=0$ in (4.3) and use [18, (14.5.17)]. This completes the proof. \square

The Gegenbauer polynomial expansion of the binomial is given in the following lemma.

Lemma 4.4. Let $p \in \mathbb{N}_0$, $\mu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $x \in \mathbb{C}$. Then

$$(z-x)^p = \frac{2^{\mu+\frac{1}{2}}}{\sqrt{\pi}}e^{i\pi(p-\mu-\frac{1}{2})}\Gamma(\mu)p!(z^2-1)^{\frac{p+\mu}{2}+\frac{1}{4}}\sum_{n=0}^p\frac{(-1)^n(n+\mu)C_n^\mu(x)}{(p-n)!\Gamma(n+p+2\mu+1)}Q_{n+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(z). \tag{4.4}$$

Proof. Starting with [9, (3.4)], (4.1), we take the limit $\nu \rightarrow -p$, producing

$$(z-x)^p = \frac{2^{\mu+\frac{1}{2}}}{\sqrt{\pi}}e^{i\pi(\mu+p+\frac{1}{2})}\Gamma(\mu)(z^2-1)^{\frac{\mu+p}{2}+\frac{1}{4}}\sum_{n=0}^{\infty}(n+\mu)C_n^\mu(x)\lim_{\nu \rightarrow -p}\frac{1}{\Gamma(\nu)}Q_{n+\mu-\frac{1}{2}}^{\nu-\mu-\frac{1}{2}}(z). \tag{4.5}$$

By using (2.11), (2.4), we have

$$\lim_{\nu \rightarrow -p}\frac{1}{\Gamma(\nu)}Q_{n+\mu-\frac{1}{2}}^{\nu-\mu-\frac{1}{2}}(z) = \begin{cases} \frac{(-1)^{n+1}e^{-2i\pi\mu}p!}{(p-n)!\Gamma(n+p+2\mu+1)}Q_{n+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(z) & \text{if } 0 \leq n \leq p, \\ 0 & \text{if } n \geq p+1. \end{cases}$$

Using this limit in (4.5) completes the proof. \square

Note that the above lemma generalizes (4.2) by the limit formula (2.21).

The Gegenbauer polynomial expansion of the logarithmic kernel is given as follows.

Lemma 4.5. Let $p \in \mathbb{N}_0$, $\mu \in \mathbb{N}$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $z-x \in \mathbb{C} \setminus (-\infty, 0]$, x lies inside the ellipse with foci at ± 1 that passes through z . Then

$$\begin{aligned}
 (z-x)^p \log(z-x) &= (z-x)^p \left(\log \frac{z + \sqrt{z^2-1}}{2} + 2H_{2p+2\mu} + H_p - H_{p+\mu} \right) \\
 &+ i(-1)^{p+\mu} \frac{2^{\mu+\frac{1}{2}}}{\sqrt{\pi}} p!(\mu-1)! (z^2-1)^{\frac{p+\mu}{2}+\frac{1}{4}} \sum_{n=0}^p \frac{(n+\mu)(-1)^n C_n^\mu(x)}{(p-n)!(p+n+2\mu)!} (H_{p+n+2\mu} + H_{p-n}) Q_{n+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(z) \\
 &+ i(-1)^{p+\mu+1} \frac{2^{\mu+\frac{1}{2}}}{\sqrt{\pi}} p!(\mu-1)! (z^2-1)^{\frac{p+\mu}{2}+\frac{1}{4}} \sum_{n=0}^{p-1} \frac{(n+\mu)(-1)^n C_n^\mu(x)}{(p+n+2\mu)!} \\
 &\quad \times \sum_{k=0}^{p-n-1} \frac{(2n+2k+2\mu+1)}{k!(p-n-k)(p+n+k+2\mu+1)} \left[1 + \frac{k!(p+n+2\mu)!}{(k+2n+2\mu)!(p-n)!} \right] Q_{n+\mu-\frac{1}{2}}^{k+n+\mu+\frac{1}{2}}(z) \\
 &+ i(-1)^{p+\mu+1} \frac{2^{\mu+\frac{1}{2}}}{\sqrt{\pi}} p!(\mu-1)! (z^2-1)^{\frac{p+\mu}{2}+\frac{1}{4}} \sum_{n=0}^p \frac{(n+\mu)(-1)^n C_n^\mu(x)}{(p-n)!} \\
 &\quad \times \sum_{k=0}^{n+\mu-1} \frac{2k+1}{(n+k+\mu)!(p+\mu-k)(p+k+\mu+1)} Q_{n+\mu-\frac{1}{2}}^{k+\frac{1}{2}}(z) \\
 &+ i(-1)^\mu \frac{2^{\mu+\frac{1}{2}}}{\sqrt{\pi}} p!(\mu-1)! (z^2-1)^{\frac{p+\mu}{2}+\frac{1}{4}} \sum_{n=p+1}^\infty \frac{(n+\mu)(n-p-1)! C_n^\mu(x)}{(p+n+2\mu)!} Q_{n+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(z). \tag{4.6}
 \end{aligned}$$

Proof. Let $\nu \in \mathbb{C}$, $\mu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $z \in \mathbb{C} \setminus (-\infty, 1]$ and $x \in \mathbb{C}$ lies inside the ellipse with foci at ± 1 that passes through z . Using (4.1) and (2.9), one has

$$(z-x)^\nu = 2^\mu \Gamma(\mu) (z^2-1)^{\frac{\nu+\mu}{2}} \sum_{n=0}^\infty (n+\mu)(-\nu)_n P_{\nu+\mu}^{-n-\mu} \left(\frac{z}{\sqrt{z^2-1}} \right) C_n^\mu(x). \tag{4.7}$$

Let $p \in \mathbb{N}_0$. One then may use the following identity

$$(z-x)^p \log(z-x) = \lim_{\nu \rightarrow 0} \frac{\partial}{\partial \nu} (z-x)^{\nu+p},$$

which upon substituting $\nu \mapsto \nu + p$ in (4.7) produces

$$(z-x)^p \log(z-x) = 2^\mu (z^2-1)^{\frac{\mu+p}{2}} \sum_{n=0}^\infty (n+\mu)(-1)^n C_n^\mu(x) \lim_{\nu \rightarrow 0} \frac{\partial}{\partial \nu} \frac{(z^2-1)^{\frac{\nu}{2}} \Gamma(\nu+p+1)}{\Gamma(\nu+p+2\mu+1)} P_{\nu+p+\mu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right).$$

Performing the derivatives, one obtains

$$\begin{aligned}
 &\frac{\partial}{\partial \nu} \frac{(z^2-1)^{\frac{\nu}{2}} \Gamma(\nu+p+1)}{\Gamma(\nu+p+2\mu+1)} P_{\nu+p+\mu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right) \\
 &= \frac{(z^2-1)^{\frac{\nu}{2}} \Gamma(\nu+p+1)}{\Gamma(\nu+p+2\mu+1)} \left(\log \sqrt{z^2-1} + \psi(p+\nu+1) - \psi(p+n+\nu+2\mu+1) \right) P_{\nu+p+\mu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right) \\
 &+ \frac{(z^2-1)^{\frac{\nu}{2}} \Gamma(\nu+p+1)}{\Gamma(\nu+p+2\mu+1)} \frac{\partial}{\partial \nu} P_{\nu+p+\mu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right),
 \end{aligned}$$

and after taking the limit one has

$$\begin{aligned} & \lim_{\nu \rightarrow 0} \frac{\partial}{\partial \nu} \frac{p!}{\Gamma(p+2\mu+1)} P_{p+\mu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right) \\ &= \frac{p!}{\Gamma(p+2\mu+1)} \left(\log \sqrt{z^2-1} + \psi(p+1) - \psi(p+n+2\mu+1) \right) P_{p+\mu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right) \\ &+ \frac{p!}{\Gamma(p+2\mu+1)} \left[\frac{\partial}{\partial \nu} P_{\nu}^{n+\mu} \left(\frac{z}{\sqrt{z^2-1}} \right) \right]_{\nu=p+\mu}. \end{aligned}$$

For general μ , the parameter derivative of the associated Legendre function of the first kind given in the above equation is not known. However for $\mu \in \mathbb{N}_0$ it is known (see (2.15)). The expansion for the logarithmic kernel for $\mu = 0$ corresponds to the Chebyshev polynomial of the first kind (see (2.21)), and therefore corresponds to (4.3). Hence from this point forward we treat $\mu \in \mathbb{N}$, and the result follows using (2.17). \square

One interesting consequence of (4.6) is the following expansion.

Corollary 4.6. *Let $m \in \mathbb{N}$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $z - x \in \mathbb{C} \setminus (-\infty, 0]$, x lies inside the ellipse with foci at ± 1 that passes through z . Then*

$$\begin{aligned} \log(z-x) &= \log \frac{z + \sqrt{z^2-1}}{2} + H_{2m} - H_m \\ &+ i(-1)^{m+1} \frac{2^{m+\frac{1}{2}}}{\sqrt{\pi}} m! (z^2-1)^{\frac{m}{2}+\frac{1}{4}} \sum_{k=0}^{m-1} \frac{2k+1}{(k+m)!(m-k)(k+m+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(z) \\ &+ i(-1)^m \frac{2^{m+\frac{1}{2}}}{\sqrt{\pi}} (m-1)! (z^2-1)^{\frac{m}{2}+\frac{1}{4}} \sum_{n=1}^{\infty} \frac{(n+m)n! C_n^m(x)}{(2m+n)!} Q_{n+m-\frac{1}{2}}^{m+\frac{1}{2}}(z). \end{aligned} \quad (4.8)$$

Proof. Let $p = 0$ in (4.6) using the duplication theorem [18, (5.5.5)], and (2.14), completes the proof. \square

Corollary 4.7. *Let $z \in \mathbb{C} \setminus (-\infty, 1]$, $z - x \in \mathbb{C} \setminus (-\infty, 0]$, x lies inside the ellipse with foci at ± 1 that passes through z . Then*

$$\log(z-x) = \log \frac{z + \sqrt{z^2-1}}{2} + \frac{1}{2} - \frac{\sqrt{z^2-1}}{z + \sqrt{z^2-1}} - 2 \sum_{n=1}^{\infty} \frac{U_n(x)}{n(n+2)} \frac{z + (n+1)\sqrt{z^2-1}}{(z + \sqrt{z^2-1})^{n+1}}.$$

Proof. Let $\mu = 1$ in (4.8), then use (2.12), (2.13). Simplification completes the proof. \square

5. Azimuthal Fourier and Gegenbauer polynomial expansions of binomial and logarithmic fundamental solutions

The behavior of a logarithmic fundamental solution of the polyharmonic equation on even-dimensional Euclidean space \mathbb{R}^d in a rotationally-invariant coordinate system and in a Vilenkin polyspherical coordinate system are given respectively by (3.16), (3.17). For rotationally-invariant coordinate systems (3.8), recall that R, R' are the cylindrical radii and ϕ, ϕ' are azimuthal angles and that χ , the toroidal parameter, is defined in (3.10). Furthermore, for the polyharmonic equation (3.2), $k \in \mathbb{N}$ is the power of the positive Laplacian. For the definitions of the special functions and numbers used in the results presented in this section, see Section 2. We now present the expression for the azimuthal Fourier expansion of a logarithmic fundamental solution of the polyharmonic equation.

Theorem 5.1. Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ with $d \in 2\mathbb{N}$, $p = k - d/2 \in \mathbb{N}_0$. Then the azimuthal Fourier expansion of a logarithmic fundamental solution of the polyharmonic equation \mathbb{V}_k^d in a rotationally-invariant coordinate system on Euclidean space \mathbb{R}^d is given by

$$\begin{aligned} \mathbb{V}_k^d(\mathbf{x}, \mathbf{x}') &= \frac{i p!}{\sqrt{2\pi}} (2RR')^p (\chi^2 - 1)^{\frac{p}{2} + \frac{1}{4}} \left[\left(\log(RR') + \log(\chi + \sqrt{\chi^2 - 1}) + 2H_{2p} - H_p - H_{\frac{d}{2} + p - 1} + H_{\frac{d}{2} - 1} \right) \right. \\ &\quad \times (-1)^{p+1} \sum_{m=0}^p \frac{\epsilon_m (-1)^m \cos(m(\phi - \phi'))}{(p-m)!(p+m)!} Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \\ &\quad + (-1)^p \sum_{m=0}^p \frac{\epsilon_m (-1)^m \cos(m(\phi - \phi'))}{(p-m)!(p+m)!} (H_{p+m} + H_{p-m}) Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \\ &\quad + (-1)^{p+1} \sum_{m=0}^{p-1} \frac{\epsilon_m (-1)^m \cos(m(\phi - \phi'))}{(p+m)!} \sum_{k=0}^{p-m-1} \frac{(2m+2k+1) \left[1 + \frac{k!(p+m)!}{(2m+k)!(p-m)!} \right]}{k!(p-m-k)(p+m+k+1)} Q_{m-\frac{1}{2}}^{m+k+\frac{1}{2}}(\chi) \\ &\quad + 2(-1)^{p+1} \sum_{m=1}^p \frac{(-1)^m \cos(m(\phi - \phi'))}{(p-m)!} \sum_{k=0}^{m-1} \frac{2k+1}{(m+k)!(p-k)(p+k+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(\chi) \\ &\quad \left. + 2 \sum_{m=p+1}^{\infty} \frac{(m-p-1)! \cos(m(\phi - \phi'))}{(p+m)!} Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \right]. \end{aligned} \tag{5.1}$$

Proof. Beginning with (3.16), applying the identities (4.2), (4.3) and simplifying, completes the proof. \square

A logarithmic fundamental solution of the polyharmonic equation expressed in a Vilenkin polyspherical coordinate system is given by (3.17). Also recall that r, r' are the hyperspherical radii and $\cos \gamma = (\mathbf{x}, \mathbf{x}') / (\|\mathbf{x}\| \|\mathbf{x}'\|)$ (3.13) is the separation angle (3.13) in a Vilenkin polyspherical coordinate system [9, Appendix B and references therein], and that $\zeta = (r^2 + r'^2) / (2rr')$ is defined in (3.12). We now give the Gegenbauer expansion for a logarithmic fundamental solution of the polyharmonic equation in Vilenkin polyspherical coordinate systems.

Theorem 5.2. Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ with $d \in 2\mathbb{N}$, $p = k - d/2 \in \mathbb{N}_0$. Then the Gegenbauer expansion of a logarithmic fundamental solution of the polyharmonic equation \mathbb{V}_k^d on even-dimensional Euclidean space \mathbb{R}^d in a Vilenkin polyspherical coordinate system is given by

$$\begin{aligned} \mathbb{V}_k^d(\mathbf{x}, \mathbf{x}') &= i \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}} p! \Gamma(\mu) (\zeta^2 - 1)^{\frac{p+\mu}{2} + \frac{1}{4}} (2rr')^p \\ &\quad \times \left[\left(\log(rr') + \log\left(\zeta + \sqrt{\zeta^2 - 1}\right) + 2H_{2p+2\mu} + H_p - H_{p+\mu} - 2\beta_{p,d} \right) \right. \\ &\quad \times (-1)^{p+\mu+1} \sum_{l=0}^p \frac{(-1)^l (l+\mu) C_l^\mu(\cos \gamma)}{(p-l)!(l+p+2\mu)!} Q_{l+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(\zeta) \\ &\quad + (-1)^{p+\mu} \sum_{l=0}^p \frac{(-1)^l (l+\mu) C_l^\mu(\cos \gamma)}{(p-l)!(l+p+2\mu)!} (H_{p+l+2\mu} + H_{p-l}) Q_{l+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(\zeta) \\ &\quad \left. + (-1)^{p+\mu+1} \sum_{l=0}^{p-1} \frac{(-1)^l (l+\mu) C_l^\mu(\cos \gamma)}{(p+l+2\mu)!} \right] \end{aligned}$$

$$\begin{aligned}
& \times \sum_{k=0}^{p-l-1} \frac{(2l+2k+2\mu+1)}{k!(p-l-k)(p+l+k+2\mu+1)} \left[1 + \frac{k!(p+l+2\mu)!}{(k+2l+2\mu)!(p-l)!} \right] Q_{l+\mu-\frac{1}{2}}^{k+l+\mu+\frac{1}{2}}(\zeta) \\
& + (-1)^{p+\mu+1} \sum_{l=0}^p \frac{(-1)^l (l+\mu) C_l^\mu(\cos \gamma)}{(p-l)!} \sum_{k=0}^{l+\mu-1} \frac{2k+1}{(l+k+\mu)!(p+\mu-k)(p+k+\mu+1)} Q_{l+\mu-\frac{1}{2}}^{k+\frac{1}{2}}(\zeta) \\
& + (-1)^\mu \sum_{l=p+1}^{\infty} \frac{(l+\mu)(l-p-1)! C_l^\mu(\cos \gamma)}{(p+l+2\mu)!} Q_{l+\mu-\frac{1}{2}}^{p+\mu+\frac{1}{2}}(\zeta) \Big]. \tag{5.2}
\end{aligned}$$

Proof. Applying the identities (4.6) and (4.4) to (3.17) completes the proof. \square

For the polyharmonic equation on even-dimensional Euclidean space \mathbb{R}^d with $k \geq \frac{d}{2}$, apart from multiplicative constants, the algebraic expression for a binomial fundamental solution of the polyharmonic equation $j_k^d: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is given by $j_k^d(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2k-d}$, with $2k-d \in 2\mathbb{N}$. By expressing j_k^d in a rotationally-invariant coordinate system one has (3.19) which leads to the following result for an azimuthal Fourier expansion of the binomial fundamental solution of the polyharmonic equation.

Theorem 5.3. *Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ with $d \in 2\mathbb{N}$, $p = k - d/2 \in \mathbb{N}_0$. Then the azimuthal Fourier expansion of a binomial fundamental solution of the polyharmonic equation j_k^d on Euclidean space \mathbb{R}^d is given by*

$$j_k^d(\mathbf{x}, \mathbf{x}') = i(-1)^{p+1} \sqrt{\frac{2}{\pi}} p! (2RR')^p (\chi^2 - 1)^{\frac{p}{2} + \frac{1}{4}} \sum_{n=0}^p \frac{(-1)^n \epsilon_n \cos(n(\phi - \phi'))}{(p-n)!(p+n)!} Q_{n-\frac{1}{2}}^{p+\frac{1}{2}}(\chi). \tag{5.3}$$

Proof. Starting with (3.19), and using the binomial expansion (4.2) with the Fourier cosine representation of the Chebyshev polynomials of the first kind completes the proof. \square

The expression j_k^d can also be represented in any Vilenkin polyspherical coordinate system (3.20) which likewise can be expanded using Gegenbauer polynomials. This is presented as in the following result.

Theorem 5.4. *Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ with $d \in 2\mathbb{N}$, $p = k - d/2 \in \mathbb{N}_0$. Then the Gegenbauer expansion of a binomial fundamental solution of the polyharmonic equation j_k^d on Euclidean space \mathbb{R}^d is given by*

$$\begin{aligned}
j_k^d(\mathbf{x}, \mathbf{x}') &= i(-1)^{p-\frac{d}{2}} \frac{2^{\frac{d-1}{2}}}{\sqrt{\pi}} p! \left(\frac{d}{2} - 2\right)! (2rr')^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{d-1}{2}} \\
& \times \sum_{n=0}^p \frac{(-1)^n (n + \frac{d}{2} - 1) C_n^{\frac{d}{2}-1}(\cos \gamma)}{(p-n)!(n+p+d-2)!} Q_{n+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta). \tag{5.4}
\end{aligned}$$

Proof. The Gegenbauer polynomial expansion of Lemma 4.4 with $\mu = d/2 - 1$, is combined with a fundamental solution written in a Vilenkin polyspherical coordinate system and using (3.14) completes the proof. \square

6. Global analysis on \mathbb{R}^d of standard and generalized Hopf Vilenkin polyspherical coordinate systems

Now we study some of the particular details which will arise in the study of binomial and logarithmic fundamental solutions of the polyharmonic equation in standard polyspherical coordinates and generalized Hopf coordinates. We will utilize the addition theorem for hyperspherical harmonics to expand the critical Gegenbauer polynomial (that with an order equal to $d/2 - 1$) over the product of normalized hyperspherical harmonics in that particular Vilenkin polyspherical coordinate system. The reason we used the critical order

$d/2 - 1$ is that these Gegenbauer polynomials with this particular order provide a basis for analytic solutions on the hypersphere \mathbf{S}_r^{d-1} . For $d \geq 3$, the addition theorem for hyperspherical harmonics is given by (for a proof see [36], [19, §10.2.1])

$$C_n^{d/2-1}(\cos \gamma) = \frac{2(d-2)\pi^{d/2}}{(2n+d-2)\Gamma(d/2)} \sum_K Y_n^K(\hat{\mathbf{x}}) \overline{Y_n^K(\hat{\mathbf{x}}')}, \tag{6.1}$$

where K stands for a set of $(d-2)$ -quantum numbers identifying harmonics for a given value of $n \in \mathbb{N}_0$, and $\cos \gamma$ is the cosine of the separation angle (3.13) between two arbitrary vectors $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$. The functions $Y_n^K : \mathbf{S}^{d-1} \rightarrow \mathbb{C}$ are the normalized hyperspherical harmonics. Normalization of the hyperspherical harmonics is achieved through the integral

$$\int_{\mathbf{S}^{d-1}} Y_n^K(\hat{\mathbf{x}}) \overline{Y_n^K(\hat{\mathbf{x}})} d\Omega = 1,$$

where $d\Omega$ is the Riemannian volume measure on \mathbf{S}^{d-1} .

First we will treat standard polyspherical coordinates and then we will treat generalized Hopf coordinates. Both of these polyspherical coordinate systems and many of their various properties are described in [9, Appendix B].

6.1. Standard polyspherical coordinates

Here we review details connected with standard Vilenkin polyspherical coordinates. These coordinates and as well the general Vilenkin polyspherical coordinates are described carefully in [9, Appendix B], and we will not depart from the description and usage of standard polyspherical coordinates described therein. Standard polyspherical coordinates are a generalization of the spherical coordinate system that is most commonly encountered in multi-dimensional calculus. What we refer to as *standard polyspherical coordinates* are given by

$$\begin{aligned} x_1 &= r \cos \theta_1, \\ x_2 &= r \sin \theta_1 \cos \theta_2, \\ x_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ &\dots\dots\dots \\ x_{d-2} &= r \sin \theta_1 \cdots \sin \theta_{d-3} \cos \theta_{d-2}, \\ x_{d-1} &= r \sin \theta_1 \cdots \sin \theta_{d-3} \sin \theta_{d-2} \cos \phi, \\ x_d &= r \sin \theta_1 \cdots \sin \theta_{d-3} \sin \theta_{d-2} \sin \phi, \end{aligned} \tag{6.2}$$

where $\theta_i \in [0, \pi]$ for $1 \leq i \leq d - 2$ and $\phi \in [-\pi, \pi)$. In standard polyspherical coordinates, the normalized hyperspherical harmonics can be written as [9, (B.19)]

$$Y_l^K(\hat{\mathbf{x}}) = \frac{e^{im\phi}}{\sqrt{2\pi}} \prod_{j=1}^{d-2} \Theta_j^d(l_j, l_{j+1}; \theta_j), \tag{6.3}$$

where [9, (B.20)]

$$\Theta_j^d(l_j, l_{j+1}; \theta_j) := \frac{\Gamma\left(l_{j+1} + \frac{d-j+1}{2}\right)}{2^{l_{j+1} + d - j - 1}} \sqrt{\frac{2^{2l_{j+1} + d - j - 1} (2l_j + d - j - 1) (l_j - l_{j+1})!}{\pi (l_j + l_{j+1} + d - j - 2)!}} \\ \times (\sin \theta_j)^{l_{j+1}} C_{l_j - l_{j+1}}^{l_{j+1} + (d-j-1)/2}(\cos \theta_j). \quad (6.4)$$

The addition theorem for hyperspherical harmonics (6.1) involves the product $Y_l^K(\mathbf{x})\overline{Y_l^K(\mathbf{x}')}$, so we introduce a convenient notation, namely $\Omega_k^d : \mathbb{N}_0 \times \mathbb{Z} \times [0, \pi]^2 \rightarrow \mathbb{R}$ is defined by

$$\Omega_k^d\left(l_k, l_{k+1}; \theta_k\right) := \Theta_k^d(l_k, l_{k+1}; \theta_k) \Theta_k^d(l_k, l_{k+1}; \theta'_k). \quad (6.5)$$

6.1.1. Standard polyspherical coordinates multi-sum reversal lemmas

An important ingredient in the production of the binomial and logarithmic addition functions is reversing the order of the multi-sums that appear when the Gegenbauer polynomials are expanded as multi-sums of hyperspherical harmonics. This allows us to compare the Fourier coefficients relying on the azimuthal quantum number m .

Lemma 6.1. *Let $p \in \mathbb{N}_0$. Then the multi-sum Y_1 , over the allowed quantum numbers for standard polyspherical hyperspherical harmonics (6.3) defined by*

$$Y_1 := \sum_{l=0}^p \sum_K = \sum_{l=0}^p \sum_{l_2=0}^l \cdots \sum_{l_{d-2}=0}^{l_{d-3}} \sum_{m=0}^{l_{d-2}}, \quad (6.6)$$

can then be re-expressed with the sum order reversed as

$$Y_1 = \sum_{m=0}^p \sum_{l_{d-2}=m}^p \cdots \sum_{l_2=l_3=l_2}^p \sum_{l=l_2}^p. \quad (6.7)$$

Proof. To reverse the multi-sum, the upper and lower bounds of the indices need to be determined. Each index of the original multi-sum can have values between 0 and p inclusively. However, the upper bounds of the original multi-sum give the following constraint:

$$0 \leq m \leq l_{d-2} \leq \dots \leq l_2 \leq l. \quad (6.8)$$

When the indices are reversed, this constraint determines the new lower bound of each of the indices. Since there are no other constraints, the upper bound for each index will be p . \square

Lemma 6.2. *Let $p \in \mathbb{N}_0$. Then the multi-sum Y_2 over quantum numbers for standard polyspherical harmonics defined by*

$$Y_2 := \sum_{l=p+1}^{\infty} \sum_K = \sum_{l=p+1}^{\infty} \sum_{l_2=0}^l \cdots \sum_{l_{d-2}=0}^{l_{d-3}} \sum_{m=0}^{l_2},$$

can then be re-expressed with the sum order reversed as

$$Y_2 = \sum_{m=0}^p \sum_{l_{d-2}=m}^{\infty} \cdots \sum_{l_2=l_3=l_2}^{\infty} \sum_{l=\max(l_2, p+1)}^{\infty} + \sum_{m=p+1}^{\infty} \sum_{l_{d-2}=m}^{\infty} \cdots \sum_{l_2=l_3=l_2}^{\infty} \sum_{l=l_2}^{\infty}. \quad (6.9)$$

Proof. First it is noted that each of the indices in the original multi-sum, except for l , will have a lower bound of 0 and an upper bound of infinity. The index l will have the same upper bound, but a lower bound of $p + 1$. Like the previous lemma, the upper bounds of the original multi-sum will give us the constraint (6.8), which determines the lower bounds of the reversed multi-sum. The combination of the following two lower bounds:

$$l_2 \leq l, \quad p + 1 \leq l,$$

yields the lower bound $\max(p + 1, l_2)$. With this, the multi-sum can be reversed:

$$Y_2 = \sum_{m=0}^{\infty} \sum_{l_{d-2}=m}^{\infty} \cdots \sum_{l_2=l_3}^{\infty} \sum_{l=\max(l_2, p+1)}^{\infty} .$$

In later theorems, the scenarios of $m \leq p$ and $m \geq p + 1$ need to be treated separately. With the index m being the first sum of the multi-sum, this split is uncomplicated. When $m \geq p + 1$, we can drop the maximum function on the lower bound of l . \square

6.2. Generalized Hopf coordinates

Generalized Hopf coordinates are a type of Vilenkin polyspherical coordinates on \mathbb{R}^d with $d = 2^q$ for $q \geq 1$. They are Vilenkin polyspherical orthogonal curvilinear coordinates with one radial coordinate $r \in [0, \infty)$, and $(d - 1)$ -angular coordinates which together parametrize points on \mathbf{S}_r^{d-1} the $(d - 1)$ -dimensional real hypersphere with radius r . Of the $(d - 1)$ -angular coordinates $(d/2 - 1)$ - ϑ coordinates take values in $[0, \frac{1}{2}\pi]$, and the other $(d/2)$ - ϕ coordinates are of azimuthal type and take values in $[-\pi, \pi)$. For a careful treatment of generalized Hopf coordinates, see [9, Appendix B].

In this paper we depart slightly from our previous description of generalized Hopf coordinates—we have adopted a *reversed* azimuthal identification for the azimuthal angles and their corresponding quantum numbers. In particular if one considers the collection of angles in generalized Hopf coordinates given by some vector of angles $\Theta := (\Theta_1, \dots, \Theta_{d-1})$ with vector of corresponding quantum numbers $\mathbf{p} := (p_1, \dots, p_{d-1})$. In our previous paper, these were ordered as

$$\Theta = (\vartheta_1, \dots, \vartheta_{d/2-1}, \phi_1, \dots, \phi_{d/2}), \quad \mathbf{p} = (l_1, \dots, l_{d/2-1}, m_1, \dots, m_{d/2}), \tag{6.10}$$

and in the current paper we order them as

$$\Theta = (\vartheta_1, \dots, \vartheta_{d/2-1}, \phi_{d/2}, \dots, \phi_1), \quad \mathbf{p} = (l_1, \dots, l_{d/2-1}, m_{d/2}, \dots, m_1) \tag{6.11}$$

(see Fig. 1 as compared to [9, Figure 5]).

These coordinates generalize two-dimensional polar coordinates (see Fig. 1a)

$$x_1 = r \cos \phi, \quad x_2 = r \sin \phi, \tag{6.12}$$

and four-dimensional Hopf coordinates (see Fig. 1b)

$$\begin{aligned} x_1 &= r \cos \vartheta \cos \phi_2, & x_2 &= r \cos \vartheta \sin \phi_2, \\ x_3 &= r \sin \vartheta \cos \phi_1, & x_4 &= r \sin \vartheta \sin \phi_1. \end{aligned} \tag{6.13}$$

See Fig. 1c for a Vilenkin tree of the $d = 8$ generalized Hopf coordinates. In general the transformation formulae to Cartesian coordinates for generalized Hopf coordinates is given by

where $\vartheta_j \in [0, \frac{1}{2}\pi]$ for $1 \leq j \leq 2^{q-1}-1$ and $\phi_k \in [-\pi, \pi)$ for $1 \leq k \leq 2^{q-1}$. Generalized Hopf coordinates are unique in that they correspond to the only trees which contain only themselves in their equivalence class (see [9, (B.2)]). These coordinate systems have separated harmonic eigenfunctions which are given in terms of complex exponentials of the azimuthal angles, and for $q \geq 2$, non-symmetric Jacobi polynomials for the quantum numbers corresponding to the ϑ -angles.

The cosine of the separation angle (3.13) in these coordinates may be given as follows. Define the symbol ${}_qG_s^r \in [-1, 1]$, where $0 \leq s \leq q$ and $1 \leq r \leq 2^q - 1$, by the recursive formula

$${}_qG_s^r = \cos \vartheta_{r-1+2^{q-s}} \cos \vartheta'_{r-1+2^{q-s}} {}_qG_{s-1}^{2r-1} + \sin \vartheta_{r-1+2^{q-s}} \sin \vartheta'_{r-1+2^{q-s}} {}_qG_{s-1}^{2r},$$

with ${}_qG_0^i = 1$. Then the cosine of the separation angle is given by

$$\cos \gamma = {}_qG_q^1.$$

Note that through the identification $\phi_k = \Theta_{d/2+1-k}$, where $1 \leq k \leq 2^{q-1}$, then ${}_qG_1^i = \cos(\phi_i - \phi'_i)$. Thus, this shows one may stop this recursion at $s = 1$.

In regard to the quantum numbers in generalized Hopf coordinates, denote the meridional quantum numbers $l_k \in \mathbb{N}_0$ and the azimuthal quantum numbers $m_j \in \mathbb{Z}$, such that $1 \leq k \leq d/2 - 1$, $1 \leq j \leq d/2$. Define $\mathcal{L} := (l_1, l_2, \dots, l_{d/2-1})$, $\mathcal{M} := (m_{d/2}, \dots, m_1)$ where $l := l_1$ and $m := m_1$. In these coordinates, it is convenient to relate the meridional quantum numbers l_k to corresponding *surrogate* quantum numbers n_k using $l_k = 2n_k + l_{2k} + l_{2k+1}$, and we define $\mathcal{N} := (n_1, n_2, \dots, n_{d/2-1})$, where $n := n_1$, and $n_k \in \mathbb{N}_0$ for all $1 \leq k \leq d/2 - 1$. For convenience define $N := \sum_{k=1}^{d/2-1} n_k$, $M := \sum_{j=1}^{d/2} m_j$. Note that one can always write

$$\begin{aligned} l_1 &= 2N + M, \\ l_2 &= 2 \sum_{j=2}^{\log_2(d/2)} \sum_{k=0}^{2^j-2-1} n_{2^{j-1}+k} + \sum_{j=d/4+1}^{d/2} m_j, \\ l_3 &= 2 \sum_{j=2}^{\log_2(d/2)} \sum_{k=0}^{2^j-2-1} n_{2^{j-2}+2^{j-1}+k} + \sum_{j=1}^{d/4} m_j. \end{aligned}$$

In generalized Hopf coordinates, the normalized hyperspherical harmonics are given by [9, (B.21)]

$$Y_l^K(\mathbf{x}) = \frac{\prod_{1 \leq j \leq d/2} \exp(im_j \phi_j)}{\sqrt{2} \pi^{d/4}} \Upsilon_1^{\log_2 d} \left(\begin{matrix} n \\ l_2, l_3 \end{matrix}; \vartheta \right) \times \dots \times \Upsilon_{d/2-1}^{\log_2 d} \left(\begin{matrix} n_{d/2-1} \\ m_2, m \end{matrix}; \vartheta_{d/2-1} \right),$$

where $\Upsilon_k^{\log_2 d} : \mathbb{N}_0^3 \times [0, \frac{1}{2}\pi] \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} \Upsilon_k^{\log_2 d} \left(\begin{matrix} n_k \\ l_{2k}, l_{2k+1} \end{matrix}; \vartheta_k \right) &:= \sqrt{\frac{(2n_k + \alpha + \beta + 1)(n_k + \alpha + \beta)! n_k!}{(n_k + \alpha)!(n_k + \beta)!}} \\ &\quad \times (\cos \vartheta_k)^{l_{2k}} (\sin \vartheta_k)^{l_{2k+1}} P_n^{(\beta, \alpha)}(\cos(2\vartheta_k)), \\ \alpha &= \alpha_k^d(l_{2k}) := l_{2k} - 1 + 2^{\log_2(d/4) - \lfloor \log_2 k \rfloor}, \\ \beta &= \beta_k^d(l_{2k+1}) := l_{2k+1} - 1 + 2^{\log_2(d/4) - \lfloor \log_2 k \rfloor}. \end{aligned} \tag{6.15}$$

Remark 6.1. Note that if $d/4 \leq k \leq d/2 - 1$ then $-1 + 2^{\log_2(d/4) - \lfloor \log_2 k \rfloor} = 0$, hence $\alpha, \beta, l_{2k}, l_{2k+1} \in \mathcal{M}$. Also if $1 \leq k \leq d/4 - 1$, then $l_{2k}, l_{2k+1} \in \mathcal{L}$.

The addition theorem for hyperspherical harmonics (6.1) involves the product $Y_l^K(\mathbf{x})\overline{Y_l^K(\mathbf{x}')}$, so we introduce a convenient notation, namely $\Psi_k^{\log_2 d} : \mathbb{N}_0^3 \times [0, \frac{1}{2}\pi]^2 \rightarrow \mathbb{R}$ is defined by

$$\Psi_k^{\log_2 d} \left(\begin{matrix} n_k & ; & \vartheta_k \\ l_{2k}, l_{2k+1} & ; & \vartheta'_k \end{matrix} \right) := \Upsilon_k^{\log_2 d} \left(\begin{matrix} n_k & ; & \vartheta_k \\ l_{2k}, l_{2k+1} & ; & \vartheta_k \end{matrix} \right) \Upsilon_k^{\log_2 d} \left(\begin{matrix} n_k & ; & \vartheta'_k \\ l_{2k}, l_{2k+1} & ; & \vartheta'_k \end{matrix} \right). \tag{6.16}$$

Theorem 6.3. *In generalized Hopf coordinates, the product of normalized harmonics which appears in the addition theorem for hyperspherical harmonics (6.1) is given by*

$$Y_l^K(\mathbf{x})\overline{Y_l^K(\mathbf{x}')} = \frac{1}{2\pi^{\frac{d}{2}}} \epsilon_m \cos(m(\phi - \phi')) \epsilon_{m_2} \cos(m_2(\phi_{m_2} - \phi'_{m_2})) \cdots \epsilon_{m_{d/2}} \cos(m_{d/2}(\phi_{d/2} - \phi'_{d/2})) \\ \times \Psi_{d/2-1}^{\log_2 d} \left(\begin{matrix} n_{d/2-1} & ; & \vartheta_{d/2-1} \\ m_2, m & ; & \vartheta'_{d/2-1} \end{matrix} \right) \times \cdots \times \Psi_1^{\log_2 d} \left(\begin{matrix} n & ; & \vartheta \\ l_2, l_3 & ; & \vartheta' \end{matrix} \right). \tag{6.17}$$

Proof. In the sequel, we sum over all quantum numbers. If the coefficients of a product of Fourier series over $m_j \in \mathcal{M}$, whose coefficients are f_{m_j} , are invariant under sign reversal transformation, namely $f_{-m_j} = f_{m_j}$, we can rewrite the product of complex exponentials as a product of trigonometric cosine functions. This is accomplished using

$$\sum_{m_j \in \mathbb{Z}} f_j \exp(im_j \psi_j) = \sum_{m_j \in \mathbb{N}_0} \epsilon_m f_j \cos(m_j \psi_j),$$

where $\epsilon_m = 2 - \delta_{m,0}$. The eigenfunctions $\Psi_k^{\log_2 d}$ are invariant under this transformation, which can be verified by applying (2.22), (2.23) for $d/4 \leq k \leq d/2 - 1$ (see Remark 6.1). \square

6.2.1. Generalized Hopf multi-sum reversal lemmas

In generalized Hopf coordinates (as well as in the large-part of Vilenkin polyspherical coordinate systems), the procedure for summing over the degenerate quantum numbers can become somewhat technical. In the specialized case of generalized Hopf coordinates we outline the procedure for describing the multi-sums over the degenerate quantum numbers and derive descriptions of those same multi-sums, but instead with the sum orders reversed.

Lemma 6.4. *Let $p \in \mathbb{N}_0$, $1 \leq j \leq d/2$, $1 \leq k \leq d/2 - 1$. Consider the multi-sum Z_1 defined by*

$$Z_1 := \sum_{l=0}^p \sum_{l_2=0}^{\infty} \cdots \sum_{l_{2q-1-1}=0}^{\infty} \sum_{m_{2q-1}=0}^{\infty} \cdots \sum_{m_2=0}^{\infty} \sum_{m=0}^p \tag{6.18}$$

over $\mathcal{L} \cup \mathcal{M}$ with $0 \leq m_j \leq p$, $0 \leq l_k \leq p$, where the sum over the $l_k \in \mathcal{L}$ quantum numbers are restricted such that

$$n_k = \frac{l_k - l_{2k} - l_{2k+1}}{2} \in \mathbb{N}_0.$$

Then the multi-sum Z_1 can be re-expressed as a multi-sum over $\mathcal{N} \cup \mathcal{M}$, with sum order reversed to obtain

$$Z_1 = \sum_{m=0}^p \cdots \sum_{m_k=0}^{p-\sum_{i=1}^{k-1} m_i} \cdots \sum_{m_{2q-1}=0}^{p-\sum_{i=1}^{2q-1-1} m_i} \left[\sum_{n_{2q-1-1}=0}^{\lfloor \frac{p-\sum_{i=1}^{2q-1-1} m_i}{2} \rfloor} \cdots \sum_{n=0}^{\lfloor \frac{p-\sum_{i=1}^{2q-1} m_i - 2\sum_{i=2}^{2q-1-1} n_i}{2} \rfloor} \right]. \tag{6.19}$$

Proof. Let q denote an integer at least 2. Put $d = 2^q$ and suppose also that p is a positive integer. Let $\Lambda \subseteq \mathbb{R}^{d-1}$ consisting of $(d-1)$ -tuples $\bar{l} = (l_1, \dots, l_{d-1})$ satisfying the following conditions: (i) for k satisfying $1 \leq k \leq d-1$, $0 \leq l_k \leq p$; (ii) for k satisfying $1 \leq k \leq \frac{d}{2} - 1$, $l_k - l_{2k} - l_{2k+1}$ is a nonnegative, even integer. Let $S \subseteq \mathbb{R}^{d-1}$ denote the convex polytope defined by the following inequalities:

$$\begin{aligned} l_1 &\leq p, \\ l_k &\geq l_{2k} + l_{2k+1} \quad (1 \leq k < \frac{d}{2}), \\ l_k &\geq 0 \quad (\frac{d}{2} \leq k < d). \end{aligned}$$

If L denotes the lattice $L \subseteq \mathbb{Z}^{d-1}$ consisting of $(d-1)$ -tuples of integers for which $l_k - l_{2k} - l_{2k+1}$ is even for $1 \leq k < \frac{d}{2}$, then $\Lambda = L \cap S$. (The polytope S is a $(d-1)$ -simplex, as shown below.) For $1 \leq k < \frac{d}{2}$, put $n_k = \frac{1}{2}(l_k - l_{2k} - l_{2k+1})$; for $\frac{d}{2} \leq k < d$, put $n_k = l_k$. The function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ taking \bar{l} to $\bar{n} = (n_1, \dots, n_{d-1})$ is an invertible linear transformation. In particular, l_1 is given in terms of the n_k 's by

$$l_1 = 2(n_1 + \dots + n_{\frac{d}{2}-1}) + n_{\frac{d}{2}} + \dots + n_{d-1}.$$

The transformation takes S to the polytope $T = \varphi(S)$ defined by

$$\begin{aligned} n_k &\geq 0 \quad (1 \leq k < d), \\ 2(n_1 + \dots + n_{\frac{d}{2}-1}) + n_{\frac{d}{2}} + \dots + n_{d-1} &\leq p. \end{aligned}$$

The polytope T is a simplex, being the subset of the first orthant in \mathbb{R}^{d-1} that is cut off by a hyperplane. Therefore the linearly equivalent set S is also a simplex. The function φ takes L to \mathbb{Z}^{d-1} and consequently it takes $\Lambda := L \cap S$ to $\Gamma := \mathbb{Z}^{d-1} \cap T$. For a set $\{x_{\bar{n}}\}$ indexed by Γ , one has $\sum_{\bar{n} \in \Gamma} x_{\bar{n}} = \sum_{\bar{l} \in \Lambda} x_{\varphi(\bar{l})}$. This completes the proof. \square

Lemma 6.5. Let $p \in \mathbb{N}_0$, $1 \leq j \leq d/2$, $1 \leq k \leq d/2 - 1$. Consider the multi-sum Z_2 defined by

$$Z_2 := \sum_{l=p+1}^{\infty} \sum_{l_2=0}^{\infty} \dots \sum_{l_{2q-1-1}=0}^{\infty} \sum_{m_{2q-1}=0}^{\infty} \dots \sum_{m_2=0}^{\infty} \sum_{m=0}^p \tag{6.20}$$

over $\mathcal{L} \cup \mathcal{M}$ with $0 \leq m_j \leq p$, $l_k \geq p + 1$, with the same restriction over the $l_k \in \mathcal{L}$ in the above lemma. Then Z_2 can be re-expressed as a multi-sum over $\mathcal{N} \cup \mathcal{M}$ with sum order reversed to obtain

$$Z_2 = \sum_{m=0}^p \sum_{m_2=0}^{\infty} \dots \sum_{m_{2q-1}=0}^{\infty} \sum_{n_{2q-1-1}=0}^{\infty} \dots \sum_{n_2=0}^{\infty} \sum_{n=\max\left(0, \left\lfloor \frac{p - \sum_{i=1}^{2q-1} m_i - 2 \sum_{i=2}^{2q-1-1} n_i}{2} \right\rfloor + 1\right)}^{\infty} \tag{6.21}$$

Proof. Let $S' \subseteq \mathbb{R}^{d-1}$ denote the convex polyhedron defined by the following inequalities:

$$\begin{aligned} l_1 &\geq p + 1, \\ l_k &\geq l_{2k} + l_{2k+1} \quad (1 \leq k < \frac{d}{2}), \\ p &\geq l_k \geq 0 \quad (\frac{d}{2} \leq k < d). \end{aligned}$$

The polyhedron $T' = \varphi(S')$ is given by the following inequalities:

$$\begin{aligned} n_k &\geq 0 & (1 \leq k < d), \\ n_k &\leq p & \left(\frac{d}{2} \leq k < d\right), \\ 2(n_1 + \cdots + n_{\frac{d}{2}-1}) + n_{\frac{d}{2}} + \cdots + n_{d-1} &\geq p + 1. \end{aligned}$$

The set $\Lambda' := S' \cap L$ is mapped to $\Gamma' := T' \cap \mathbb{Z}^{d-1}$ by φ . For a set $\{x_{\bar{n}}\}$ indexed by Γ' , one has $\sum_{\bar{n} \in \Gamma'} x_{\bar{n}} = \sum_{\bar{l} \in \Lambda'} x_{\varphi(\bar{l})}$. This completes the proof. \square

Lemma 6.6. *Let $m, p, k \in \mathbb{N}_0$ such that $p = k - d/2 \in \mathbb{N}_0$, $d \in 2\mathbb{N}$ and $1 \leq k \leq d/2 - 1$. Consider the following multi-sum over $\mathcal{L} \cup \mathcal{M}$ with $m \geq p + 1$:*

$$Z_3 := \sum_{l=p+1}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_{d/2-1}=0}^{\infty} \sum_{m_{d/2}=0}^{\infty} \cdots \sum_{m_2=0}^{\infty} \sum_{m=p+1}^{\infty}, \quad (6.22)$$

with the same restriction over the $l_k \in \mathcal{L}$ in the above lemmas. Then the multi-sum Z_3 can be re-expressed as a multi-sum over $\mathcal{N} \cup \mathcal{M}$ with sum order reversed to obtain

$$Z_3 = \sum_{m=p+1}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{d/2}=0}^{\infty} \sum_{n_{d/2-1}=0}^{\infty} \cdots \sum_{n=0}^{\infty}. \quad (6.23)$$

Proof. Let $S'' \subseteq \mathbb{R}^{d-1}$ denote the convex polyhedron defined by the following inequalities:

$$\begin{aligned} l_k &\geq l_{2k} + l_{2k+1}, & (1 \leq k < \frac{d}{2}), \\ l_k &\geq 0, & (\frac{d}{2} \leq k < d), \\ l_{d-1} &\geq p + 1. \end{aligned}$$

The polyhedron $T'' = \varphi(S'')$ is given by the following inequalities:

$$\begin{aligned} n_k &\geq 0 & (1 \leq k < d), \\ n_{d-1} &\geq p + 1. \end{aligned}$$

The set $\Lambda'' := S'' \cap L$ is mapped to $\Gamma'' := T'' \cap \mathbb{Z}^{d-1}$ by φ . For a set $\{x_{\bar{n}}\}$ indexed by Γ'' , one has

$$\sum_{\bar{n} \in \Gamma''} x_{\bar{n}} = \sum_{\bar{l} \in \Lambda''} x_{\varphi(\bar{l})}.$$

This completes the proof. \square

7. Binomial and logarithmic addition theorems for the azimuthal Fourier coefficients

In even-dimensional Euclidean space \mathbb{R}^d , the kernels for a fundamental solution of the polyharmonic equation are represented by a function of the distance between two points (3.6), $\|\mathbf{x} - \mathbf{x}'\|$. In rotationally-invariant coordinates systems, one may represent the distance between the source and observation points in terms of the difference between azimuthal coordinates, namely (3.9). In Vilenkin's polyspherical coordinates, which are rotationally-invariant, one may also represent this distance in terms of the separation angle, namely (3.11). In a previous publication [9], addition theorems which arise when a fundamental solution of the polyharmonic equation is a power-law in Euclidean space with odd-dimensions and for even-dimensions when $1 \leq k < d/2$ were derived. In this manuscript, we treat the even-dimensional case for $k \geq d/2$, in

which the functional dependence is either logarithmic or of binomial form. By considering the equivalent azimuthal $\phi - \phi'$ and separation angle γ , Fourier and Gegenbauer expansions respectively of the kernels for the polyharmonic equation in even-dimensional space, one can derive addition theorems for the azimuthal Fourier coefficients.

The procedure for developing the addition theorems in this section is as follows. Let $p = k - d/2 \in \mathbb{N}_0$, with $f_p : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $g_p : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\} \rightarrow \mathbb{R}$, defined such that $f_p(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2p}$, $g_p(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2p} \log \|\mathbf{x} - \mathbf{x}'\|$. First, express f_p, g_p in terms of their azimuthal separation angle Fourier cosine series using (4.2), (4.3). Call these the left-hand sides. Then express f_p, g_p in terms of their separation angle Gegenbauer expansions given using (4.4), (4.6) with $\mu = d/2 - 1$. Call these the right-hand sides. By using the addition theorem for hyperspherical harmonics (6.1), we can expand the right-hand sides in terms of a product of separable harmonics in a chosen Vilenkin polyspherical coordinate system. Since Vilenkin's polyspherical coordinate systems are rotationally-invariant, one of the coordinates will correspond to the chosen azimuthal separation angle which has been expanded about on the left-hand side of the azimuthal Fourier expansion. To obtain the addition theorem, one must re-arrange the multi-sum expression which arises on the right-hand side so that the outermost sum is the sum over the relevant azimuthal quantum number. Addition theorems are simply derived by comparing the azimuthal Fourier coefficients on both sides.

In order to obtain binomial and logarithmic addition theorems in a Vilenkin polyspherical coordinate system, we relate respectively $\|\mathbf{x} - \mathbf{x}'\|^{2p}$ and $\|\mathbf{x} - \mathbf{x}'\|^{2p} \log \|\mathbf{x} - \mathbf{x}'\|$, in terms of their Fourier cosine series over the azimuthal separation angle, and their Gegenbauer polynomial expansions over the separation angle (3.13). These equalities reduce respectively to

$$(\chi - \cos(\phi - \phi'))^p = \left(\frac{rr'}{RR'}\right)^p (\zeta - \cos \gamma)^p, \tag{7.1}$$

and

$$\begin{aligned} &\log(2RR') (\chi - \cos(\phi - \phi'))^p + (\chi - \cos(\phi - \phi'))^p \log(\chi - \cos(\phi - \phi')) \\ &= \left(\frac{rr'}{RR'}\right)^p (\log(2rr') (\zeta - \cos \gamma)^p + (\zeta - \cos \gamma)^p \log(z - \cos \gamma)). \end{aligned} \tag{7.2}$$

7.1. Addition theorems in standard polyspherical coordinates

The binomial addition theorems in standard polyspherical coordinates are given by the following two theorems.

Theorem 7.1. *Let $m \in \mathbb{N}_0$, $p = k - d/2 \in \mathbb{N}_0$, $d \in 2\mathbb{N}$, $0 \leq m \leq p$. Then*

$$\begin{aligned} Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) &= (-1)^{\frac{d}{2}-1+m} (2\pi)^{\frac{d}{2}-1} (\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} (p-m)!(p+m)! \left(\frac{rr'}{RR'}\right)^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'}\right)^{p+\frac{d-1}{2}} \\ &\times \sum_{l_{d-2}=m}^p \Omega_{d-2}^d \left(l_{d-2}, m; \theta_{d-2}\right) \cdots \sum_{l=l_2}^p \Omega_1^d \left(l, l_2; \theta'\right) \frac{(-1)^l}{(p-l)!(l+p+d-2)!} Q_{l+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta). \end{aligned}$$

Proof. This equality can be found by comparing the binomial expansions of Theorems 5.3 and 5.4. The Gegenbauer polynomials in Theorem 5.4 can be expanded with the aid of (6.1). The normalized hyperspherical harmonics can be written in standard polyspherical coordinates, as seen in (6.3). Using the concise notation of (6.5), the Gegenbauer expansion of (6.1) can be written as follows:

$$C_l^{d/2-1}(\cos \gamma) = \frac{(d-2)\pi^{\frac{d}{2}-1}}{(2l+d-2)\Gamma(d/2)} \sum_K e^{im(\phi-\phi')} \Omega_{d-2}^d \left(l_{d-2}, m; \begin{matrix} \theta_{d-2} \\ \theta'_{d-2} \end{matrix} \right) \cdots \Omega_1^d \left(l, l_2; \begin{matrix} \theta \\ \theta' \end{matrix} \right). \tag{7.3}$$

Now the expansion of Theorem 5.4 will contain a multi-sum. Our goal is to compare the Fourier coefficients of the binomial expansions, this multi-sum will need to be reversed. This reversal is shown in Lemma 6.1. This allows us to compare the Fourier coefficients, and complete the proof. \square

Theorem 7.2. *Let $m \in \mathbb{N}_0, k \in \mathbb{N}, p = k - d/2 \in \mathbb{N}_0, d = 4$ and $0 \leq m \leq p$. Then*

$$\begin{aligned} Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) &= (-1)^{1-m} (\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} (p-m)!(p+m)! \left(\frac{rr'}{RR'} \right)^p \left(\frac{r_{>}^2 - r_{\leq}^2}{2rr'} \right)^{p+\frac{3}{2}} \\ &\times \sum_{l_2=m}^p \frac{2^{2l_2} (l_2!)^2 (2l_2+1)(l_2-m)!}{(l_2+m)!} (\sin \theta \sin \theta')^{l_2} P_{l_2}^m(\cos \theta_2) P_{l_2}^m(\cos \theta'_2) \\ &\times \sum_{l=l_2}^p \frac{(-1)^l (2l+2)(l-l_2)!}{(p-l)!(l+p+2)!(l+l_2+1)!} C_{l-l_2}^{l_2+1}(\cos \theta) C_{l-l_2}^{l_2+1}(\cos \theta') Q_{l+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta). \end{aligned}$$

Proof. To prove this, take the result of Theorem 7.1 and use $d = 4$. This simplifies the multi-sum of Ω functions:

$$\sum_{l_{d-2}} \Omega_{d-2}^d \left(l_{d-2}, m; \begin{matrix} \theta_{d-2} \\ \theta'_{d-2} \end{matrix} \right) \cdots \sum_l \Omega_1^d \left(l, l_2; \begin{matrix} \theta \\ \theta' \end{matrix} \right) \rightarrow \sum_{l_2} \Omega_2^4 \left(l_2, m; \begin{matrix} \theta_2 \\ \theta'_2 \end{matrix} \right) \sum_l \Omega_1^4 \left(l, l_2; \begin{matrix} \theta \\ \theta' \end{matrix} \right). \tag{7.4}$$

By (6.5), we know the Ω -functions are products of Θ -functions, which are defined in (6.4). A couple of the Gegenbauer polynomials that appear can be rewritten as Ferrers functions using (2.19). This allows (7.4) to be written as:

$$\begin{aligned} \sum_{l_2} \Omega_2^4 \left(l_2, m; \begin{matrix} \theta_2 \\ \theta'_2 \end{matrix} \right) \sum_l \Omega_1^4 \left(l, l_2; \begin{matrix} \theta \\ \theta' \end{matrix} \right) &= \frac{1}{\pi} \sum_{l_2} \frac{2^{2l_2} (l_2!)^2 (2l_2+1)(l_2-m)!}{(l_2+m)!} \\ &\times (\sin \theta \sin \theta')^{l_2} P_{l_2}^m(\cos \theta_2) P_{l_2}^m(\cos \theta'_2) \sum_l \frac{(l+1)(l-l_2)!}{(l+l_2+1)!} C_{l-l_2}^{l_2+1}(\cos \theta) C_{l-l_2}^{l_2+1}(\cos \theta'). \end{aligned} \tag{7.5}$$

Then simplification completes the proof. \square

In standard polyspherical coordinates, one has the following logarithmic addition theorem for $0 \leq m \leq p$.

Theorem 7.3. *Let $m \in \mathbb{N}_0, k \in \mathbb{N}, p = k - d/2 \in \mathbb{N}_0, d \in 2\mathbb{N}$ and $0 \leq m \leq p$. Then*

$$\begin{aligned} &\frac{1}{(p+m)!(p-m)!} \left(\log(RR') + \log(\chi + \sqrt{\chi^2 - 1}) + 2H_{2p} - H_{p+m} - H_{p-m} \right) Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \\ &+ \frac{1}{(p+m)!} \sum_{k=0}^{p-m-1} \frac{(2m+2k+1)}{k!(p-m-k)(p+m+k+1)} \left[1 + \frac{k!(p+m)!}{(2m+k)!(p-m)!} \right] Q_{m-\frac{1}{2}}^{m+k+\frac{1}{2}}(\chi) \\ &+ \frac{1}{(p-m)!} \sum_{k=0}^{m-1} \frac{2k+1}{(m+k)!(p-k)(p+k+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(\chi) \\ &= \left(\frac{rr'}{RR'} \right)^p (-1)^{m+d/2-1} (2\pi)^{d/2-1} \frac{(\zeta^2 - 1)^{\frac{p}{2} + \frac{d-1}{4}}}{(\chi^2 - 1)^{\frac{p}{2} + \frac{1}{4}}} \end{aligned}$$

$$\begin{aligned}
 & \times \left\{ \sum_{l_{d-2}=m}^p \Omega_{d-2}^d \left(l_{d-2}, m; \theta_{d-2} \right) \cdots \sum_{l_2=l_2}^p \Omega_1^d \left(l, l_2; \theta' \right) \right. \\
 & \times \left[\frac{(-1)^l Q_{l+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta)}{(p-l)!(p+l+d-2)!} \left(\log(rr') + \log(\zeta + \sqrt{\zeta^2 - 1}) + 2H_{2p+d-2} + H_p - H_{p+d/2-1} - H_{p+l+d-2} - H_{p-l} \right) \right. \\
 & + \frac{(-1)^l}{(p+l+d-2)!} \sum_{k=0}^{p-l-1} \frac{(2l+2k+d-1)}{k!(p-l-k)(p+l+k+d-1)} \left[1 + \frac{k!(p+l+d-2)!}{(k+2l+d-2)!(p-l)!} \right] Q_{l+\frac{d-1}{2}}^{k+l+\frac{d-1}{2}}(\zeta) \\
 & + \frac{(-1)^l}{(p-l)!} \sum_{k=0}^{l+d/2-2} \frac{2k+1}{(l+d/2+k-1)!(p+d/2-k-1)(p+d/2+k)} Q_{l+\frac{d-3}{2}}^{k+\frac{1}{2}}(\zeta) \left. \right] \\
 & + \sum_{l_{d-2}=m}^{\infty} \Omega_{d-2}^d \left(l_{d-2}, m; \theta_{d-2} \right) \cdots \sum_{l_2=l_3}^{\infty} \Omega_2^d \left(l_2, l_3; \theta'_2 \right) \\
 & \times \left. \sum_{l=\max(p+1, l_2)}^{\infty} \frac{(-1)^{p+1}(l-p-1)!}{(p+l+d-2)!} \Omega_1^d \left(l, l_2; \theta' \right) Q_{l+\frac{d-1}{2}}^{p+\frac{d-1}{2}}(\zeta) \right\}. \tag{7.6}
 \end{aligned}$$

Proof. To prove this theorem, we take the Fourier and Gegenbauer expansions of logarithmic fundamental solutions, seen in Theorems 5.1 and 5.2 respectively. For the Gegenbauer expansion, the substitution of $\mu = d/2 - 1$ is used. Again, the Gegenbauer polynomials can be rewritten as in (7.3). The Gegenbauer expansion contains three different multi-sums, one contains the sum of l from zero to p , the next contains the sum of l from zero to $p + 1$, and the last contains the sum of l from $p + 1$ to infinity. The first two of these sums can be reversed as in Lemma 6.1, where the later p can be replaced with $p + 1$. For the final multi-sum, Lemma 6.2 is used. Since only the region $0 \leq m \leq p$ is required, only the first term of the split multi-sum is used. When comparing the Fourier coefficients of the expansions, only those terms where $0 \leq m \leq p$ are used. Thus only the first two parts of the above multi-sum are used. \square

The following corollary results from substituting $d = 4$ in the above logarithmic addition theorem (7.6).

Corollary 7.4. *Let $0 \leq m \leq p$, $p = k - d/2 \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $d = 4$. Then*

$$\begin{aligned}
 & \frac{1}{(p+m)!(p-m)!} \left(\log(RR') + \log(\chi + \sqrt{\chi^2 - 1}) + 2H_{2p} - H_{p+m} - H_{p-m} \right) Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \\
 & + \frac{1}{(p+m)!} \sum_{k=0}^{p-m-1} \frac{(2m+2k+1)}{k!(p-m-k)(p+m+k+1)} \left[1 + \frac{k!(p+m)!}{(2m+k)!(p-m)!} \right] Q_{m-\frac{1}{2}}^{m+k+\frac{1}{2}}(\chi) \\
 & + \frac{1}{(p-m)!} \sum_{k=0}^{m-1} \frac{2k+1}{(m+k)!(p-k)(p+k+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(\chi) \\
 & = 2(-1)^{m+1} \left(\frac{rr'}{RR'} \right)^p \frac{(\zeta^2 - 1)^{\frac{p}{2} + \frac{3}{4}}}{(\chi^2 - 1)^{\frac{p}{2} + \frac{1}{4}}} \left\{ \sum_{l_2=m}^p \frac{(2l_2+1)(l_2-m)!}{(l_2+m)!} P_{l_2}^m(\cos \theta_2) P_{l_2}^m(\cos \theta'_2) 2^{2l_2} (l_2!)^2 (\sin \theta \sin \theta')^{l_2} \right. \\
 & \times \sum_{l=l_2}^p \frac{(-1)^l (l+1)(l-l_2)!}{(l+l_2+1)!} C_{l-l_2}^{l_2+1}(\cos \theta) C_{l-l_2}^{l_2+1}(\cos \theta') \\
 & \times \left[\frac{1}{(p-l)!(p+l+2)!} \left(\log(rr') + \log(\zeta + \sqrt{\zeta^2 - 1}) + 2H_{2p+2} + H_p - H_{p+1} - H_{p+l+2} - H_{p-l} \right) Q_{l+\frac{3}{2}}^{p+\frac{3}{2}}(\zeta) \right. \\
 & \left. \left. + \frac{1}{(p+l+2)!} \sum_{k=0}^{p-l-1} \frac{(2l+2k+3)}{k!(p-l-k)(p+l+k+3)} \left[1 + \frac{k!(p+l+2)!}{(k+2l+2)!(p-l)!} \right] Q_{l+\frac{1}{2}}^{k+l+\frac{3}{2}}(\zeta) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{(p-l)!} \sum_{k=0}^l \frac{2k+1}{(l+k+1)!(p-k+1)(p+k+2)} Q_{l+\frac{1}{2}}^{k+\frac{1}{2}}(\zeta) \Big] \\
 & + (-1)^{p+1} \sum_{l_2=m}^{\infty} \frac{(2l_2+1)(l_2-m)!}{(l_2+m)!} P_{l_2}^m(\cos \theta_2) P_{l_2}^m(\cos \theta'_2) 2^{2l_2} (l_2!)^2 (\sin \theta \sin \theta')^{l_2} \\
 & \times \sum_{l=\max(l_2, p+1)}^{\infty} \frac{(l+1)(l-l_2)!}{(l+l_2+1)!} C_{l-l_2}^{l_2+1}(\cos \theta) C_{l-l_2}^{l_2+1}(\cos \theta') \frac{(l-p-1)!}{(p+l+2)!} Q_{l+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta) \Big\}. \tag{7.7}
 \end{aligned}$$

Proof. To prove this, take the result of Theorem 7.3 for the $d = 4$ case. The Ω -functions can be simplified as in the proof of Theorem 7.2, as seen in (7.5). This completes the proof. \square

In standard polyspherical coordinates, one has the following logarithmic addition theorem for $m \geq p + 1$.

Theorem 7.5. Let $m \in \mathbb{N}_0$, $p = k - d/2 \in \mathbb{N}_0$, $d \in 2\mathbb{N}$, $m \geq p + 1$. Then

$$\begin{aligned}
 Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) &= (-1)^{\frac{d}{2}-1} (2\pi)^{\frac{d}{2}-1} (\zeta^2 - 1)^{\frac{p}{2} + \frac{d-1}{4}} (\chi^2 - 1)^{-\frac{p}{2} - \frac{1}{4}} \frac{(p+m)!}{(m-p-1)!} \left(\frac{rr'}{RR'} \right)^p \\
 &\times \sum_{l_{d-2}=p+1}^{\infty} \Omega_{d-2}^d \left(l_{d-2}, m; \begin{matrix} \theta_{d-2} \\ \theta'_{d-2} \end{matrix} \right) \cdots \sum_{l=l_2}^{\infty} \Omega_1^d \left(l, l_2; \begin{matrix} \theta \\ \theta' \end{matrix} \right) \frac{(l-p-1)!}{(p+l+d-2)!} Q_{l+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta). \tag{7.8}
 \end{aligned}$$

Proof. Again, this proof starts with equating the logarithmic expansions of Theorems 5.1 and 5.2. With the Gegenbauer expansion, the polynomials are rewritten as seen in (7.3). The Fourier coefficients of interest only appear in the multi-sum that has l summing over $p + 1$ to infinity. This multi-sum can be seen in Lemma 6.2. The second term of this multi-sum contains values of m that are greater than p , which are the terms needed for the comparison to complete the proof. \square

Corollary 7.6. Let $m \in \mathbb{N}_0$, $p = k - d/2 \in \mathbb{N}_0$, $d = 4$, $m \geq p + 1$. Then

$$\begin{aligned}
 Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) &= -2(\zeta^2 - 1)^{\frac{p}{2} + \frac{3}{4}} (\chi^2 - 1)^{-\frac{p}{2} - \frac{1}{4}} \frac{(p+m)!}{(m-p-1)!} \left(\frac{rr'}{RR'} \right)^p \\
 &\times \sum_{l_2=p+1}^{\infty} \frac{2^{2l_2} (l_2!)^2 (2l_2+1)(l_2-m)!}{(l_2+m)!} \\
 &\times (\sin \theta \sin \theta')^{l_2} P_{l_2}^m(\cos \theta_2) P_{l_2}^m(\cos \theta'_2) \sum_{l=l_2}^{\infty} \frac{(l+1)(l-l_2)!}{(l+l_2+1)!} C_{l-l_2}^{l_2+1}(\cos \theta) C_{l-l_2}^{l_2+1}(\cos \theta') Q_{l-\frac{1}{2}}^{p+\frac{3}{2}}(\zeta). \tag{7.9}
 \end{aligned}$$

Proof. Start with (7.8) and set $d = 4$. The Ω -functions can be simplified as in (7.5), which completes the proof. \square

7.2. Addition theorems in generalized Hopf coordinates

In generalized Hopf coordinates, one has the following binomial addition theorem valid for $0 \leq m \leq p$.

Theorem 7.7. Let $p, m \in \mathbb{N}_0$, $0 \leq m \leq p$, $d = 2^q$, $q \geq 2$. Then

$$Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) = (-1)^{m+d/2-1} 2^{d/2-1} (p-m)!(p+m)! (\chi^2 - 1)^{-\frac{p}{2} - \frac{1}{4}} \left(\frac{rr'}{RR'} \right)^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{d-1}{2}}$$

$$\begin{aligned}
 & \times \sum_{m_2=0}^{p-m} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) \cdots \sum_{m_k=0}^{p-\sum_{i=1}^{k-1} m_i} \cdots \sum_{m_{d/2}=0}^{p-\sum_{i=1}^{d/2-1} m_i} \epsilon_{m_{d/2}} \cos(m_{d/2}(\phi_{d/2} - \phi'_{d/2})) \\
 & \times (-1)^M \sum_{n_{d/2-1}=0}^{\lfloor \frac{p-M}{2} \rfloor} \Psi_{d/2-1}^{\log_2 d} \left(\begin{matrix} n_{d/2-1} \\ m_2, m \end{matrix}; \vartheta_{d/2-1} \right) \cdots \sum_{n=0}^{\lfloor \frac{p-M-2\sum_{i=2}^{d/2-1} n_i}{2} \rfloor} \Psi_1^{\log_2 d} \left(\begin{matrix} n \\ l_2, l_3 \end{matrix}; \vartheta' \right) \\
 & \times \frac{1}{(p-2N-M)!(p+2N+M+d-2)!} Q_{2N+M+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta). \tag{7.10}
 \end{aligned}$$

Proof. Start with the Gegenbauer expansion of a fundamental solution given by Theorem 5.4. The hyper-spherical harmonics contained within can be expanded with the aid of (6.1) and Theorem 6.3 to be

$$\begin{aligned}
 C_l^{d/2-1}(\cos \gamma) &= \frac{(d-2)}{(2l+d-2)\Gamma(d/2)} \\
 & \times \sum_K \epsilon_m \cos(m(\phi - \phi')) \epsilon_{m_2} \cos(m_2(\phi_{m_2} - \phi'_{m_2})) \cdots \epsilon_{m_{d/2}} \cos(m_{d/2}(\phi_{d/2} - \phi'_{d/2})) \\
 & \times \Psi_{d/2-1}^{\log_2 d} \left(\begin{matrix} n_{d/2-1} \\ m_2, m \end{matrix}; \vartheta_{d/2-1} \right) \times \cdots \times \Psi_1^{\log_2 d} \left(\begin{matrix} n \\ l_2, l_3 \end{matrix}; \vartheta' \right). \tag{7.11}
 \end{aligned}$$

The multi-sum in (7.11) can be combined with the sum in (5.4), and with the ability to reverse this multi-sum given by Lemma 6.4 yields:

$$\begin{aligned}
 \mathfrak{h}_k^d(\mathbf{x}, \mathbf{x}') &= \frac{2^{\frac{d}{2}-\frac{1}{2}}}{\sqrt{\pi}} e^{i\pi(p-\frac{d}{2}+\frac{1}{2})} p!(2rr')^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{d-1}{2}} \sum_{m=0}^{p-m} \epsilon_m \cos(m(\phi - \phi')) \\
 & \times \sum_{m_2=0}^{p-m} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) \cdots \sum_{m_k=0}^{p-\sum_{i=1}^{k-1} m_i} \cdots \sum_{m_{d/2}=0}^{p-\sum_{i=1}^{d/2-1} m_i} \epsilon_{m_{d/2}} \cos(m_{d/2}(\phi_{d/2} - \phi'_{d/2})) \\
 & \times (-1)^M \sum_{n_{d/2-1}=0}^{\lfloor \frac{p-M}{2} \rfloor} \Psi_{d/2-1}^{\log_2 d} \left(\begin{matrix} n_{d/2-1} \\ m_2, m \end{matrix}; \vartheta_{d/2-1} \right) \cdots \sum_{n=0}^{\lfloor \frac{p-M-2\sum_{i=2}^{d/2-1} n_i}{2} \rfloor} \Psi_1^{\log_2 d} \left(\begin{matrix} n \\ l_2, l_3 \end{matrix}; \vartheta' \right) \\
 & \times \frac{1}{(p-2N-M)!(2N+M+p+d-2)!} Q_{l+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta). \tag{7.12}
 \end{aligned}$$

The Fourier coefficients of this can be compared with the Fourier expansion of a fundamental solution given in Theorem 5.3. This completes the proof \square

The simplest example of a binomial addition theorem for $0 \leq m \leq p$ in generalized Hopf coordinates occurs in four dimensions.

Theorem 7.8. Let $p, m \in \mathbb{N}_0$, $0 \leq m \leq p$, $p = k - d/2 \in \mathbb{N}_0$ and $d = 4$. Then

$$Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) = -2(p-m)!(p+m)!(\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} \left(\frac{rr'}{RR'} \right)^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{3}{2}} (\sin \vartheta \sin \vartheta')^m$$

$$\begin{aligned} & \times \sum_{m_2=0}^{p-m} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) (-1)^{m_2} (\cos \vartheta \cos \vartheta')^{m_2} \sum_{n=0}^{\lfloor \frac{p-m-m_2}{2} \rfloor} P_n^{(m, m_2)}(\cos(2\vartheta)) P_n^{(m, m_2)}(\cos(2\vartheta')) \\ & \times \frac{(2n + m + m_2 + 1)(n + m + m_2)! n!}{(p - 2n - m - m_2)!(p + 2n + m + m_2 + 2)!(n + m)!(n + m_2)!} Q_{2n+m+m_2+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta). \end{aligned} \tag{7.13}$$

Proof. Starting with Theorem 7.7, let $d = 4$. From this we obtain the following:

$$\begin{aligned} Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) &= -2(p - m)!(p + m)!(\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} \left(\frac{rr'}{RR'} \right)^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{3}{2}} \\ & \times \sum_{m_2=0}^{p-m} (-1)^{m_2} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) \sum_{n=0}^{\lfloor \frac{p-m-m_2}{2} \rfloor} \Psi_1^2 \left(\begin{matrix} n & \vartheta \\ m, m_2 & \vartheta' \end{matrix} \right) \\ & \times \frac{1}{(p - 2n - m - m_2)!(p + 2n + m + m_2 + 2)!} Q_{2n+m+m_2+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta). \end{aligned} \tag{7.14}$$

Then by (6.15), (6.16), the function denoted by Ψ can be written in terms of Jacobi polynomials:

$$\begin{aligned} \Psi_1^2 \left(\begin{matrix} n & \vartheta \\ m, m_2 & \vartheta' \end{matrix} \right) &= \frac{(2n + m + m_2 + 1)(n + m + m_2)! n!}{(n + m)!(n + m_2)!} (\sin \vartheta \sin \vartheta')^m (\cos \vartheta \cos \vartheta')^{m_2} \\ & \times P_n^{(m, m_2)}(\cos(2\vartheta)) P_n^{(m, m_2)}(\cos(2\vartheta')). \end{aligned} \tag{7.15}$$

Using this identity with (7.14) completes the proof. \square

In generalized Hopf coordinates, one has the following logarithmic addition theorem valid for $0 \leq m \leq p$.

Theorem 7.9. Let $0 \leq m \leq p$, $d = 2^q$, $q \geq 2$ and $p = k - d/2 \in \mathbb{N}_0$. Then

$$\begin{aligned} & \frac{1}{(p + m)!(p - m)!} \left(\log(RR') + \log(\chi + \sqrt{\chi^2 - 1}) + 2H_{2p} - H_{p+m} - H_{p-m} \right) Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \\ & + \frac{1}{(p + m)!} \sum_{k=0}^{p-m-1} \frac{(2m + 2k + 1)}{k!(p - m - k)(p + m + k + 1)} \left[1 + \frac{k!(p + m)!}{(2m + k)!(p - m)!} \right] Q_{m-\frac{1}{2}}^{m+k+\frac{1}{2}}(\chi) \\ & + \frac{1}{(p - m)!} \sum_{k=0}^{m-1} \frac{2k + 1}{(m + k)!(p - k)(p + k + 1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(\chi) \\ & = (-1)^{m+\frac{d}{2}-1} 2^{\frac{d}{2}-1} \left(\frac{rr'}{RR'} \right)^p (\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{d-1}{2}} \\ & \times \left\{ \sum_{m_2=0}^{p-m} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) \cdots \sum_{m_j=0}^{p-\sum_{i=1}^{j-1} m_i} \epsilon_{m_j} \cos(m_j(\phi_j - \phi'_j)) \cdots \sum_{m_{\frac{d}{2}}=0}^{p-\sum_{i=1}^{\frac{d}{2}-1} m_i} \epsilon_{m_{\frac{d}{2}}} \cos(m_{\frac{d}{2}}(\phi_{\frac{d}{2}} - \phi'_{\frac{d}{2}})) \right\} \\ & \times (-1)^M \sum_{n_{\frac{d}{2}-1}=0}^{\lfloor \frac{p-M}{2} \rfloor} \Psi_{\frac{d}{2}-1}^{\log_2 d} \left(\begin{matrix} n_{\frac{d}{2}-1} & \vartheta_{\frac{d}{2}-1} \\ m_2, m & \vartheta'_{\frac{d}{2}-1} \end{matrix} \right) \cdots \sum_{n=0}^{\lfloor \frac{p+2n-2N-M}{2} \rfloor} \Psi_1^{\log_2 d} \left(\begin{matrix} n & \vartheta \\ l_2, l_3 & \vartheta' \end{matrix} \right) \end{aligned}$$

$$\begin{aligned}
 & \times \left(\frac{2 \log r_{>} + 2H_{2p+d-2} + H_p - H_{p+\frac{d}{2}-1} - H_{p+M+2N+d-2} - H_{p-2N-M}}{(p-2N-M)!(p+2N+M+d-2)!} Q_{2N+M+\frac{d-1}{2}}^{p+\frac{d-1}{2}}(\zeta) \right. \\
 & + \frac{1}{(p+2N+M+d-2)!} \sum_{k=0}^{p-2N-M-1} \frac{(4N+2M+2k+d-1)}{k!(p-k-2N-M)(p+k+2N+M+d-1)} \\
 & \quad \times \left[1 + \frac{k!(p+2N+M+d-2)!}{(k+4N+2M+d-2)!(p-2N-M)!} \right] Q_{2N+M+\frac{d-1}{2}}^{k+2N+M+\frac{d-1}{2}}(\zeta) \\
 & + \frac{1}{(p-2N-M)!} \sum_{k=0}^{2N+M+\frac{d}{2}-2} \frac{2k+1}{(2N+M+k+\frac{d}{2}-1)!(p+\frac{d}{2}-k-1)(p+k+\frac{d}{2})} Q_{2N+M+\frac{d-3}{2}}^{k+\frac{1}{2}}(\zeta) \left. \right) \\
 & + (-1)^{p+1} \sum_{m_2=0}^{\infty} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) \cdots \sum_{m_{\frac{d}{2}}=0}^{\infty} \epsilon_{m_{\frac{d}{2}}} \cos(m_{\frac{d}{2}}(\phi_{\frac{d}{2}} - \phi'_{\frac{d}{2}})) \\
 & \times \sum_{n_{\frac{d}{2}-1}=0}^{\infty} \Psi_{\frac{d}{2}-1}^{\log_2 d} \left(\begin{matrix} n_{\frac{d}{2}-1} \\ m_2, m \end{matrix}; \vartheta_{\frac{d}{2}-1} \right) \cdots \sum_{n_2=0}^{\infty} \Psi_2^{\log_2 d} \left(\begin{matrix} n_2 \\ l_4, l_5 \end{matrix}; \vartheta_2 \right) \\
 & \times \left. \sum_{n=\max(0, \lfloor \frac{p-2N-M}{2} \rfloor + 1)}^{\infty} \frac{(2N+M-p-1)!}{(2N+M+p+d-2)!} \Psi_1^{\log_2 d} \left(\begin{matrix} n \\ l_2, l_3 \end{matrix}; \vartheta' \right) Q_{2N+M+\frac{d-1}{2}}^{p+\frac{d-1}{2}}(\zeta) \right\}. \tag{7.16}
 \end{aligned}$$

Proof. Again, we start by comparing the Fourier and Gegenbauer expansions of the logarithmic fundamental solutions. For the Gegenbauer expansion, replace μ with $\frac{d}{2} - 1$, use (3.14) and the following relation:

$$\log(rr') + \log(\zeta + \sqrt{\zeta^2 - 1}) = 2 \log r_{>}.$$

The Gegenbauer expansion of Theorem 5.2 can be rewritten in Hopf coordinates by use of (7.11). Next, we consider the multi-sums. The first sums of $\sum_{l=0}^p \sum_K$, are dealt with by Lemma 6.4. The later sum uses Lemma 6.5. Using the identity $l = 2N + M$ and compare the Fourier coefficients to complete the proof. \square

The simplest example of a logarithmic addition theorem for $0 \leq m \leq p$ in generalized Hopf coordinates occurs in four dimensions.

Corollary 7.10. *Let $0 \leq m \leq p$, $m \in \mathbb{N}_0$, $p = k - d/2 \in \mathbb{N}_0$ where $d = 4$. Then*

$$\begin{aligned}
 & \frac{1}{(p+m)!(p-m)!} \left(\log(RR') + \log(\chi + \sqrt{\chi^2 - 1}) + 2H_{2p} - H_{p+m} - H_{p-m} \right) Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) \\
 & + \frac{1}{(p+m)!} \sum_{k=0}^{p-m-1} \frac{(2m+2k+1)}{k!(p-m-k)(p+m+k+1)} \left[1 + \frac{k!(p+m)!}{(2m+k)!(p-m)!} \right] Q_{m-\frac{1}{2}}^{m+k+\frac{1}{2}}(\chi) \\
 & + \frac{1}{(p-m)!} \sum_{k=0}^{m-1} \frac{2k+1}{(m+k)!(p-k)(p+k+1)} Q_{m-\frac{1}{2}}^{k+\frac{1}{2}}(\chi) \\
 & = -2 \left(\frac{rr'}{RR'} \right)^p (\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{3}{2}} (\sin \vartheta \sin \vartheta')^m \\
 & \times \left\{ \sum_{m_2=0}^{p-m} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) (\cos \vartheta \cos \vartheta')^{m_2} (-1)^{m_2} \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{n=0}^{\lfloor \frac{p-m-m_2}{2} \rfloor} \frac{(2n+m+m_2+1)(n+m+m_2)!n!}{(n+m)!(n+m_2)!} P_n^{(m,m_2)}(\cos(2\vartheta)) P_n^{(m,m_2)}(\cos(2\vartheta')) \\
 & \times \left(\frac{2 \log r_{>} + 2H_{2p+2} + H_p - H_{p+1} - H_{p+2n+m+m_2+2} - H_{p-2n-m-m_2}}{(p-2n-m-m_2)!(p+2n+m+m_2+2)!} Q_{2n+m+m_2+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta) \right. \\
 & + \frac{1}{(p+2n+m+m_2+2)!} \sum_{k=0}^{p-2n-m-m_2-1} \frac{(4n+2m+2m_2+2k+3)}{k!(p-k-2n-m-m_2)(p+k+2n+m+m_2+3)} \\
 & \quad \times \left[1 + \frac{k!(p+2n+m+m_2+2)!}{(k+4n+2m+2m_2+2)!(p-2n-m-m_2)!} \right] Q_{2n+m+m_2+\frac{1}{2}}^{k+2n+m+m_2+\frac{3}{2}}(\zeta) \\
 & + \frac{1}{(p-2n-m-m_2)!} \sum_{k=0}^{2n+m+m_2} \frac{2k+1}{(2n+m+m_2+k+1)!(p-k+1)(p+k+2)} Q_{2n+m+m_2+\frac{1}{2}}^{k+\frac{1}{2}}(\zeta) \left. \right) \\
 & + (-1)^{p+m+1} \sum_{m_2=0}^{\infty} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) (\cos \vartheta \cos \vartheta')^{m_2} \\
 & \times \sum_{n=\max(0, \lfloor \frac{p-m-m_2}{2} \rfloor + 1)}^{\infty} \frac{(2n+m+m_2+1)(n+m+m_2)!(2n+m+m_2-p-1)!n!}{(n+m)!(n+m_2)!(2n+m+m_2+p+2)!} \\
 & \quad \times P_n^{(m,m_2)}(\cos(2\vartheta)) P_n^{(m,m_2)}(\cos(2\vartheta')) Q_{2n+m+m_2+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta) \left. \right\}.
 \end{aligned}$$

Proof. When $d = 4$, note that $N = n$ and $M = m + m_2$. The functions of the Hopf coordinate system can be simplified as in Theorem 7.8. \square

The general logarithmic addition theorem for $m \geq p + 1$ in generalized Hopf coordinates is given as follows.

Theorem 7.11. *Let $p, m \in \mathbb{N}_0$, $m \geq p + 1$, $d = 2^q$, $q \geq 2$. Then*

$$\begin{aligned}
 Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) &= (-2)^{\frac{d}{2}-1} 2^{\frac{d}{2}-1} \frac{(p+m)!}{(m-p-1)!} (\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} \left(\frac{rr'}{RR'} \right)^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{d-1}{2}} \\
 & \times \sum_{m_2=0}^{\infty} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) \times \cdots \times \sum_{m_{\frac{d}{2}}=0}^{\infty} \epsilon_{m_{\frac{d}{2}}} \cos(m_{\frac{d}{2}}(\phi_{\frac{d}{2}} - \phi'_{\frac{d}{2}})) \sum_{n_{\frac{d}{2}-1}=0}^{\infty} \Psi_{\frac{d}{2}-1}^{\log_2 d} \left(\begin{matrix} n_{\frac{d}{2}-1} \\ m_2, m \end{matrix}; \begin{matrix} \vartheta_{\frac{d}{2}-1} \\ \vartheta'_{\frac{d}{2}-1} \end{matrix} \right) \\
 & \times \cdots \times \sum_{n=0}^{\infty} \Psi_1^{\log_2 d} \left(\begin{matrix} n \\ l_2, l_3 \end{matrix}; \vartheta \right) \frac{(2N + M - p - 1)!}{(p + 2N + M + d - 2)!} Q_{2N+M+\frac{d-3}{2}}^{p+\frac{d-1}{2}}(\zeta). \tag{7.17}
 \end{aligned}$$

Proof. This is similar to the proof of Theorem 7.9. For this case only the Fourier coefficients where $m \geq p + 1$ are used, therefore Lemma 6.6 is needed. Simplifying the expression completes the proof. \square

The simplest example of a logarithmic addition theorem for $m \geq p + 1$ in generalized Hopf coordinates occurs in four dimensions.

Theorem 7.12. *Let $p, m \in \mathbb{N}_0$, $m \geq p + 1$. Then*

$$Q_{m-\frac{1}{2}}^{p+\frac{1}{2}}(\chi) = -2 \frac{(p+m)!}{(m-p-1)!} (\chi^2 - 1)^{-\frac{p}{2}-\frac{1}{4}} \left(\frac{rr'}{RR'} \right)^p \left(\frac{r_{>}^2 - r_{<}^2}{2rr'} \right)^{p+\frac{3}{2}} (\sin \vartheta \sin \vartheta')^m$$

$$\begin{aligned}
 & \times \sum_{m_2=0}^{\infty} \epsilon_{m_2} \cos(m_2(\phi_2 - \phi'_2)) (\cos \vartheta \cos \vartheta')^{m_2} \sum_{n=0}^{\infty} P_n^{(m, m_2)}(\cos(2\vartheta)) P_n^{(m, m_2)}(\cos(2\vartheta')) \\
 & \times \frac{(2n + m + m_2 + 1)(2n + m + m_2 - p - 1)!(n + m + m_2)!n!}{(p + 2n + m + m_2 + 2)!(n + m)!(n + m_2)!} Q_{2n+m+m_2+\frac{1}{2}}^{p+\frac{3}{2}}(\zeta). \tag{7.18}
 \end{aligned}$$

Proof. Simplifying as in Theorem 7.9 for $d = 4$ completes the proof. \square

8. Conclusion and future outlook

With the publication of this work, we have fully-implemented the procedure which appeared for the first time in the Cohl mathematics thesis [5, §§6.3-4] for computing azimuthal Fourier expansions of fundamental solutions of the polyharmonic operator $(-\Delta)^k$ on even dimensional Euclidean space \mathbb{R}^{2n} where the power of the Laplacian is restricted to $k \geq n$. This effort illustrates the geometric notion concerning rotationally-invariant coordinate systems which solve the underlying partial differential equation using separation and \mathcal{R} -separation of variables (for both types we henceforward refer to as separable).

For a given linear partial differential operator which admits separable solutions in a given coordinate system on a Riemannian manifold, in principle, one may expand a fundamental solution of that partial differential operator in terms of the separated eigenfunctions, which provide a basis for solutions to the partial differential equation, over their corresponding degenerate quantum numbers. As a function of the dimension of the space, these systems are numerous! For the three-dimensional Laplace equation, there are 17 conformally inequivalent separable coordinate systems. As far as the authors are aware, there has been no comprehensive study of the exact number of inequivalent separable Laplace coordinate systems as a function of dimension, but one can find at least an extreme lower bound by examining for instance Vilenkin’s polyspherical coordinates. Remember, in three-dimensions, spherical coordinates represent only one in the 17 separable coordinate systems. In [9] it was shown that the sequence of numbers of inequivalent Vilenkin polyspherical coordinate systems of this type is for $d = 2, \dots, 13$ given by 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, 983, and the sequence in general is given by the Wedderburn–Etherington numbers (see for instance the On-Line Encyclopedia of Integer Sequences [30, A001190]) and for $d = 34$, the number of inequivalent Vilenkin polyspherical coordinate systems is 44, 214, 569, 100. Therefore the growth of the numbers of inequivalent coordinate systems as a function of dimension in this case is exponential. In every one of these coordinate systems, there exist special functions which arise out of this procedure. It is mystifying to imagine their full quality, especially considering some of separable solutions to the three-dimensional problem for Laplace’s equation, most likely, have never been computed (take for instance, the special functions which arise in the asymmetric cyclidic coordinate systems which are solutions of a second order differential equation with five regular singularities [15–17]).

Now the problem at hand is the determination of addition theorems for the azimuthal Fourier coefficients in rotationally-invariant coordinate systems. Examining the particular case of the three-dimensional Laplace equation, 9 of the 17 conformally inequivalent coordinate systems are rotationally-invariant about an axis of symmetry (spherical, circular cylindrical, parabolic, oblate spheroidal, prolate spheroidal, toroidal, flat-ring cyclidic, flat-disk cyclidic, bi-cyclidic). For Laplace’s equation, the azimuthal Fourier component of its fundamental solution was computed in [14]. To date, the various addition theorems in these coordinate systems, have not been computed, we are close! In fact, in each of these coordinate systems, there will be (at most) two different addition theorems since there are in general different harmonic solutions corresponding to both coordinate equals constant surfaces. The special functions which arise in separation of variables in these coordinate systems include Legendre functions, Bessel functions, Lamé functions and Lamé-Wangerin functions (see the discussions in [3]).

For the polyharmonic equation in Euclidean space \mathbb{R}^d , as of the publication of this paper, the identification of the azimuthal Fourier components are now known for all powers of the Laplacian operator and for all

dimensions. These are given in terms of associated Legendre functions of the second kind with argument given by a hyper-toroidal function of all space. So addition theorems arising from comparing the azimuthal Fourier components with the eigenfunction expansions in terms of the described separated systems is clearly obtainable. All that remains is to obtain the separated eigenfunction expansions and integrals in the various separable coordinate systems and to compute the full expansion of a fundamental solution in that given coordinate system. We have now computed a smattering of all the possible addition theorems (we've only computed a vanishingly small number of possible Vilenkin polyspherical coordinate systems), the world is at hand to investigate their remainder, not to mention all those coordinate systems where harmonic (or polyharmonic) solutions are not straightforwardly computable, of which there are many.

There are some future steps that we would like take to further the investigation of this rich special functions problem. These include the corresponding study for the polyharmonic operator (natural powers of the Laplace-Beltrami operator) in spaces of constant curvature [7,12,13] and beyond such as in the rank-one symmetric spaces. The rank-one symmetric spaces are the real, complex, quaternionic and octonionic (Cayley) plane hyperbolic and projective spaces. These spaces are isotropic (two-point homogeneous Riemannian) so their fundamental solutions satisfy ordinary differential equations and are therefore straightforward to compute. In these spaces it is known that separable solutions exist, so it is compelling to obtain some example addition theorems in their rotationally-invariant coordinate systems for their harmonic problem. Of course, other intriguing problems include the application of this problem while meandering into other linear partial differential operators (as well as their natural powers) such as Helmholtz [10], heat and wave operators in Euclidean or curved isotropic spaces.

Acknowledgments

We would like to thank A.F.M. Tom ter Elst, Matthew Auger and Radosław Szmytkowski for valuable discussions. H.S. Cohl would also like to thank Ernie Kalnins for originally pointing out the existence and utility of Vilenkin's polyspherical coordinates.

References

- [1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, U.S. Government Printing Office, Washington, D.C., 1972.
- [2] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Encyclopedia of Mathematics and Its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
- [3] L. Bi, H.S. Cohl, H. Volkmer, Peanut harmonic expansion for a fundamental solution of Laplace's equation in flat-ring coordinates, *Anal. Math.* (2022), in press.
- [4] J.B. Boyling, Green's functions for polynomials in the Laplacian, *Z. Angew. Math. Phys.* 47 (3) (1996) 485–492.
- [5] H.S. Cohl, Fourier and Gegenbauer expansions for fundamental solutions of the Laplacian and powers in \mathbf{R}^d and \mathbf{H}^d , PhD thesis, The University of Auckland, Auckland, New Zealand, 2010, xiv+190 pages.
- [6] H.S. Cohl, On parameter differentiation for integral representations of associated Legendre functions, in: Special Issue on Symmetry, Separation, Super-Integrability and Special Functions (S^4), *SIGMA* 7 (2011) 050.
- [7] H.S. Cohl, Opposite antipodal fundamental solution of Laplace's equation in hyperspherical geometry, *SIGMA* 7 (108) (2011) 14.
- [8] H.S. Cohl, Fourier expansions for logarithmic fundamental solutions of the polyharmonic equation, *J. Class. Anal.* 2 (2) (2013) 107–127.
- [9] H.S. Cohl, Fourier, Gegenbauer and Jacobi expansions for a power-law fundamental solution of the polyharmonic equation and polyspherical addition theorems, *SIGMA* 9 (042) (2013) 26.
- [10] H.S. Cohl, T.H. Dang, T.M. Dunster, Fundamental solutions and Gegenbauer expansions of Helmholtz operators in Riemannian spaces of constant curvature, in: Special Issue on Orthogonal Polynomials, Special Functions and Applications (OPSFA14), *SIGMA* 14 (2018) 136.
- [11] H.S. Cohl, D.E. Dominici, Generalized Heine's identity for complex Fourier series of binomials, *Proc. R. Soc. A* 467 (2011) 333–345.
- [12] H.S. Cohl, E.G. Kalnins, Fourier and Gegenbauer expansions for a fundamental solution of the Laplacian in the hyperboloid model of hyperbolic geometry, *J. Phys. A, Math. Theor.* 45 (14) (2012) 145206.
- [13] H.S. Cohl, R.M. Palmer, Fourier and Gegenbauer expansions for a fundamental solution of Laplace's equation in hyperspherical geometry, in: Special Issue on Exact Solvability and Symmetry Avatars in Honour of Luc Vinet, *SIGMA* 11 (2015) 015.

- [14] H.S. Cohl, J.E. Tohline, A compact cylindrical Green's function expansion for the solution of potential problems, *Astrophys. J.* 527 (1999) 86–101.
- [15] H.S. Cohl, H. Volkmer, Publisher's Note: "Separation of variables in an asymmetric cyclidic coordinate system", *J. Math. Phys.* 54 (7) (2013) 079904.
- [16] H.S. Cohl, H. Volkmer, Separation of variables in an asymmetric cyclidic coordinate system, *J. Math. Phys.* 54 (6) (2013) 063513.
- [17] H.S. Cohl, H. Volkmer, Expansions for a fundamental solution of Laplace's equation on \mathbb{R}^3 in 5-cyclidic harmonics, in: *Special Issue: Dedicated to the Memory of Frank Olver*, *Anal. Appl.* 12 (6) (2014) 613–633.
- [18] F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller, B.V. Saunders, H.S. Cohl, M.A. McClain (Eds.), *NIST Digital Library of Mathematical Functions Release 1.1.6 of 2022-06-30*, <https://dlmf.nist.gov/>.
- [19] U. Fano, A.R.P. Rau, *Symmetries in Quantum Physics*, Academic Press Inc., San Diego, CA, 1996.
- [20] I.M. Gel'fand, G.E. Shilov, *Generalized Functions, Vol. 1: Properties and Operations*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 (1997). Translated from the Russian by Eugene Saletan.
- [21] B.T. Helenbrook, Mesh deformation using the biharmonic operator, *Int. J. Numer. Methods Eng.* 56 (7) (2003) 1007–1021.
- [22] B.J. Kirby, *Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices*, Cambridge University Press, Cambridge, 2010.
- [23] W.M. Lai, D. Rubin, E. Kreml, *Introduction to continuum mechanics, revised edition, SI/metric units*, Pergamon, Oxford, UK, 1978.
- [24] S.A. Lurie, V.V. Vasiliev, *The Biharmonic Problem in the Theory of Elasticity*, Gordon and Breach Publishers, Luxembourg, 1995.
- [25] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, third enlarged edition, *Die Grundlehren der Mathematischen Wissenschaften*, vol. 52, Springer-Verlag, New York, Inc., New York, 1966.
- [26] W. Miller Jr., *Symmetry and Separation of Variables*, *Encyclopedia of Mathematics and Its Applications*, vol. 4, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1977, With a foreword by Richard Askey.
- [27] J. Monterde, H. Ugail, On harmonic and biharmonic Bézier surfaces, *Comput. Aided Geom. Des.* 21 (7) (2004) 697–715.
- [28] M. Nicolescu, *Les Fonctions Polyharmoniques*, Hermann, Paris, 1936.
- [29] L. Schwartz, *Théorie des distributions. Tome I, Actualités Sci.Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9*, Hermann, Paris, 1950.
- [30] N.J.A. Sloane, S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press Inc., San Diego, CA, 1995. With a separately available computer disk.
- [31] S.L. Sobolev, *Cubature Formulas and Modern Analysis: An Introduction*, Oxonian, New Delhi, 1992.
- [32] G. Szegő, *Orthogonal Polynomials, revised ed.*, American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, Providence, R.I., 1959.
- [33] R. Szymtkowski, On the derivative of the associated Legendre function of the first kind of integer order with respect to its degree (with applications to the construction of the associated Legendre function of the second kind of integer degree and order), *J. Math. Chem.* 49 (7) (June 2011) 1436–1477.
- [34] H. Ugail, *Partial Differential Equations for Geometric Design*, Springer-Verlag, London, 2011.
- [35] D. Vautherin, Hartree-Fock calculations with Skyrme's interaction. II. Axially deformed nuclei, *Phys. Rev. C* 7 (1973) 296–316.
- [36] Z.Y. Wen, J. Avery, Some properties of hyperspherical harmonics, *J. Math. Phys.* 26 (3) (1985) 396–403.