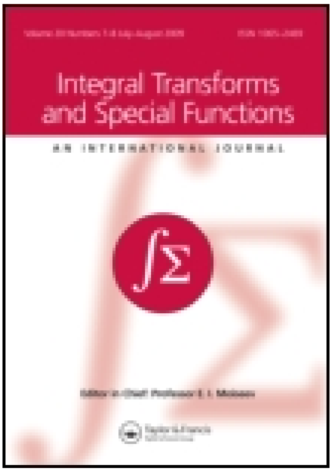

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Open problems

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Open problems

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The following open problems were presented at OPSFA-12, Sousse, Tunisia, March 29, 2013. Christian Berg, University of Copenhagen, has collected these open problems.

Mohamed J. Atia: Conjecture about a symmetric system of orthogonal polynomials

We consider the weight $w(x) = |x + \frac{1}{2}|/\sqrt{1+x} + |x - \frac{1}{2}|/\sqrt{1-x}$ on $[-1, 1]$, and claim that there exits a sequence of polynomials $\{P_n\}_{n\geq 0}$ orthogonal with respect to $w(x)$ on $[-1, 1]$ and which fulfils

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x \\ &P_{n+2}(x) = xP_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0 \end{aligned} \tag{1}$$

such that

$$\gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{1}{4}, \quad \gamma_3 = \frac{7}{30}, \quad \gamma_4 = \frac{4}{15},$$

$$\gamma_3 + \gamma_4 = \frac{1}{2},$$

$$\gamma_5 = \frac{1}{4}, \quad \gamma_6 = \frac{12}{49}, \quad \gamma_7 = \frac{25}{98},$$

$$\gamma_6 + \gamma_7 = \frac{1}{2},$$

$$\gamma_8 = \frac{1}{4}, \quad \gamma_9 = \frac{3187}{12870}, \quad \gamma_{10} = \frac{1624}{6435},$$

$$\gamma_9 + \gamma_{10} = \frac{1}{2},$$

that is to say

$$\gamma_1 = \frac{1}{2}, \quad \gamma_{3n+2} = \frac{1}{4}, \quad \gamma_{3n+3} + \gamma_{3n+4} = \frac{1}{2}.$$

- This conjecture was presented, first, in Leuven, Belgium 2009, second, in Madrid, Spain 2011 and, third, in Sousse, Tunisia 2013.
- The weight $w(x)$ is the solution of a second-order differential equation that is why we call the sequence $\{P_n\}_{n\geq 0}$ a sequence of second category.

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References

[1] Atia MJ, Leffet S. On orthogonal polynomials: semiclassical and of second category. Integral Transforms Spec Funct. doi:10.1080/10652469.2011.553612

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Howard Cohl: Hypergeometric series

Given a hypergeometric series

$$ {}_{p+1}F_p\left(\begin{matrix} a_0,\ldots,a_p \\ b_1,\ldots,b_p \end{matrix};x\right), $$

where the parameters in the hypergeometric series satisfy the relations

$$ a_0+1=a_1+b_1=\cdots=a_p+b_p, $$

then the series is called well-poised. If one has a well-poised hypergeometric series, where there exists some $1\le i\le p$, so that

$$ a_i=b_i+1=\frac{a_0}{2}+1, $$

with $a_0\in\mathbb{C}\setminus\{0,-2,-4,\ldots\}$, then the hypergeometric series is called very-well-poised. I have a terminating very-well-poised ${}_9F_8(1)$ hypergeometric series,

$$ {}_9F_8\left(\begin{matrix} a,a,a,-n,-n,\frac{1}{2},2a+n,2a+n,\frac{a}{2}+1 \\ 1,1,1+n+a,1+n+a,a+\frac{1}{2},1-n-a,1-n-a,\frac{a}{2} \end{matrix};1\right). $$

Does anybody have any ideas on how I can sum this in closed form?

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Anbu Swaminathan: Pick functions

The function which is analytic in the slit domain $\mathbb{C}\backslash[1,\infty)$ and whose Taylor coefficients are the Hausdorff moment sequences also belong to the set of Pick functions [1] which are related to moment problems. I am interested in the following problem.

Problem 1 To characterize the members of this class of Pick functions.

The generalized polylogarithm [3] is defined by the normalized function $\Phi_{p,q}(a,b;z)=\sum_{k=1}^{\infty}((1+a)^p(1+b)^q/(k+a)^p(k+b)^q)z^k$, $|z|<1$, where, $k+a\neq 0$, $k+b\neq 0$, p,q are complex numbers such that $\operatorname{Re} p>0$, $\operatorname{Re} q>0$. Note that for $a=b=0$, $p+q=r\in\mathbb{C}$, $\operatorname{Re} r>0$, $\Phi_{p,q}(a,b;z)=L_r(z)$, is the polylogarithm or Jonquière′s function that further reduces to Riemann zeta function for $z=-1$. Numerical evidences suggest the following conjecture.

Conjecture 1 For $0 \le p_1 + q_1 \le p_2 + q_2$, the ratio

$$\frac{\Phi_{p_1,q_1}(a_1, b_1; z)}{\Phi_{p_2,q_2}(a_1, b_1; z)}$$

belongs to the class of Pick functions.

Note that for $p_1 + q_1 = a$ and $p_2 + q_2 = b$, $0 \le a \le b$, with $a_1 = b_1 = a_2 = b_2 = 0$, this result that $L_a(z)/L_b(z)$ is in the class of Pick functions, is obtained in [4] using duality techniques of function theory. It would be interesting to see if the ratio given in the conjecture has the g-fraction as there exist a relation between the completely monotone sequences and the g-fractions, see [6]. Hence, it is important to identify the corresponding g-sequence for this ratio. Particular cases of this conjecture are proved in [5]. Further generalized polylogarithms are particular cases of the well-known generalized hypergeometric function [3] and since certain ratios of Gaussian hypergeometric functions [2] and their q-analgoues [5] can be written in terms of g-fractions, I propose

Problem 2 Characterize the ratios of generalized (basic) hypergeometric functions so that the corresponding g-fractions lead to the geometric properties of Pick functions.

References

[1] Berg C, Pedersen HL. Pick functions related to the gamma function. Rocky Mountain J Math. 2002;32(2):507–525.
[2] Küstner R. On the order of starlikeness of the shifted Gauss hypergeometric function. J Math Anal Appl. 2007;334(2):1363–1385.
[3] Mondal SR, Swaminathan A. Geometric properties of generalized polylogarithms. Integral Transforms Spec Funct. 2010;21(9):691–701.
[4] Roth O, Ruscheweyh S, Salinas L. A note on generating functions for Hausdorff moment sequences. Proc Am Math Soc. 2008;136(9):3171–3176.
[5] Swaminathan A, Tsygvintsev A. Pick functions and chain sequences for hypergeometric type functions. Preprint.
[6] Wall HS. Analytic theory of continued fractions. New York: D. Van Nostrand Company; 1948.

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Walter Van Assche: Minkowski's question mark function

In 1904, Hermann Minkowski introduced an interesting function, which he called the question mark function and he denoted its values by $?(x)$. This notation with a question mark is somewhat confusing, so instead I will denote the function by q and I will only consider it on the interval $[0, 1]$. If $0 < x < 1$, then we can write x as a regular continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, \qquad a_i \in \mathbb{N} \cup \{0\}.$$

The Minkowski question mark function at x is then defined as

$$q(x) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+a_2+\cdots+a_k}}.$$

If x is a rational number, then the continued fraction is terminating and $q(x)$ is given by a finite sum (see Figure 1). By setting $q(0) = 0$ and $q(1) = 1$ one can show that $q : [0,1] \to [0,1]$ is a continuous and increasing function, so that q is a probability distribution function on $[0,1]$ which defines a probability measure on $[0,1]$. This distribution function has the property that $q'(x) = 0$ almost everywhere on $[0,1]$ so that the corresponding measure is singular and continuous. See [7] for some information about this function and references.

The definition via regular continued fractions is not convenient for integration. A more convenient way is to define it as a fixed point of an iterated function system with two rational functions. One has

$$q(x) = \begin{cases} \frac{1}{2}q\left(\frac{x}{1-x}\right), & 0 \le x \le \frac{1}{2}, \\ 1 - \frac{1}{2}q\left(\frac{1-x}{x}\right), & \frac{1}{2} \le x \le 1 \end{cases}$$

and one can show that the sequence of probability distribution functions $(q_n)_{n\in\mathbb{N}}$, with

$$q_n(x) = \begin{cases} \frac{1}{2}q_{n-1}\left(\frac{x}{1-x}\right), & 0 \le x \le \frac{1}{2}, \\ 1 - \frac{1}{2}q_{n-1}\left(\frac{1-x}{x}\right), & \frac{1}{2} \le x \le 1, \end{cases}$$

and q_0 any probability distribution on $[0,1]$, converges uniformly to Minkowski's question mark function. This allows us to compute integrals by a limit procedure

$$\int_0^1 f(x)\,\mathrm{d}q(x) = \lim_{n\to\infty} \int_0^1 f(x)\,\mathrm{d}q_n(x).$$

Salem's problem

A first open problem is about the Fourier coefficients of Minkowski's question mark function:

$$\alpha_n = \int_0^1 \mathrm{e}^{2\mathrm{i}n\pi x}\,\mathrm{d}q(x).$$

The Riemann–Lebesgue lemma tells us that Fourier coefficients of an absolutely continuous measure on $[0,1]$ tend to zero. The Minkowski question mark function is singularly continuous, so one cannot use the Riemann–Lebesgue lemma. Nevertheless, the support of q is the full interval $[0,1]$ and it was proved by Salem [5] that q is Hölder continuous of order $\alpha = \log 2/(2\log((\sqrt{5}+1)/2)) = 0.7202$. Furthermore, Salem showed that

$$\frac{1}{n}\sum_{k=0}^{n} \alpha_k = \mathcal{O}(n^{-\alpha/2}),$$

so that α_n converges to zero on the average and there is the possibility that $\alpha_n \to 0$.

This is the problem posed by Raphaël Salem [5,1]:

> Do the Fourier coefficients of the Minkowski question mark function converge to 0? If not, what are $\liminf_{n\to\infty} \alpha_n$ and $\limsup_{n\to\infty} \alpha_n$?

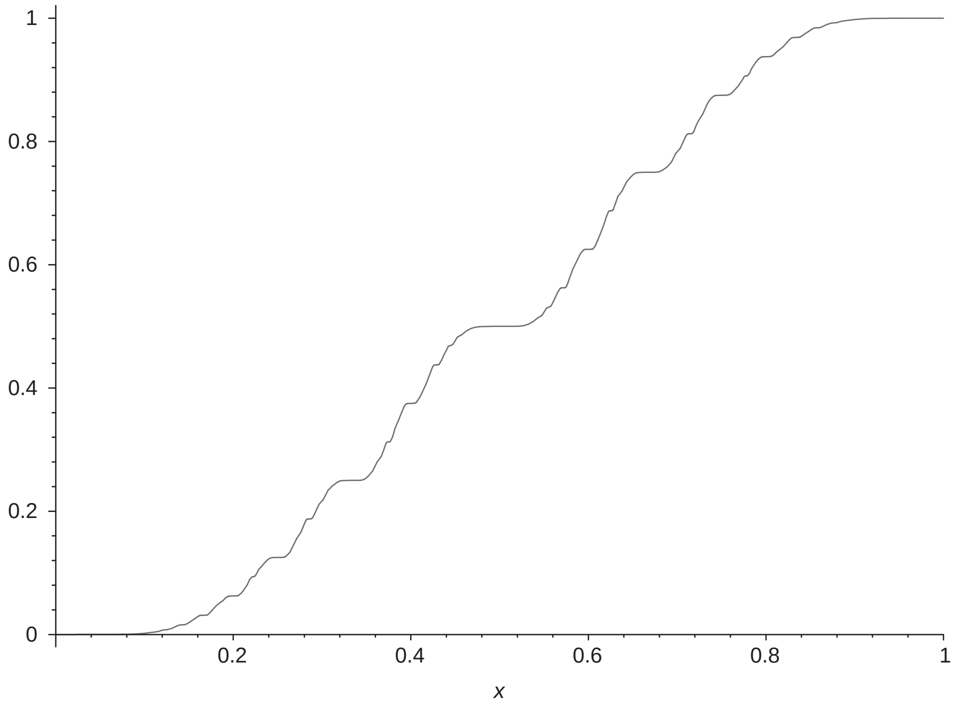

Figure 1. The Minkowski question mark function.

Orthogonal polynomials

Consider the orthonormal polynomials for the Minkowski question mark function:

$$\int_0^1 p_n(x)p_m(x)\,\mathrm{d}q(x) = \delta_{m,n}$$

with recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geq 0,$$

with $p_0 = 1$ and $p_{-1} = 0$. The symmetry of q around the point $\frac{1}{2}$ implies that $b_n = \frac{1}{2}$ for all $n \in \mathbb{N}$.

Rakhmanov's theorem [4] tells us that for an absolutely continuous measure μ on $[0,1]$ for which $\mu' > 0$ almost everywhere, one has $a_n \to \frac{1}{4}$ as $n \to \infty$. In our case $q' = 0$ almost everywhere, so one cannot use Rakhmanov's theorem to deduce the asymptotic behaviour of the recurrence coefficients. However, it is known (see, e.g., [2,3,6]) that there exist discrete measures and singular continuous measures (on $[0,1]$) for which the recurrence coefficients have the behaviour $b_n \to \frac{1}{2}$ and $a_n \to \frac{1}{4}$ as $n \to \infty$, so that they are in the Nevai class (for the interval $[0,1]$). The open problem is whether the Minkowski question mark function is such a singular continuous function for which the recurrence coefficients are in the Nevai class (for the interval $[0,1]$), i.e.

Is $\lim_{n\to\infty} a_n = \frac{1}{4}$? If not, then what are $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$?

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References

[1] Alkauskas G. The Minkowski $?(x)$ function and Salem's problem. C R Math Acad Sci Paris. 2012;350(3–4):137–140.
[2] Lubinsky DS. Singularly continuous measures in Nevai's class M. Proc Am Math Soc. 1991;111:413–420.
[3] Van Assche W, Magnus AP. Sieved orthogonal polynomials and discrete measures with jumps dense in an interval. Proc Am Math Soc. 1989;106:163–173.
[4] Rakhmanov EA. On the asymptotics of the ratio of orthogonal polynomials. Mat Sb. 1977;103(145):237–252 (in Russian); translated in Math. USSR Sb. 1977;32:199–213. Rakhmanov EA. On the asymptotics of the ratio of orthogonal polynomials, II. Mat. Sb. 1982;118(160):104–117 (in Russian); translated in Math. USSR Sb. 1983;47:105–117.
[5] Salem R. On some singular monotonic functions which are strictly increasing. Trans Am Math Soc. 1943;53:427–439.
[6] Totik V. Orthogonal polynomials with ratio asymptotics. Proc Am Math Soc. 1992;114:491–495.
[7] http://en.wikipedia.org/wiki/Minkowski's_question_mark_function (March 29, 2013).

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