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Eigenfunction expansions for a fundamental solution of Laplace's equation on $\mathbf{R}^3$ in parabolic and elliptic cylinder coordinates

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Abstract
A fundamental solution of Laplace's equation in three dimensions is expanded in harmonic functions that are separated in parabolic or elliptic cylinder coordinates. There are two expansions in each case which reduce to expansions of the Bessel functions $J_0(kr)$ or $K_0(kr)$, $r^2 = (x-x_0)^2+(y-y_0)^2$, in parabolic and elliptic cylinder harmonics. Advantage is taken of the fact that $K_0(kr)$ is a fundamental solution and $J_0(kr)$ is the Riemann function of partial differential equations on the Euclidean plane.

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1. Introduction

A fundamental solution of Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{1}$$

is given by (apart from a multiplicative factor of 4π)

$$U(\mathbf{x}, \mathbf{x}_0) = \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|}, \quad \text{where } \mathbf{x} = (x, y, z) \neq \mathbf{x}_0 = (x_0, y_0, z_0), \tag{2}$$

and $\|\mathbf{x} - \mathbf{x}_0\|$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{x}_0$. In many applications it is required to expand a fundamental solution in the form of a series or an integral, in terms of solutions of (1) that are separated in suitable curvilinear coordinates. Examples of such applications include electrostatics, magnetostatics, quantum direct and exchange Coulomb interactions, Newtonian gravity, potential flow and steady state heat transfer. Morse and

Feshbach [19] (see also [11]; [14]; [10]) provide a list of such expansions for various coordinate systems but the formulas for several coordinate systems are missing. It is the goal of this paper to provide these expansions for parabolic and elliptic cylinder coordinates. Although these expansions are partially known from [3] and [13] for parabolic cylinder coordinates, and from [17] for elliptic cylinder coordinates, we found it desirable to investigate these expansions in a systematic fashion and provide direct proofs for them based on eigenfunction expansions.

There will be two expansions for both of these coordinate systems. The first expansion for a cylindrical coordinate system on $\mathbf{R}^3$ starts from the known formula in terms of the integral of Lipschitz (see [27, section 13.2]; [6, (8)])

$$\frac{1}{\|\mathbf{x}-\mathbf{x_0}\|} = \int_0^\infty J_0(k\sqrt{(x-x_0)^2+(y-y_0)^2})\,\mathrm{e}^{-k|z-z_0|}\,\mathrm{d}k. \tag{3}$$

The Bessel function J_ν can be defined by (see for instance (10.2.2) in [20])

$$J_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-z^2/4)^n}{n!\Gamma(\nu+n+1)}. \tag{4}$$

Note that $W_k : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$, defined by

$$W_k(x, y, x_0, y_0) := J_0(kr), \tag{5}$$

where $r^2 := (x-x_0)^2 + (y-y_0)^2$, solves the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0. \tag{6}$$

Therefore, in a cylindrical coordinate system on $\mathbf{R}^3$ involving the Cartesian coordinate z, the first expansion (3) reduces to expanding W_k from (5) in terms of solutions of (6) that are separated in a curvilinear coordinate system on the plane.

The second expansion for a cylindrical coordinate system on $\mathbf{R}^3$ is based on the known formula given in terms of the Lipschitz–Hankel integral (see [27, section 13.21]; [6, (9)])

$$\frac{1}{\|\mathbf{x}-\mathbf{x_0}\|} = \frac{2}{\pi}\int_0^\infty K_0(k\sqrt{(x-x_0)^2+(y-y_0)^2})\cos k(z-z_0)\,\mathrm{d}k, \tag{7}$$

where $K_\nu : (0,\infty) \to \mathbf{R}$ (cf (10.32.9) in [20]), the modified Bessel function of the second kind (Macdonald's function), of order $\nu \in \mathbf{R}$, is defined by

$$K_\nu(z) := \int_0^\infty \mathrm{e}^{-z\cosh t}\cosh(\nu t)\,\mathrm{d}t. \tag{8}$$

Now $V_k : \mathbf{R}^2 \times \mathbf{R}^2 \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{R}^2\} \to (0,\infty)$, defined by

$$V_k(x, y, x_0, y_0) := K_0(kr) \tag{9}$$

solves the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - k^2 U = 0. \tag{10}$$

In a cylindrical coordinate system on $\mathbf{R}^3$ involving the Cartesian coordinate z, the second expansion (7) reduces to expanding V_k in terms of solutions of (10) that are separated in curvilinear coordinates on the plane.

For the benefit of the reader, it is to be noted that in this paper, the most important expansion formulas associated with a fundamental solution of Laplace's equation on $\mathbf{R}^3$ in parabolic and elliptic cylinder coordinates are listed as follows. There are those given in terms of Hermite functions for the order zero modified Bessel function of the second kind, namely theorems 2.2, 2.3; those given in terms of the (modified) parabolic cylinder functions for the order zero Bessel function of the first kind, namely theorems 3.2, 3.3; those given in terms of Mathieu functions for the order zero Bessel function of the first kind, namely theorems 4.2, 4.3; and those given in terms of modified Mathieu functions for the order zero modified Bessel function of the second kind, namely theorems 5.2, 5.3.

The paper is organized as follows. In section 2 we derive the desired expansion of $K_0(kr)$ in parabolic cylinder coordinates. This expansion is given in terms of series over Hermite functions. In section 3 we obtain an integral representation for $J_0(kr)$ in terms of separated solutions of (10) in terms of (modified) parabolic cylinder functions. We show how these results are based on a general expansion theorem in terms of solutions of the differential equation

$$-u'' - \tfrac{1}{4}\xi^2 u = \lambda u, \tag{11}$$

where ξ is the independent variable. The above equation can be viewed as a quantum mechanical inverted harmonic oscillator at energy λ (see for instance [2]). In sections 4 and 5 we derive the fundamental solution expansions in elliptic cylinder coordinates for $J_0(kr)$ and $K_0(kr)$, respectively.

Throughout this paper we rely on the following definitions. The set of natural numbers is given by $\mathbf{N} := \{1, 2, 3, \ldots\}$, the set $\mathbf{N}_0 := \{0, 1, 2, \ldots\} = \mathbf{N} \cup \{0\}$ and the set $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$. The set $\mathbf{R}$ represents the real numbers and the set $\mathbf{C}$ represents the complex numbers.

2. Expansion of $K_0(kr)$ for parabolic cylinder coordinates

Parabolic coordinates on the plane (ξ, η) (see figure 1, and for instance chapter 10 in [13]) are connected to Cartesian coordinates (x, y) by

$$x = \tfrac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta, \tag{12}$$

where $\xi \in \mathbf{R}$ and $\eta \in [0, \infty)$. To simplify notation we will first set $k = 1$ in (9). Then V_1 satisfies

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - U = 0 \quad \text{if } (x, y) \neq (x_0, y_0). \tag{13}$$

Let (ξ, η), (ξ_0, η_0) be parabolic coordinates on $\mathbf{R}^2$ for (x, y) and (x_0, y_0), respectively. Then V_1 transforms to

$$v(\xi, \eta, \xi_0, \eta_0) := K_0(r(\xi, \eta, \xi_0, \eta_0)), \qquad (\xi, \eta) \neq \pm(\xi_0, \eta_0)$$

and $r(\xi, \eta, \xi_0, \eta_0)$ is defined by

$$r^2 = \tfrac{1}{4}\left[(\xi + \xi_0)^2 + (\eta + \eta_0)^2\right]\left[(\xi - \xi_0)^2 + (\eta - \eta_0)^2\right], \qquad r > 0. \tag{14}$$

Here and in the following we allow all $\xi, \eta, \xi_0, \eta_0 \in \mathbf{R}$ such that $(\xi, \eta) \neq \pm(\xi_0, \eta_0)$. From (13) or by direct computation we obtain that v solves the equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - (\xi^2 + \eta^2)u = 0. \tag{15}$$

One can view this equation as a quantum mechanical simple harmonic oscillator with zero energy. Separating variables $u(\xi, \eta) = u_1(\xi)u_2(\eta)$ in (15), we obtain the ordinary differential equations

$$u_1'' + (2n + 1 - \xi^2)u_1 = 0, \tag{16}$$

$$u_2'' - (2n + 1 + \eta^2)u_2 = 0, \tag{17}$$

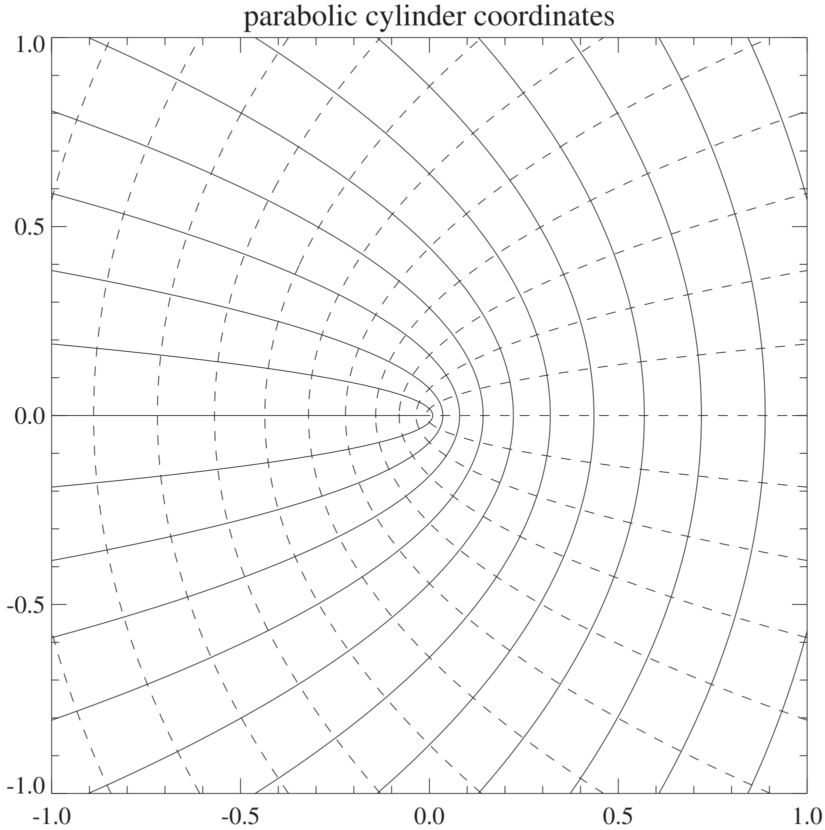

Figure 1. The (x, y) plane of parabolic cylinder coordinates on $\mathbf{R}^3$. The curves of constant ξ (solid) and η (dashed) both represent parabolic cylinders (parabolas extended infinitely in the positive and negative z directions).

where we will use only $n \in \mathbf{N}_0$. One can view these equations as quantum mechanical simple harmonic oscillators in one-dimension with positive and negative energies respectively.

Equations (16), (17) have the general solutions

$$u_1(\xi) = c_1 \mathrm{e}^{-\xi^2/2} H_n(\xi) + c_2\, \mathrm{e}^{\xi^2/2} H_{-n-1}(\mathrm{i}\xi),$$
$$u_2(\eta) = c_3\, \mathrm{e}^{\eta^2/2} H_n(i\eta) + c_4\, \mathrm{e}^{-\eta^2/2} H_{-n-1}(\eta),$$

where $H_\nu : \mathbf{C} \to \mathbf{C}$ is the Hermite function which can be defined in terms of Kummer's function of the first kind M as (cf (10.2.8) in [13])

$$H_\nu(z) := \frac{2^\nu \sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)} M\left(-\frac{\nu}{2}, \frac{1}{2}, z^2\right) - \frac{2^{\nu+1}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)} z M\left(\frac{1-\nu}{2}, \frac{3}{2}, z^2\right),$$

and

$$M(a, b, z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{18}$$

(see for instance (13.2.2) in [20]). Note that the Kummer function of the first kind is entire in z and a, and is a meromorphic function of b. The Hermite function is an entire function of both z and ν. If $\nu = n \in \mathbf{N}_0$ then $H_\nu(z)$ reduces to the Hermite polynomial of degree n.

Arguing as in [26, theorem 1.11], we obtain the following integral representation for solutions of (15).

Theorem 2.1. *Let $u \in C^2(\mathbf{R}^2)$ be a solution of (15). Let $(\xi_0, \eta_0) \in \mathbf{R}^2$, and let C be a closed rectifiable curve on $\mathbf{R}^2$ which does not pass through $\pm(\xi_0, \eta_0)$, and let $n^\pm$ be the winding number of C with respect to $\pm(\xi_0, \eta_0)$. Then we have*

$$2\pi\left[n^+u(\xi_0, \eta_0) + n^-u(-\xi_0, -\eta_0)\right] = \int_C (u\partial_2 v - v\partial_2 u)d\xi + (v\partial_1 u - u\partial_1 v)\,\mathrm{d}\eta,$$

where ∂_1, ∂_2 denote partial derivatives with respect to ξ, η, respectively.

This is a generalization of Green's representation formula (see for instance [12, p 60] and [9, (6.25)]) with the right-hand side being the boundary integral of $u_n v - uv_n$, where the subscript n indicates the normal derivative on C. Note that v is a fundamental solution of (15) which has logarithmic singularities at the points $\pm(\xi_0, \eta_0)$.

We apply theorem 2.1 to the solution

$$u(\xi, \eta) = \mathrm{e}^{-\xi^2/2}H_n(\xi)\mathrm{e}^{\eta^2/2}H_n(\mathrm{i}\eta),$$

of (15), and for C we take the positively oriented boundary of the rectangle $|\xi| \leqslant \xi_1, |\eta| \leqslant \eta_1$, where $|\xi_0| < \xi_1, |\eta_0| < \eta_1$, so $n^+ = n^- = 1$. Then let $\xi_1 \to \infty$ and note that the integrals over the vertical sides converge to 0. The integrals over the horizontal sides of the rectangle give the same contribution because the integrand changes sign when (ξ, η) is replaced by $(-\xi, -\eta)$. Therefore, we obtain, for $\eta = \eta_1 > |\eta_0|$,

$$2\pi u(\xi_0, \eta_0) = \int_{-\infty}^{\infty} (v(\xi, \eta, \xi_0, \eta_0)\partial_2 u(\xi, \eta) - u(\xi, \eta)\partial_2 v(\xi, \eta, \xi_0, \eta_0))\,\mathrm{d}\xi. \tag{19}$$

We expand v as a function of ξ in an orthogonal series of functions $\mathrm{e}^{-\xi^2/2}H_n(\xi)$, $n \in \mathbf{N}_0$, so that the coefficients

$$f_n(\eta, \xi_0, \eta_0) := \int_{-\infty}^{\infty} v(\xi, \eta, \xi_0, \eta_0)\,\mathrm{e}^{-\xi^2/2}H_n(\xi)\,\mathrm{d}\xi, \tag{20}$$

are to be evaluated.

Set $f(\eta) = f_n(\eta, \xi_0, \eta_0)$ and $g(\eta) = \mathrm{e}^{\eta^2/2}H_n(\mathrm{i}\eta)$. Then after substituting the above particular u, (19) becomes

$$2\pi u(\xi_0, \eta_0) = g'(\eta)f(\eta) - g(\eta)f'(\eta). \tag{21}$$

By differentiating both sides of (21) with respect to η we see that f satisfies (17). Since $f(\eta)$ goes to 0 as $\eta \to \infty$, it follows that

$$f(\eta) = c\mathrm{e}^{-\eta^2/2}H_{-n-1}(\eta),$$

where c is a constant. Going back to (21), we find that

$$2\pi u(\xi_0, \eta_0) = cW,$$

where $W = i^n$ is the (constant) Wronskian of $\mathrm{e}^{-\eta^2/2}H_{-n-1}(\eta)$ and $g(\eta)$. Therefore, if $\eta > |\eta_0|$, we obtain

$$f_n(\eta, \xi_0, \eta_0) = 2\pi(-i)^n\,\mathrm{e}^{-\xi_0^2/2}H_n(\xi_0)\,\mathrm{e}^{\eta_0^2/2}H_n(\mathrm{i}\eta_0)\,\mathrm{e}^{-\eta^2/2}H_{-n-1}(\eta). \tag{22}$$

By taking limits, we see that (22) remains true when $\eta = |\eta_0|$. Expanding $\xi \mapsto v(\xi, \eta, \xi_0, \eta_0)$ in a series of Hermite functions [23, theorem 9.1.6] we obtain the following result.

Theorem 2.2. *Let $\xi, \eta, \xi_0, \eta_0 \in \mathbf{R}$ with $|\eta_0| \leqslant \eta$ such that $(\xi, \eta) \neq \pm(\xi_0, \eta_0)$. Then*

$$K_0(r(\xi, \eta, \xi_0, \eta_0)) = \sqrt{\pi}\, \mathrm{e}^{(\eta_0^2-\xi_0^2-\eta^2-\xi^2)/2} \sum_{n=0}^{\infty} \frac{(-i)^n}{2^{n-1} n!} H_n(\xi) H_{-n-1}(\eta) H_n(\xi_0) H_n(\mathrm{i}\eta_0),$$

where r is defined by (14).

The special case $\xi_0 = \eta_0 = 0$ of theorem 2.2 can be found in [13, problem 7, p 298]. If we multiply each ξ, η, ξ_0, η_0 by $\sqrt{k}$ then we obtain an expansion for $K_0(kr)$. Inserting this expansion into (7) yields our main result.

Theorem 2.3. *Let $\mathbf{x}$, $\mathbf{x}_0$ be distinct points on $\mathbf{R}^3$ with parabolic cylinder coordinates (ξ, η, z) and (ξ_0, η_0, z_0), respectively. If $\eta_{\lessgtr} := \substack{\min \\ \max}\{\eta, \eta_0\}$ then*

$$\frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-k(\xi^2+\eta_>^2+\xi_0^2-\eta_<^2)/2} \cos k(z - z_0)$$
$$\times \sum_{n=0}^{\infty} \frac{(-i)^n}{2^{n-1} n!} H_n(\sqrt{k}\xi) H_{-n-1}(\sqrt{k}\eta_>) H_n(\sqrt{k}\xi_0) H_n(\mathrm{i}\sqrt{k}\eta_<)\, \mathrm{d}k.$$

If we interchange the order of the infinite series and the definite integral in the above expression we obtain the following theorem.

Corollary 2.4. *Under the same assumptions as in theorem 2.3 but with $\eta_0 \neq \eta$, we have*

$$\frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-i)^n}{2^{n-1} n!} \int_0^\infty \mathrm{e}^{-k/2(\xi^2+\eta_>^2+\xi_0^2-\eta_<^2)}$$
$$\times H_n(\sqrt{k}\xi) H_{-n-1}(\sqrt{k}\eta_>) H_n(\sqrt{k}\xi_0) H_n(\mathrm{i}\sqrt{k}\eta_<) \cos k(z - z_0)\, \mathrm{d}k.$$

Proof. We want to justify the interchange of sum and integral in theorem 2.3 based on the Beppo Levi theorem [21, p 36]: Let $f_n : (0, \infty) \to \mathbf{R}$, $n \in \mathbf{N}_0$, be a sequence of integrable functions. If

$$\int_0^\infty \sum_{n=0}^{\infty} |f_n(x)|\, \mathrm{d}x < \infty$$

then

$$\int_0^\infty \sum_{n=0}^{\infty} f_n(x)\, \mathrm{d}x = \sum_{n=0}^{\infty} \int_0^\infty f_n(x)\, \mathrm{d}x.$$

Let $\xi, \xi_0, \eta, \eta_0 \in \mathbf{R}$ with $0 \leqslant \eta_0 < \eta$ (the case $0 \leqslant \eta < \eta_0$ can be treated the same way.) Note that $H_{-n-1}(\eta) > 0$ and $(-i)^n H_n(\mathrm{i}\eta_0) \geqslant 0$. Therefore, using first the inequality $2ab \leqslant ta^2 + t^{-1}b^2$ for $a, b \in \mathbf{R}$, $t > 0$, and then theorem 2.2,

$$\sum_{n=0}^{\infty} \left| \frac{(-i)^n}{2^{n-1} n!} H_n(\xi) H_{-n-1}(\eta) H_n(\xi_0) H_n(\mathrm{i}\eta_0) \right|$$
$$\leqslant \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-i)^n}{2^{n-1} n!} \left(\mathrm{e}^{(\xi_0^2-\xi^2)/2} \{H_n(\xi)\}^2 + \mathrm{e}^{(\xi^2-\xi_0^2)/2} \{H_n(\xi_0)\}^2 \right) H_{-n-1}(\eta) H_n(\mathrm{i}\eta_0)$$
$$= \frac{1}{2\sqrt{\pi}} \mathrm{e}^{(\xi_0^2+\eta^2+\xi^2-\eta_0^2)/2} (K_0(r(\xi, \eta, \xi, \eta_0) + K_0(r(\xi_0, \eta, \xi_0, \eta_0)).$$

Noting that

$$\{r(\xi,\eta,\xi,\eta_0)\}^2 = \tfrac{1}{4}\left(4\xi^2+(\eta+\eta_0)^2\right)(\eta-\eta_0)^2 \geqslant \tfrac{1}{4}(\eta^2-\eta_0^2)^2,$$

and using [16, p 151],

$$K_0(x) < K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}, \quad x>0,$$

we obtain

$$\sqrt{\pi}\mathrm{e}^{(\eta_0^2-\xi_0^2-\xi^2-\eta^2)/2}\sum_{n=0}^{\infty}\left|\frac{(-i)^n}{2^{n-1}n!}H_n(\xi)H_{-n-1}(\eta)H_n(\xi_0)H_n(\mathrm{i}\eta_0)\right|$$
$$\leqslant K_{1/2}\left(\frac{1}{2}(\eta^2-\eta_0^2)\right) = \sqrt{\frac{\pi}{\eta^2-\eta_0^2}}\mathrm{e}^{-(\eta^2-\eta_0^2)/2}.$$

If we multiply each ξ,η,ξ_0,η_0 by $\sqrt{k}$ and integrate on $k\in(0,\infty)$ we see that the assumption of the Beppo Levi theorem is satisfied. $\square$

The interesting question arises whether the integrals appearing in corollary 2.4 can be evaluated in closed form.

3. Expansion of $J_0(kr)$ for parabolic cylinder coordinates

Transforming equation (6) to parabolic coordinates (12) we obtain

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+k^2(\xi^2+\eta^2)u=0.$$

In order to simplify notation we will (temporarily) set $k=\frac{1}{2}$ and $\zeta=\mathrm{i}\eta$ with ζ real. Thus we consider

$$\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\zeta^2}+\frac{1}{4}(\xi^2-\zeta^2)u=0, \quad \xi,\zeta\in\mathbf{R}. \tag{23}$$

If $u_1(\xi)$ and $u_2(\zeta)$ are solutions of the ordinary differential equation (11) for some λ then $u(\xi,\zeta)=u_1(\xi)u_2(\zeta)$ solves (23).

The function $W_{1/2}$ from (5) transformed to (ξ,ζ) becomes

$$w(\xi,\zeta,\xi_0,\zeta_0)=J_0\left(\tfrac{1}{2}\tilde{r}(\xi,\zeta,\xi_0,\zeta_0)\right), \tag{24}$$

where $\tilde{r}^2$ is a symmetric polynomial defined by

$$4\tilde{r}^2 := \left[(\xi-\xi_0)^2-(\zeta-\zeta_0)^2\right]\left[(\xi+\xi_0)^2-(\zeta+\zeta_0)^2\right]$$
$$= 8\xi\xi_0\zeta\zeta_0+\xi^4+\xi_0^4+\zeta^4+\zeta_0^4-2\xi_0^2\zeta^2-2\xi_0^2\zeta_0^2-2\zeta^2\zeta_0^2-2\xi^2\xi_0^2-2\xi^2\zeta^2-2\xi^2\zeta_0^2,$$

and J_0 is the order zero Bessel function of the first kind (see (4)). For fixed $\zeta,\xi_0,\zeta_0\in\mathbf{R}$ consider the function $f:\mathbf{R}\to\mathbf{R}$ defined by

$$f(\xi):=w(\xi,\zeta,\xi_0,\zeta_0). \tag{25}$$

We wish to expand this function in terms of (modified) parabolic cylinder harmonics according to a general expansion theorem that is derived in the following subsection.

3.1. Spectral theory of (modified) parabolic cylinder harmonics—a singular Sturm–Liouville problem

We discuss the Sturm–Liouville problem

$$-u'' - \tfrac{1}{4}x^2 u = \lambda u, \quad -\infty < x < \infty, \tag{26}$$

involving the spectral parameter λ subject to

$$u \in L^2(-\infty, \infty).$$

By replacing $\frac{1}{4}x^2$ by $-\frac{1}{4}x^2$, we obtain the equation describing the harmonic oscillator whose eigenfunctions (Hermite functions) and the corresponding spectral theory is well-known. The spectral problem associated with (26) is far less known. We will refer to these solutions of Laplace's equation as (modified) parabolic cylinder harmonics, because (1) they are different from those parabolic cylinder harmonics usually encountered (those given in terms of Hermite functions) and (2) they are not given by Hermite functions with argument multiplied by i, as usually is the convention when defining modified solutions. In fact, these solutions are given by the usual parabolic cylinder functions with argument multiplied by $\exp(\pm \mathrm{i}\pi/4)$. A discussion of differential equation (26) and its solutions can be found in [18] and in section 8.2 of [8] (see also [15], [28], [4] and [7]).

We will follow chapter 9 in [5]. First note that, by [5, corollary 2, p 231], equation (26) is in the limit-point case at $x = \pm\infty$. Therefore, one can apply section 5 of chapter 9 in [5].

For $\lambda, x \in \mathbf{C}$ we define the functions $u_1(\lambda, x)$, $u_2(\lambda, x)$ as the solutions of (26) uniquely determined by the initial conditions

$$u_1(\lambda, 0) = u_2'(\lambda, 0) = 1, \quad u_1'(\lambda, 0) = u_2(\lambda, 0) = 0.$$

These functions may be expressed in terms of Kummer's function of the first kind (18)

$$u_1(\lambda, x) = \mathrm{e}^{-\frac{\mathrm{i}}{4}x^2} M\left(\frac{1}{4} + \frac{\mathrm{i}}{2}\lambda, \frac{1}{2}, \frac{\mathrm{i}}{2}x^2\right), \tag{27}$$

$$u_2(\lambda, x) = \mathrm{e}^{-\frac{\mathrm{i}}{4}x^2} x M\left(\frac{3}{4} + \frac{\mathrm{i}}{2}\lambda, \frac{3}{2}, \frac{\mathrm{i}}{2}x^2\right). \tag{28}$$

For $x > 0$, the function

$$\begin{aligned} u_3(\lambda, x) &= \mathrm{e}^{-\frac{\mathrm{i}}{4}x^2} U\left(\frac{1}{4} + \frac{\mathrm{i}}{2}\lambda, \frac{1}{2}, \frac{\mathrm{i}}{2}x^2\right) \\ &= \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4} + \frac{\mathrm{i}}{2}\lambda\right)} u_1(\lambda, x) - (1+\mathrm{i}) \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{4} + \frac{\mathrm{i}}{2}\lambda\right)} u_2(\lambda, x) \end{aligned} \tag{29}$$

is another solution of (26). Here the Kummer function of the second kind $U : \mathbf{C} \times \mathbf{C} \times (\mathbf{C} \setminus (-\infty, 0]) \to \mathbf{C}$ can be defined as (see [20, (13.2.42)])

$$U(a, b, z) := \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z).$$

Except when $z = 0$, each branch of U is entire in a and b. We assume that $U(a, b, z)$ has its principal value. The asymptotic behavior of the Kummer function of the second kind [20, (13.2.6)] shows that $u_3(\lambda, \cdot) \in L^2(0, \infty)$ provided that $\Im\lambda < 0$. Since (26) is in the limit-point case at $+\infty$, u_3 is the only solution with this property except for a constant factor. The Titchmarsh–Weyl functions $m_{\pm\infty}(\lambda)$, $\Im\lambda \neq 0$, are defined by the property that $u_1(\lambda, x) + m_{\pm\infty}(\lambda) u_2(\lambda, x)$ is square-integrable at $x = \pm\infty$. Therefore,

$$m_\infty(\lambda) = \begin{cases} -(1+\mathrm{i}) \dfrac{\Gamma\left(\frac{3}{4} + \frac{\mathrm{i}}{2}\lambda\right)}{\Gamma\left(\frac{1}{4} + \frac{\mathrm{i}}{2}\lambda\right)} & \text{if } \Im\lambda < 0, \\[2ex] -(1-\mathrm{i}) \dfrac{\Gamma\left(\frac{3}{4} - \frac{\mathrm{i}}{2}\lambda\right)}{\Gamma\left(\frac{1}{4} - \frac{\mathrm{i}}{2}\lambda\right)} & \text{if } \Im\lambda > 0. \end{cases}$$

By symmetry, we have

$$m_{-\infty}(\lambda) = -m_\infty(\lambda).$$

Using the notation of [5, theorem 5.1, p 251], namely

$$M_{11}(\lambda) = \frac{1}{m_{-\infty}(\lambda) - m_\infty(\lambda)},$$

$$M_{12}(\lambda) = M_{21}(\lambda) = \frac{m_{-\infty}(\lambda) + m_\infty(\lambda)}{2(m_{-\infty}(\lambda) - m_\infty(\lambda))},$$

$$M_{22}(\lambda) = \frac{m_{-\infty}(\lambda)m_\infty(\lambda)}{m_{-\infty}(\lambda) - m_\infty(\lambda)},$$

one obtains

$$M_{11}(\lambda) = \frac{-1}{2m_\infty(\lambda)}, \quad M_{12} = M_{21} = 0, \quad M_{22}(\lambda) = \frac{m_\infty(\lambda)}{2}.$$

Now using [5, p 250, last line], we find that, for $\lambda \in \mathbf{R}$,

$$\rho_1'(\lambda) := \rho_{11}'(\lambda) = \frac{1}{\pi} \lim_{\epsilon\to 0+} \Im\,(M_{11}(\lambda + \mathrm{i}\epsilon)) = \frac{\mathrm{e}^{\frac{\pi}{2}\lambda}}{4\sqrt{2}\pi^2} \left|\Gamma\left(\tfrac14 + \tfrac{\mathrm{i}}{2}\lambda\right)\right|^2,$$

since [20, (5.4.5)]

$$\Gamma\left(\tfrac14 + \mathrm{i}y\right)\Gamma\left(\tfrac34 - \mathrm{i}y\right) = \frac{\pi\sqrt{2}}{\cosh(\pi y) + \mathrm{i}\sinh(\pi y)}.$$

Moreover, $\rho_{12}(\lambda) = \rho_{21}(\lambda) = 0$ and

$$\rho_2'(\lambda) := \rho_{22}'(\lambda) = \frac{1}{\pi} \lim_{\epsilon\to 0+} \Im\,(M_{22}(\lambda + \mathrm{i}\epsilon)) = \frac{\mathrm{e}^{\frac{\pi}{2}\lambda}}{2\sqrt{2}\pi^2} \left|\Gamma\left(\tfrac34 + \tfrac{\mathrm{i}}{2}\lambda\right)\right|^2.$$

Since the ρ-functions are real-analytic functions (with no jumps), we see that the spectrum of (26) is the whole real line $\mathbf{R}$ and there are no eigenvalues. The latter also follows from the known asymptotic behavior of the solutions of (26) (see [20, chapter 12]).

Using a variant of Stirling's formula (see (5.11.9) in [20])

$$|\Gamma(x + \mathrm{i}y)| \sim \sqrt{2\pi}|y|^{x-1/2}\mathrm{e}^{-\pi|y|/2} \quad \text{as } x, y \in \mathbf{R},\ |y| \to \infty, \tag{30}$$

we can determine the asymptotic behavior of ρ_j', namely

$$\rho_1'(\lambda) \sim \frac{1}{2\pi}|\lambda|^{-1/2}\mathrm{e}^{\pi(\lambda-|\lambda|)/2} \quad \text{as } |\lambda| \to \infty, \tag{31}$$

$$\rho_2'(\lambda) \sim \frac{1}{2\pi}|\lambda|^{1/2}\mathrm{e}^{\pi(\lambda-|\lambda|)/2} \quad \text{as } |\lambda| \to \infty. \tag{32}$$

Applying [5, theorem 5.2, p 251] to the analysis above, we obtain the following result on the spectral resolution associated with equation (26).

Theorem 3.1. *For a given function $f \in L^2(\mathbf{R})$, form the functions*

$$g_j(\lambda) = \int_{-\infty}^{\infty} u_j(\lambda, x) f(x)\,\mathrm{d}x, \quad j = 1, 2,\ \lambda \in \mathbf{R}. \tag{33}$$

Then

$$g_j \in L^2(\mathbf{R}, \rho_j), \quad j = 1, 2,$$

or, equivalently,

$$\int_{-\infty}^{\infty} |g_j(\lambda)|^2 \rho_j'(\lambda)\,\mathrm{d}\lambda < \infty, \quad j = 1, 2.$$

The function f can be represented in the form

$$f(x) = \sum_{j=1}^{2} \int_{-\infty}^{\infty} u_j(\lambda, x) g_j(\lambda) \rho_j'(\lambda)\, \mathrm{d}\lambda. \tag{34}$$

Moreover, we have Parseval's equation

$$\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = \sum_{j=1}^{2} \int_{-\infty}^{\infty} |g_j(\lambda)|^2 \rho_j'(\lambda)\, \mathrm{d}\lambda.$$

Equations (33), (34) establish a one-to-one correspondence between f and (g_1, g_2). The integrals appearing in (33), (34) have to be interpreted in the L^2-sense. For instance, (33) means that

$$\int_{-n}^{n} u_j(\lambda, x) f(x)\, \mathrm{d}x$$

converges to $g_j(\lambda)$ in $L^2(\mathbf{R}, \rho_j)$ as $n \to \infty$. Of course, if

$$\int_{-\infty}^{\infty} \left| u_j(\lambda, x) f(x) \right| \mathrm{d}x < \infty \quad \text{for every } \lambda \in \mathbf{R},$$

then (33) is also true pointwise.

3.2. Applying the spectral theory to the expansion of $J_0(kr)$ in parabolic cylinder coordinates

Since $f(\xi) = O(|\xi|^{-1})$ in (25) as $|\xi| \to \infty$, we have that $f \in L^2(\mathbf{R})$. Therefore, we can expand f using theorem 3.1, according to (34). For $\lambda \in \mathbf{R}$, we form the integrals

$$g_j(\lambda, \zeta, \xi_0, \zeta_0) = \int_{-\infty}^{\infty} w(\xi, \zeta, \xi_0, \zeta_0) u_j(\lambda, \xi)\, \mathrm{d}\xi, \qquad j = 1, 2. \tag{35}$$

These are absolutely convergent integrals because $f(\xi) = O(|\xi|^{-1})$ and $u_j(\lambda, \xi) = O(|\xi|^{-1/2})$ as $|\xi| \to \infty$.

Since $w(\xi, \zeta, \xi_0, \zeta_0)$ from (24) solves (23) for fixed (ξ_0, ζ_0), and $u_j(\lambda, \xi)$ from (27), (28) solves (26), it follows from differentiation under the integral sign followed by integration by parts (see for instance [22, Satz 8, p 26]) that g_j solves (26) as a function of ζ. Since the function w is symmetric in its four variables, it is also true that g_j solves (26) as a function of ξ_0 for fixed ζ_0, ζ and as a function of ζ_0 for fixed ξ_0, ζ. From these properties of g_j it follows easily that there are functions $c_{jk\ell m} : \mathbf{R} \to \mathbf{R}$ with $j, k, \ell, m = 1, 2$, depending on λ but not on ζ, ξ_0, ζ_0 such that

$$g_j(\lambda, \zeta, \xi_0, \zeta_0) = \sum_{k,\ell,m=1}^{2} c_{jk\ell m}(\lambda) u_k(\lambda, \zeta) u_\ell(\lambda, \xi_0) u_m(\lambda, \zeta_0). \tag{36}$$

This formula holds for all $\lambda, \zeta, \xi_0, \zeta_0 \in \mathbf{R}$. Substituting $\zeta = \xi_0 = \zeta_0 = 0$ in (35), (36), we obtain

$$c_{j111}(\lambda) = \int_{-\infty}^{\infty} J_0\left(\frac{1}{4}\xi^2\right) u_j(\lambda, \xi)\, \mathrm{d}\xi.$$

If $j = 2$, we integrate over an odd function, so $c_{2111}(\lambda) = 0$. By differentiating (36) with respect to ζ and/or ξ_0 and/or ζ_0 and then substituting $\zeta = \xi_0 = \zeta_0 = 0$ we find (after some calculations)

$$c_{jk\ell m}(\lambda) = \begin{cases} c_1(\lambda) & \text{if } j = k = \ell = m = 1, \\ c_2(\lambda) & \text{if } j = k = \ell = m = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Here $c_j : \mathbf{R} \to \mathbf{R}$ for $j = 1, 2$ is given by (see appendix A)

$$c_1(\lambda) := \int_{-\infty}^{\infty} J_0\left(\frac{1}{4}\xi^2\right) u_1(\lambda, \xi)\, \mathrm{d}\xi = \frac{2\sqrt{2}\pi\, \mathrm{e}^{-\pi\lambda/2}}{\cosh(\pi\lambda)|\Gamma\left(\frac{3}{4} + \frac{\mathrm{i}\lambda}{2}\right)|^2}, \tag{37}$$

$$c_2(\lambda) := -\int_{-\infty}^{\infty} \xi^{-1} J_1\left(\frac{1}{4}\xi^2\right) u_2(\lambda, \xi)\, \mathrm{d}\xi = \frac{-4\sqrt{2}\pi\, \mathrm{e}^{-\pi\lambda/2}}{\cosh(\pi\lambda)|\Gamma\left(\frac{1}{4} + \frac{\mathrm{i}\lambda}{2}\right)|^2}. \tag{38}$$

According to (34),

$$w(\xi, \zeta, \xi_0, \zeta_0) = \sum_{j=1}^{2} \int_{-\infty}^{\infty} c_j(\lambda)\rho_j'(\lambda) u_j(\lambda, \xi) u_j(\lambda, \zeta) u_j(\lambda, \xi_0) u_j(\lambda, \zeta_0)\, \mathrm{d}\lambda. \tag{39}$$

Note that

$$c_1(\lambda)\rho_1'(\lambda) = \frac{1}{2\pi\cosh(\pi\lambda)} \left| \frac{\Gamma\left(\frac{1}{4} + \mathrm{i}\frac{\lambda}{2}\right)}{\Gamma\left(\frac{3}{4} + \mathrm{i}\frac{\lambda}{2}\right)} \right|^2 = \frac{1}{4\pi^3} \left|\Gamma\left(\frac{1}{4} + \frac{\mathrm{i}\lambda}{2}\right)\right|^4, \tag{40}$$

$$c_2(\lambda)\rho_2'(\lambda) = -\frac{2}{\pi\cosh(\pi\lambda)} \left| \frac{\Gamma\left(\frac{3}{4} + \mathrm{i}\frac{\lambda}{2}\right)}{\Gamma\left(\frac{1}{4} + \mathrm{i}\frac{\lambda}{2}\right)} \right|^2 = \frac{-1}{\pi^3} \left|\Gamma\left(\frac{3}{4} + \frac{\mathrm{i}\lambda}{2}\right)\right|^4, \tag{41}$$

where we used [20, (5.4.4), (5.5.5)]. It follows from (30) that

$$c_1(\lambda)\rho_1'(\lambda) \sim \frac{1}{\pi|\lambda|\cosh(\pi\lambda)} \quad \text{as } |\lambda| \to \infty, \tag{42}$$

$$c_2(\lambda)\rho_2'(\lambda) \sim -\frac{|\lambda|}{\pi\cosh(\pi\lambda)} \quad \text{as } |\lambda| \to \infty. \tag{43}$$

It is known (see [1, (8.2.5)]) that, for fixed $x \in \mathbf{C}$, there is a constant C such that $u_j(\lambda, x) = O(\mathrm{e}^{C|\lambda|^{1/2}})$. It follows from (42), (43), that the integrands in (39) decay exponentially and therefore the corresponding integrals are absolutely convergent. By the identity theorem for analytic functions we see that equation (39) is true for all $\xi, \zeta, \xi_0, \zeta_0 \in \mathbf{C}$.

After setting $\zeta = \mathrm{i}\eta$ and $\zeta_0 = \mathrm{i}\eta_0$ in (39), one obtains the following result.

Theorem 3.2. *Let $\xi, \eta, \xi_0, \eta_0 \in \mathbf{R}$. Then*

$$J_0\left(\frac{1}{2}r(\xi, \eta, \xi_0, \eta_0)\right) = \sum_{j=1}^{2} \int_{-\infty}^{\infty} c_j(\lambda)\rho_j'(\lambda) u_j(\lambda, \xi) u_j(\lambda, \mathrm{i}\eta) u_j(\lambda, \xi_0) u_j(\lambda, \mathrm{i}\eta_0)\, \mathrm{d}\lambda,$$

where r is given by (14) and $c_j(\lambda)\rho_j'(\lambda)$ is given by (40), (41).

In the special case $\xi_0 = \eta_0 = 0$ (or correspondingly $\xi = \eta = 0$), theorem 3.2 can be found in [3, (16), p 175]. Of course, if we multiply each ξ, η, ξ_0, η_0 by $\sqrt{2k}$ we get the expansion of $J_0(k\, r(\xi, \eta, \xi_0, \eta_0))$. This leads to the $J_0(kr)$ expansion of a fundamental solution for the three-dimensional Laplace equation in parabolic cylindrical coordinates.

Theorem 3.3. *Let $\mathbf{x}$, $\mathbf{x}_0$ be distinct points on $\mathbf{R}^3$ with parabolic cylinder coordinates (ξ, η, z) and (ξ_0, η_0, z_0), respectively. Then*

$$\frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} = \sum_{j=1}^{2} \int_0^{\infty} \int_{-\infty}^{\infty} c_j(\lambda)\rho_j'(\lambda) \times u_j(\lambda, \sqrt{2k}\xi) u_j(\lambda, \mathrm{i}\sqrt{2k}\eta) u_j(\lambda, \sqrt{2k}\xi_0) u_j(\lambda, \mathrm{i}\sqrt{2k}\eta_0)\, \mathrm{e}^{-k|z-z_0|}\, \mathrm{d}\lambda\, \mathrm{d}k.$$

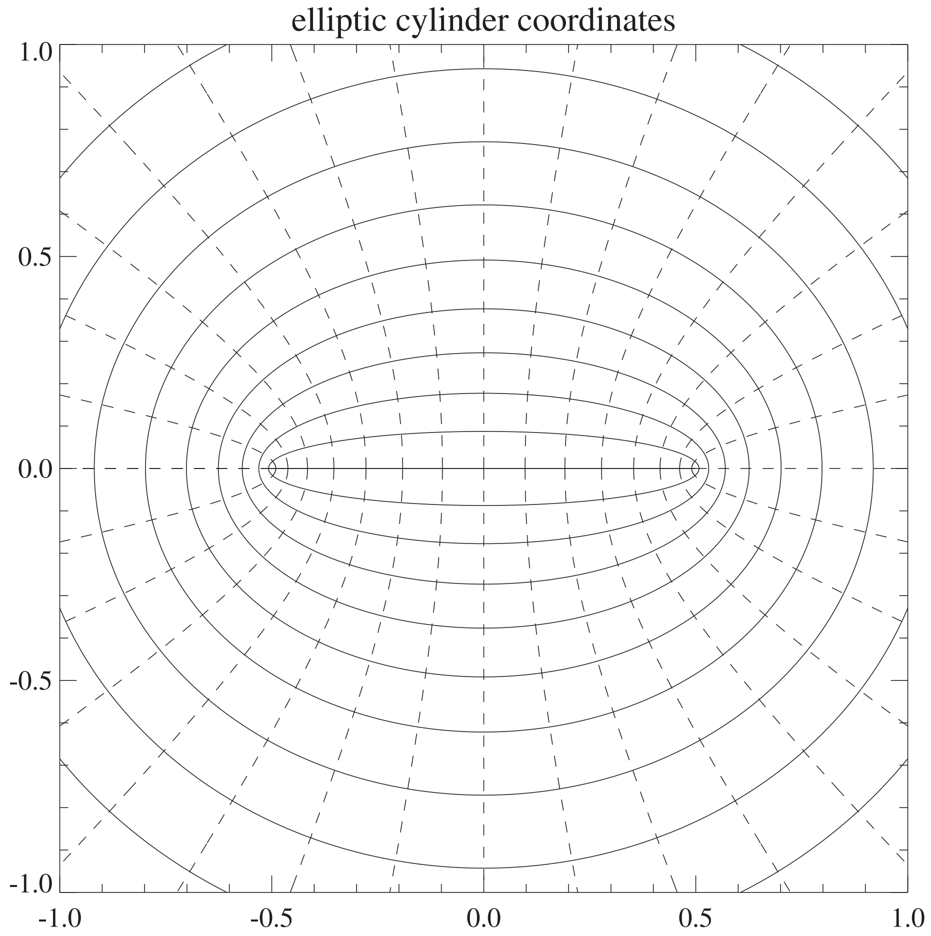

Figure 2. The (x, y) plane of elliptic cylinder coordinates (with $c = 1/2$) on $\mathbf{R}^3$. The curves of constant ξ (solid) represent elliptic cylinders (ellipses with foci at $\pm c$ extended infinitely in the positive and negative z directions). The curves of constant η (dashed) represent hyperbolic cylinders (hyperbolas with foci at $\pm c$ extended infinitely in the positive and negative z directions).

4. Expansion of $J_0(kr)$ for elliptic cylinder coordinates

Consider equation (6) for $k > 0$ and elliptic coordinates on the plane (see figure 2), defined by

$$x = c\cosh\xi\,\cos\eta, \quad y = c\sinh\xi\,\sin\eta, \tag{44}$$

where $\xi \in [0, \infty)$, $\eta \in \mathbf{R}$, and $c > 0$. Transforming $u(\xi, \eta) = U(x, y)$ we obtain

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} + k^2c^2(\cosh^2\xi - \cos^2\eta)u = 0. \tag{45}$$

Separating variables $u(\xi, \eta) = u_1(\xi)u_2(\eta)$, leads to

$$-u_1''(\xi) + (\lambda - 2q\cosh 2\xi)u_1(\xi) = 0, \tag{46}$$

$$u_2''(\eta) + (\lambda - 2q\cos 2\eta)u_2(\eta) = 0, \tag{47}$$

where $q = \frac{1}{4}c^2k^2 > 0$.

For $q \in \mathbf{R}$, Mathieu's equation (47) is a Hill's differential equation with period π; see [20] or [17]. Here q is positive but in the next section q will be negative. As a Hill's equation, Mathieu's equation (47) admits nontrivial 2π-periodic solutions if and only if λ is equal to one of its eigenvalues $a_n(q)$, $n \in \mathbf{N}_0$ or $b_n(q)$, $n \in \mathbf{N}$. If $\lambda = a_n(q)$, then Mathieu's equation has an even 2π-periodic solution $\mathrm{ce}_n(\eta, q)$, and if $\lambda = b_n(q)$ then Mathieu's equation has an odd 2π-periodic solution $\mathrm{se}_n(\eta, q)$. These functions are normalized according to

$$\int_0^{2\pi} \mathrm{ce}_n^2(\eta, q)\, \mathrm{d}\eta = \int_0^{2\pi} \mathrm{se}_n^2(\eta, q)\, \mathrm{d}\eta = \pi.$$

Moreover,

$$\mathrm{ce}_n(\eta + \pi, q) = (-1)^n \mathrm{ce}(\eta, q), \quad \mathrm{se}_n(\eta + \pi, q) = (-1)^n \mathrm{se}(\eta, q).$$

Note that all solutions of Mathieu's equation are entire functions of η.

For these 2π-periodic solutions of the Mathieu equation, we have the following expansion theorem (see for instance [20, section 28.11]).

Theorem 4.1. *Let $f(z)$ be a 2π-periodic function that is analytic in an open doubly-infinite strip S that contains the real axis. Then*

$$f(z) = \alpha_0 \mathrm{ce}_0(z, q) + \sum_{n=1}^{\infty} (\alpha_n \mathrm{ce}_n(z, q) + \beta_n \mathrm{se}_n(z, q)), \tag{48}$$

where

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{ce}_n(x, q) dx, \quad \beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{se}_n(x, q)\, \mathrm{d}x.$$

The series (48) converges absolutely and uniformly on any compact subset of the strip S.

Let (x_0, y_0) and (x, y) be points on $\mathbf{R}^2$ with distance r. Let (x_0, y_0), (x, y) have elliptic coordinates (ξ_0, η_0) and (ξ, η), respectively. Then

$$r^2 = c^2 \left[(\cosh \xi \cos \eta - \cosh \xi_0 \cos \eta_0)^2 + (\sinh \xi \sin \eta - \sinh \xi_0 \sin \eta_0)^2 \right]. \tag{49}$$

Clearly, $J_0(kr)$ as a function of (ξ, η) solves (45). We substitute $\zeta = \mathrm{i}\xi$ and $\zeta_0 = \mathrm{i}\xi_0$. Then (45) changes to

$$\frac{\partial^2 u}{\partial \zeta^2} - \frac{\partial^2 u}{\partial \eta^2} + k^2 c^2 (\cos^2 \eta - \cos^2 \zeta) u = 0, \tag{50}$$

and $J_0(kr)$ is transformed to

$$w(\zeta, \eta, \zeta_0, \eta_0) = J_0(k\tilde{r}),$$

where

$$\begin{aligned} \tilde{r}^2 &= c^2 [(\cos \zeta \cos \eta - \cos \zeta_0 \cos \eta_0)^2 - (\sin \zeta \sin \eta - \sin \zeta_0 \sin \eta_0)^2] \\ &= c^2 (\cos(\zeta - \eta) - \cos(\zeta_0 - \eta_0))(\cos(\zeta + \eta) - \cos(\zeta_0 + \eta_0)). \end{aligned}$$

The function $w(\zeta, \eta, \zeta_0, \eta_0)$ is an analytic function on $\mathbf{C}^4$ and it solves equation (50) as a function of (ζ, η). It is the Riemann function [9] of this differential equation because $w(\zeta, \eta, \zeta_0, \eta_0) = 1$ if $\zeta - \zeta_0 = \pm(\eta - \eta_0)$. For fixed η, ζ_0, η_0 we wish to expand the function $\zeta \mapsto w(\zeta, \eta, \zeta_0, \eta_0)$ in a series of Mathieu functions according to theorem 4.1. To this end we have to evaluate the integral

$$\int_{-\pi}^{\pi} w(\zeta, \eta, \zeta_0, \eta_0) \mathrm{ce}_n(\zeta, q)\, \mathrm{d}\zeta,$$

and a similar integral with ce_n replaced by se_n. Using Riemann's method of integration (see section 4.4 in [9]) applied to a pentagonal curve, it has been shown in [25] that

$$\int_{-\pi}^{\pi} w(\zeta,\eta,\zeta_0,\eta_0)\mathrm{ce}_n(\zeta,q)\,\mathrm{d}\zeta = \mu_n(q)\mathrm{ce}_n(\eta,q)\mathrm{ce}_n(\zeta_0,q)\mathrm{ce}_n(\eta_0,q), \quad (51)$$

$$\int_{-\pi}^{\pi} w(\zeta,\eta,\zeta_0,\eta_0)\mathrm{se}_n(\zeta,q)\,\mathrm{d}\zeta = \nu_n(q)\mathrm{se}_n(\eta,q)\mathrm{se}_n(\zeta_0,q)\mathrm{se}_n(\eta_0,q). \quad (52)$$

There do not exist explicit formulas for the quantities $\mu_n(q)$ and $\nu_n(q)$ but they can be determined as follows. Mathieu's equation (47) with $\lambda = a_n(q)$, $n \in \mathbf{N}_0$, has the solution $u_1(\eta) = \mathrm{ce}_n(\eta,q)$. We choose a second linear independent solution $u_2(\eta)$. Then there is σ such that

$$u_2(\eta+\pi) = \sigma u_1(\eta) + (-1)^n u_2(\eta)$$

and

$$\mu_n(q) = \frac{2(-1)^n\sigma}{W[u_1,u_2]}, \quad (53)$$

where $W[u_1,u_2]$ denotes the Wronskian of u_1 and u_2. Similarly, Mathieu's equation (47) with $\lambda = b_n(q)$, $n \in \mathbf{N}$, has the solution $u_3(\eta) = \mathrm{se}_n(\eta,q)$. We choose a second linear independent solution $u_4(\eta)$. Then there is τ such that

$$u_4(\eta+\pi) = \tau u_3(\eta) + (-1)^n u_4(\eta)$$

and

$$\nu_n(q) = \frac{2(-1)^n\tau}{W[u_3,u_4]}. \quad (54)$$

Now applying theorem 4.1 and substituting $\zeta = i\xi$, $\zeta_0 = i\xi_0$, we obtain the following result.

Theorem 4.2. *Let $\xi,\eta,\xi_0,\eta_0 \in \mathbf{C}$, and let $k > 0$, $c > 0$, $q = \frac{1}{4}c^2k^2$. Then*

$$J_0(kr) = \frac{1}{\pi}\sum_{n=0}^{\infty}\mu_n(q)\mathrm{ce}_n(\mathrm{i}\xi,q)\mathrm{ce}_n(\eta,q)\mathrm{ce}_n(\mathrm{i}\xi_0,q)\mathrm{ce}_n(\eta_0,q)$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\nu_n(q)\mathrm{se}_n(\mathrm{i}\xi,q)\mathrm{se}_n(\eta,q)\mathrm{se}_n(\mathrm{i}\xi_0,q)\mathrm{se}_n(\eta_0,q),$$

where r is given by (49).

Theorem 4.2 agrees with expansion (23) (for $j = 1$ and $\nu = 0$), section 2.66 in Meixner and Schäfke [17], who have a slightly different notation. They use $\mathrm{me}_n(z,q)$, $n \in \mathbf{Z}$, where

$$\mathrm{me}_n(z,q) := \sqrt{2}\mathrm{ce}_n(z,q) \quad \text{if } n \in \mathbf{N}_0,$$
$$\mathrm{me}_{-n}(z,q) := -\sqrt{2}\mathrm{i}\mathrm{se}_n(z,q) \quad \text{if } n \in \mathbf{N}.$$

Moreover, the coefficients $\mu_n(q)$ and $\nu_n(q)$ are represented in a different form. The proof of theorem 4.2 based on Riemann's method of integration appears to be new.

We now use (3) to obtain our final result in this section.

Theorem 4.3. *Let $\mathbf{x}$, $\mathbf{x}_0$ be distinct points on $\mathbf{R}^3$ with elliptic cylinder coordinates (ξ,η,z) and (ξ_0,η_0,z_0), respectively. Then*

$$\frac{1}{\|\mathbf{x}-\mathbf{x}_0\|} = \frac{1}{\pi}\int_0^\infty\sum_{n=0}^{\infty}\mu_n(q)\mathrm{ce}_n(\mathrm{i}\xi,q)\mathrm{ce}_n(\eta,q)\mathrm{ce}_n(\mathrm{i}\xi_0,q)\mathrm{ce}_n(\eta_0,q)\,\mathrm{e}^{-k|z-z_0|}\,\mathrm{d}k$$
$$+\frac{1}{\pi}\int_0^\infty\sum_{n=1}^{\infty}\nu_n(q)\mathrm{se}_n(\mathrm{i}\xi,q)\mathrm{se}_n(\eta,q)\mathrm{se}_n(\mathrm{i}\xi_0,q)\mathrm{se}_n(\eta_0,q)\mathrm{e}^{-k|z-z_0|}\,\mathrm{d}k,$$

where $q = \frac{1}{4}c^2k^2$.

5. Expansion of $K_0(kr)$ for elliptic cylinder coordinates

Consider equation (10) for $k > 0$. Transforming to elliptic coordinates (44), we obtain

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - k^2 c^2(\cosh^2 \xi - \cos^2 \eta)u = 0. \tag{55}$$

Separating variables $u(\xi, \eta) = u_1(\xi)u_2(\eta)$, leads again to (46), (47) but now $q = -\frac{1}{4}c^2k^2$ is negative.

We will need the following solutions of the modified Mathieu equation (46) when $q < 0$; see [20, section 28.20]. Set $q = -h^2$ with $h > 0$. For $n \in \mathbf{N}_0$, $\mathrm{Ie}_n(\xi, h)$ is the even solution of (46) with $\lambda = a_n(q)$ with asymptotic behavior

$$\mathrm{Ie}_n(\xi, h) \sim I_n(2h \cosh \xi) \quad \text{as } \xi \to +\infty,$$

while $\mathrm{Ke}_n(\xi, h)$ is the recessive solution determined by

$$\mathrm{Ke}_n(\xi, h) \sim K_n(2h \cosh \xi) \quad \text{as } \xi \to +\infty,$$

where $I_n(z) := i^{-n}J_n(\mathrm{i}z)$ and $K_n(z)$ are the modified Bessel functions of the first [20, (10.27.6)] and second kinds, respectively, with integer order n (see (4), (8)). Similarly, for $n \in \mathbf{N}$, $\mathrm{Io}_n(\xi, h)$ is the odd solution of (46) with $\lambda = b_n(q)$ with asymptotic behavior

$$\mathrm{Io}_n(\xi, h) \sim I_n(2h \cosh \xi) \quad \text{as } \xi \to +\infty,$$

while $\mathrm{Ko}_n(\xi, h)$ is the recessive solution determined by

$$\mathrm{Ko}_n(\xi, h) \sim K_n(2h \cosh \xi) \quad \text{as } \xi \to +\infty.$$

For fixed $\xi, \xi_0, \eta_0 \in \mathbf{R}$ we wish to expand the function

$$v(\xi, \eta, \xi_0, \eta_0) := K_0(kr(\xi, \eta, \xi_0, \eta_0))$$

with r given by (49) into a series of periodic Mathieu functions according to theorem 4.1. The corresponding integrals appearing in the expansion will be computed based on the observation that $(\xi, \eta) \mapsto v(\xi, \eta, \xi_0, \eta_0)$ is a fundamental solution of (55). In fact, it is a solution of (55) and it has logarithmic singularities at the points $\pm(\xi_0, \eta_0 + 2m\pi)$, where m is any integer. Arguing as in [26, theorem 1.11], we have the following representation theorem for a solution of (55).

Theorem 5.1. *Let $u \in C^2(\mathbf{R}^2)$ be a solution of (55). Let $(\xi_0, \eta_0) \in \mathbf{R}^2$, and let C be a closed rectifiable curve on $\mathbf{R}^2$ which does not pass through any of the points $\pm(\xi_0, \eta_0 + 2m\pi)$, $m \in \mathbf{Z}$. Let $n_m^{\pm}$ be the winding number of C with respect to $\pm(\xi_0, \eta_0 + 2m\pi)$. Then we have*

$$2\pi \sum_m \left[n_m^+ u(\xi_0, \eta_0 + 2m\pi) + n_m^- u(-\xi_0, -\eta_0 - 2m\pi)\right]$$
$$= \int_C (u\partial_2 v - v\partial_2 u)d\xi + (v\partial_1 u - u\partial_1 v)\,\mathrm{d}\eta,$$

where ∂_1, ∂_2 denote partial derivatives with respect to ξ, η, respectively.

In theorem 5.1 we choose

$$u(\xi, \eta) = u_1(\xi)u_2(\eta),$$

where

$$u_1(\xi) = \mathrm{Ke}_n(\xi, h), \quad u_2(\eta) = \mathrm{ce}_n(\eta, q).$$

Let $\xi_0 > 0$ and $\eta_0 \in \mathbf{R}$. We take the curve C to be the positively oriented boundary of the rectangle $\xi_1 \leqslant \xi \leqslant \xi_2$, $\eta_0 - \pi \leqslant \eta \leqslant \eta_0 + \pi$, where $|\xi_1| < \xi_0 < \xi_2$. Consider the line integral

$\int_C$ in theorem 5.1. Since u_2 has period 2π, the line integrals along the horizontal segments of C cancel each other. When $\xi_2 \to +\infty$ the asymptotic behavior of $u_1(\xi)$ shows that the integral along the right-hand vertical segment of C tends to 0 as $\xi_2 \to +\infty$. Therefore, setting

$$f(\xi) = \int_{-\pi}^{\pi} v(\xi, \eta, \xi_0, \eta_0) u_2(\eta)\, \mathrm{d}\eta = \int_{\eta_0-\pi}^{\eta_0+\pi} v(\xi, \eta, \xi_0, \eta_0) u_2(\eta)\, \mathrm{d}\eta$$

for $|\xi| < \xi_0$, one obtains

$$2\pi u_1(\xi_0) u_2(\eta_0) = u_1(\xi) f'(\xi) - u_1'(\xi) f(\xi). \tag{56}$$

We now argue as in section 2. By differentiating (56) with respect to ξ, we find that $f(\xi)$ satisfies the modified Mathieu equation (46). It is easy to see that $f(\xi)$ is an even function, so $f(\xi) = c u_3(\xi)$, where $u_3(\xi) = \mathrm{Ie}_n(\xi, h)$ and c is a constant. Then (56) implies that

$$2\pi u_1(\xi_0) u_2(\eta_0) = c W[u_1, u_3].$$

Since $W[u_1, u_3] = 1$,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} v(\xi, \eta, \xi_0, \eta_0) u_2(\eta)\, \mathrm{d}\eta = 2 u_1(\xi_0) u_2(\eta_0) u_3(\xi) \quad \text{if } |\xi| < \xi_0. \tag{57}$$

By the same reasoning, we see that (57) is also true when $u_1(\xi) = \mathrm{Ko}_n(\xi, h)$, $u_2(\eta) = \mathrm{se}_n(\eta, q)$, $u_3(\xi) = \mathrm{Io}_n(\xi, h)$. By taking limits, we can also allow $|\xi| = \xi_0$.

Expanding the function $\eta \mapsto v(\xi, \eta, \xi_0, \eta_0)$ according to theorem 4.1, the following result is obtained. Strictly speaking we can use theorem 4.1 only if $|\xi| < \xi_0$. Otherwise we apply [17, Satz 16, p 128].

Theorem 5.2. *Let* $\xi, \eta, \xi_0, \eta_0 \in \mathbf{R}$ *such that* $|\xi| \leqslant \xi_0$, $\eta - \eta_0 \notin 2\pi\mathbf{Z}$, *and let* $k > 0$, $c > 0$, $q = -\frac{1}{4}c^2k^2$, $h = \frac{1}{2}ck$. *Then*

$$K_0(kr) = 2\sum_{n=0}^{\infty} \mathrm{Ie}_n(\xi, h)\mathrm{ce}_n(\eta, q)\mathrm{Ke}_n(\xi_0, h)\mathrm{ce}_n(\eta_0, q)$$
$$+2\sum_{n=1}^{\infty} \mathrm{Io}_n(\xi, h)\mathrm{se}_n(\eta, q)\mathrm{Ko}_n(\xi_0, h)\mathrm{se}_n(\eta_0, q),$$

where r *is given by (49).*

Theorem 5.2 agrees with [17, section 2.66] although our notation and proof are different. Inserting this result in (7), we obtain our final result.

Theorem 5.3. *Let* $\mathbf{x}$, $\mathbf{x}_0$ *be distinct points on* $\mathbf{R}^3$ *with elliptic cylinder coordinates* (ξ, η, z) *and* (ξ_0, η_0, z_0), *respectively. If* $\xi_{\lessgtr} := {\min \atop \max}\{\xi, \xi_0\}$ *then*

$$\frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} = \frac{4}{\pi}\int_0^{\infty} \sum_{n=0}^{\infty} \mathrm{Ie}_n(\xi_<, h)\mathrm{ce}_n(\eta, q)\mathrm{Ke}_n(\xi_>, h)\mathrm{ce}_n(\eta_0, q)\cos k(z - z_0)\, \mathrm{d}k$$
$$+\frac{4}{\pi}\int_0^{\infty} \sum_{n=1}^{\infty} \mathrm{Io}_n(\xi_<, h)\mathrm{se}_n(\eta, q)\mathrm{Ko}_n(\xi_>, h)\mathrm{se}_n(\eta_0, q)\cos k(z - z_0)\, \mathrm{d}k,$$

where $q = -\frac{1}{4}c^2k^2$, $h = \frac{1}{2}ck$.

As a final comment, we should mention that our theorems 2.2, 3.2, 4.2, 5.2, and therefore their corollaries 2.3, 3.3, 4.3, 5.3, are based on the spectral theorem for certain Sturm–Liouville problems and that all these expansion theorems represent infinite sums, except for theorem 3.2 which is expressed in terms of an improper integral. In this one case the spectrum is continuous, as opposed to all the other cases where the spectrum is discrete. Continuous spectra for Sturm–Liouville problems is also known to occur in connection with harmonic expansions in circular cylinder coordinates and in rotationally-invariant parabolic coordinates (see for instance [19, p 1263, 1298]).

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Appendix A. Integrals for the (modified) parabolic cylinder harmonic expansion of $J_0(kr)$

The following formulas are valid for $\Im\lambda < 0$:

$$I_1 := \int_0^\infty J_0\left(\frac{1}{4}\xi^2\right) u_3(\lambda, \xi)\, \mathrm{d}\xi = \frac{1}{2}\sqrt{\pi}(1-\mathrm{i})\frac{G_1}{G_2^2}, \tag{A.1}$$

$$I_2 := \int_0^\infty \xi^{-1} J_1\left(\frac{1}{4}\xi^2\right) u_3(\lambda, \xi)\, \mathrm{d}\xi = -\sqrt{\pi}\left(\lambda\frac{1}{G_2} + 2\mathrm{i}\frac{G_2}{G_1^2}\right), \tag{A.2}$$

where

$$G_1 = G_1(\lambda) = \Gamma\left(\frac{1}{4} + \frac{\mathrm{i}}{2}\lambda\right),$$

$$G_2 = G_2(\lambda) = \Gamma\left(\frac{3}{4} + \frac{\mathrm{i}}{2}\lambda\right).$$

We believe that these integrals may be known but do not have a reference. We will derive (A.2). In (29) we use the integral representation [20, (13.4.4)]

$$\Gamma(a)U(a, b, z) = \int_0^\infty \mathrm{e}^{-zt} t^{a-1}(1+t)^{b-a-1}\, \mathrm{d}t, \quad \Re z, \Re a > 0.$$

Substituting $4s = \xi^2$ and changing the order of integration, one obtains

$$I_2 = \frac{1}{2G_1}\int_0^\infty t^{-\frac{3}{4}+\frac{\mathrm{i}}{2}\lambda}(1+t)^{-\frac{3}{4}-\frac{\mathrm{i}}{2}\lambda}\int_0^\infty s^{-1}J_1(s)\mathrm{e}^{-\mathrm{i}s(2t+1)}\, \mathrm{d}s\, \mathrm{d}t. \tag{A.3}$$

From [27, p 405], we have, for $t > 0$,

$$\int_0^\infty s^{-1}J_1(s)\cos(s(2t+1))\, \mathrm{d}s = 0,$$

$$\int_0^\infty s^{-1}J_1(s)\sin(s(2t+1))\, \mathrm{d}s = t + (t+1) - 2\sqrt{t}\sqrt{t+1}.$$

Substituting these formulas in (A.3), we can evaluate I_2 using three times the formula for the beta function [20, (5.12.3)]

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^\infty t^{z-1}(1+t)^{-z-w}\, \mathrm{d}t, \quad \Re z, \Re w > 0.$$

This gives (A.2).

The proof of (A.1) is similar, but in (29) one should first use [20, (13.2.40)]

$$U(a, b, z) = z^{1-b}U(1+a-b, 2-b, z).$$

The formulas (A.1), (A.2) remain valid for real λ. By separating real and imaginary parts, we obtain for $\lambda \in \mathbf{R}$,

$$\int_0^\infty J_0\left(\frac{1}{4}\xi^2\right) u_1(\lambda, \xi)\, \mathrm{d}\xi = \frac{\Re(G_1\overline{G_2}) + \Im(G_1\overline{G_2})}{|G_2|^2}, \tag{A.4}$$

$$\int_0^\infty J_0\left(\frac{1}{4}\xi^2\right) u_2(\lambda,\xi)\,\mathrm{d}\xi = \frac{1}{2}\left|\frac{G_1}{G_2}\right|^2, \tag{A.5}$$

$$\int_0^\infty \xi^{-1} J_1\left(\frac{1}{4}\xi^2\right) u_1(\lambda,\xi)\,\mathrm{d}\xi = 2\left|\frac{G_2}{G_1}\right|^2 - \lambda, \tag{A.6}$$

$$\int_0^\infty \xi^{-1} J_1\left(\frac{1}{4}\xi^2\right) u_2(\lambda,\xi)\,\mathrm{d}\xi = 2\frac{\Re(G_1\overline{G_2}) + \Im(G_1\overline{G_2})}{|G_1|^2}. \tag{A.7}$$

We may use

$$\Re(G_1\overline{G_2}) + \Im(G_1\overline{G_2}) = \frac{\pi\sqrt{2}\mathrm{e}^{-\frac{1}{2}\pi\lambda}}{\cosh(\pi\lambda)}.$$

Formulas (A.4), (A.7) give us the integrals (37), (38) noting that we integrate even functions in (37), (38).

Appendix B. The Riemann method of integration revisited

When we compare the proofs of the main theorems 2.2, 3.2, 4.2, 5.2, we notice that in each case certain integrals had to be evaluated. For instance, in section 2 we evaluated the integral (20). In sections 2 and 5 we applied integral formulas of Green's type involving the fundamental solutions of certain elliptic partial differential equations. In section 4 we used the Riemann method of integration involving the Riemann function of a certain hyperbolic partial differential equation. The obvious question arises whether the integrals (A.4)–(A.7) can also be evaluated by the Riemann method of integration.

The function $w(\xi,\zeta,\xi_0,\zeta_0)$ (24) as a function of (ξ,ζ) is a solution of the partial differential equation (23) and it satisfies the condition $w(\xi,\zeta,\xi_0,\zeta_0) = 1$ if $\xi-\xi_0 = \pm(\zeta-\zeta_0)$. This shows that w is the Riemann function of (23).

The Riemann method of integration applied to the partial differential equation (23) as in [24] gives, for all $\zeta,\xi_0,\zeta_0\in\mathbf{R}$,

$$\begin{aligned}2u(\xi_0,\zeta_0) &= u(\zeta-\zeta_0+\xi_0,\zeta) + u(-\zeta+\zeta_0+\xi_0,\zeta)\\ &\quad - \int_{-\zeta+\zeta_0+\xi_0}^{\zeta-\zeta_0+\xi_0} [w(\xi,\zeta,\xi_0,\zeta_0)\partial_2 u(\xi,\zeta) - \partial_2 w(\xi,\zeta,\xi_0,\zeta_0)u(\xi,\zeta)]\,\mathrm{d}\xi,\end{aligned} \tag{B.1}$$

where $u\in C^2(\mathbf{R}^2)$ is a solution of (23). This formula for $\xi_0 = \zeta = 0$ and $u(\xi,\zeta) = u_1(\lambda,\xi)u_2(\lambda,\zeta)$ with u_1,u_2 from (27), (28) (after replacing ζ_0 by ζ), implies that

$$\int_{-\zeta}^{\zeta} J_0\left(\frac{1}{4}(\xi^2-\zeta^2)\right) u_1(\lambda,\xi)\,\mathrm{d}\xi = 2u_2(\lambda,\zeta). \tag{B.2}$$

This equation allows us to transform the even solution u_1 into the odd solution u_2 of equation (26). One can prove (B.2) directly by denoting the left-hand side of (B.2) by $f(\zeta)$ and then showing that f is an odd solution of (26) with $f'(0) = 2$.

By differentiating (B.1) with $u(\xi,\zeta) = u_1(\lambda,\xi)u_2(\lambda,\zeta)$ first with respect to ξ_0,ζ_0, we obtain (after a lengthy calculation) that

$$\int_{-\zeta}^{\zeta} \frac{\xi\zeta}{\xi^2-\zeta^2} J_1\left(\frac{1}{4}(\xi^2-\zeta^2)\right) u_2(\lambda,\xi)\,\mathrm{d}\xi = 2u_1(\lambda,\zeta) - 2u_2'(\lambda,\zeta).$$

This formula can also be proved directly.

Let $h : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by

$$h(\xi,\zeta) := \begin{cases} J_0\left(\frac{1}{4}(\xi^2-\zeta^2)\right) & \text{if } |\xi| < |\zeta|, \\ 0 & \text{otherwise.} \end{cases}$$

For fixed ζ this is an even function in $L^2(\mathbf{R})$ which can be expanded according to theorem 3.1 (without knowing $c_j(\lambda)$), so that

$$\operatorname{sign}(\zeta)h(\xi,\zeta) = 2\int_{-\infty}^{\infty} u_1(\lambda,\xi)u_2(\lambda,\zeta)\rho_1'(\lambda)\,\mathrm{d}\lambda.$$

If $\xi = 0$, we obtain

$$\operatorname{sign}(\zeta)J_0\left(\frac{1}{4}\zeta^2\right) = 2\int_{-\infty}^{\infty} u_2(\lambda,\zeta)\rho_1'(\lambda)\,\mathrm{d}\lambda.$$

By theorem 3.1, this formulas allows us to conclude

$$\int_0^{\infty} J_0\left(\frac{1}{4}\zeta^2\right)u_2(\lambda,\zeta)\,\mathrm{d}\zeta = \frac{\rho_1'(\lambda)}{\rho_2'(\lambda)},$$

and this is in agreement with (A.5).

By using the Riemann method of integration, we have only been partially successful in obtaining the integrals from appendix A. We pose as a problem for the reader to obtain all these integrals using this method.

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