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Symmetry of terminating basic hypergeometric series representations of the Askey–Wilson polynomials

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ABSTRACT

In this paper, we explore the symmetric nature of the terminating basic hypergeometric series representations of the Askey–Wilson polynomials and the corresponding terminating basic hypergeometric transformations that these polynomials satisfy. In particular we identify and classify the set of 4 and 7 equivalence classes of terminating balanced ${}_4\phi_3$ and terminating very-well-poised ${}_8W_7$ basic hypergeometric series which are connected with the Askey–Wilson polynomials. We study the inversion properties of these equivalence classes and also identify the connection of both sets of equivalence classes with the symmetric group S_6, the symmetry group of the terminating balanced ${}_4\phi_3$. We then use terminating balanced ${}_4\phi_3$ and terminating very-well poised ${}_8W_7$ transformations to give a broader interpretation of Watson's q-analog of Whipple's theorem and its converse.

1. Introduction

The Askey–Wilson polynomials $p_n(x;\mathbf{a}|q)$ [11, §14.1], symmetric in four free parameters, are at the top of the q-Askey scheme and all polynomials within the q-Askey scheme can be written as either a specialization or limit of the Askey–Wilson polynomials. The Askey–Wilson polynomials have terminating basic hypergeometric representations. From these representations, one can derive transformation formulas for terminating basic hypergeometric functions.

In order to study the symmetry properties of the terminating basic hypergeometric functions which appear in the series representations of the Askey–Wilson polynomials, a detailed parametric connection between them was provided in [4, Corollary 3]. However, there were some typographical errors in that result

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and some representations which arise by inversion were inadvertently left off. An attempt to remedy this was executed in [3].

The results presented in this paper provide a framework for future work on a symmetry analysis of terminating basic hypergeometric functions. The terminating symmetry analysis is more complicated than that for the nonterminating case [18]. Therefore, it is not surprising that the symmetry classes for terminating basic hypergeometric functions are not connected by the known nonterminating transformations (see Figs. 1, 2, 3). In this paper, for the first time, we present the full symmetry structure of the terminating ${}_8W_7$ representations for the Askey–Wilson polynomials and a detailed connection with the terminating balanced ${}_4\phi_3$ representations.

2. Preliminaries

We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$, and we use the sets $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the integers, real numbers and complex numbers respectively, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $\mathbb{C}^\dagger := \mathbb{C}^* \setminus \{z \in \mathbb{C} : |z| = 1\}$. We also adopt the following notation and conventions. Let $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$, $b, a_k \in \mathbb{C}$, $k = 1, 2, 3, 4$. Define $\mathbf{a} + b := \{a_1 + b, a_2 + b, a_3 + b, a_4 + b\}$, $a_{12} := a_1a_2$, $a_{13} := a_1a_3$, $a_{23} := a_2a_3$, $a_{123} := a_1a_2a_3$, $a_{1234} := a_1a_2a_3a_4$, etc. Throughout the paper, we assume that the empty sum vanishes and the empty product is unity.

Definition 1. Throughout this paper we adopt the following conventions for succinctly writing elements of lists. To indicate sequential positive and negative elements, we write

$$\pm a := \{a, -a\}.$$

We also adopt an analogous notation

$$\mathrm{e}^{\pm i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\}.$$

In the same vein, consider the numbers $f_s \in \mathbb{C}$ with $s \in \mathcal{S} \subset \mathbb{N}$, with $\mathcal{S}$ finite. Then, the notation $\{f_s\}$ represents the set of all complex numbers f_s such that $s \in \mathcal{S}$. Furthermore, consider some $p \in \mathcal{S}$, then the notation $\{f_s\}_{s\neq p}$ represents the sequence of all complex numbers f_s such that $s \in \mathcal{S}\setminus\{p\}$. In addition, for the empty list, $n = 0$, we take

$$\{a_1, ..., a_n\} := \emptyset.$$

Consider $q \in \mathbb{C}^\dagger$. Define the sets $\Omega_q^n := \{q^{-k} : n, k \in \mathbb{N}_0,\ 0 \le k \le n-1\}$, $\Omega_q := \Omega_q^\infty = \{q^{-k} : k \in \mathbb{N}_0\}$. In order to obtain our derived identities, we rely on properties of the q-Pochhammer symbol (q-shifted factorial). For any $n \in \mathbb{N}_0$, $a, q \in \mathbb{C}$, the q-Pochhammer symbol is defined as

$$(a; q)_n := (1 - a)(1 - aq)\cdots(1 - aq^{n-1}), \quad n \in \mathbb{N}_0. \tag{1}$$

One may also define

$$(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n), \tag{2}$$

where $|q| < 1$. We will also use the common notational product conventions

$$(a_1, ..., a_k; q)_b := (a_1; q)_b \cdots (a_k; q)_b.$$

The following properties for the q-Pochhammer symbol can be found in Koekoek et al. [11, (1.8.7), (1.8.10-11), (1.8.14), (1.8.19), (1.8.21-22)], namely for appropriate values of $q, a \in \mathbb{C}^*$ and $n, k \in \mathbb{N}_0$:

$$(a; q^{-1})_n = (a^{-1}; q)_n (-a)^n q^{-\binom{n}{2}}. \tag{3}$$

The basic hypergeometric series, which we will often use, is defined for $q, z \in \mathbb{C}^*$, such that $|q|, |z| < 1$, $s, r \in \mathbb{N}_0$, $b_j \notin \Omega_q$, $j = 1, ..., s$, as [11, (1.10.1)]

$${}_r\phi_s\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r; q)_k}{(q, b_1, ..., b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k. \tag{4}$$

Note that we refer to a basic hypergeometric series as ℓ-balanced if $q^\ell a_1 \cdots a_r = b_1 \cdots b_s$, and balanced (Saalschützian) if $\ell = 1$. A basic hypergeometric series ${}_{r+1}\phi_r$ is well-poised if the parameters satisfy the relations

$$q a_1 = b_1 a_2 = b_2 a_3 = \cdots = b_r a_{r+1}.$$

It is very-well-poised if in addition, $\{a_2, a_3\} = \pm q\sqrt{a_1}$.

Similarly for terminating basic hypergeometric series which appear in basic hypergeometric orthogonal polynomials, one has

$${}_r\phi_s\left(\begin{matrix} q^{-n}, a_1, ..., a_{r-1} \\ b_1, ..., b_s \end{matrix}; q, z\right) := \sum_{k=0}^{n} \frac{(q^{-n}, a_1, ..., a_{r-1}; q)_k}{(q, b_1, ..., b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, \tag{5}$$

where $b_j \notin \Omega_q^n$, $j = 1, ..., s$. Define the very-well-poised basic hypergeometric series ${}_{r+1}W_r$ [7, (2.1.11)]

$${}_{r+1}W_r(b; a_4, ..., a_{r+1}; q, z) := {}_{r+1}\phi_r\left(\begin{matrix} b, \pm q\sqrt{b}, a_4, ..., a_{r+1} \\ \pm\sqrt{b}, \frac{qb}{a_4}, ..., \frac{qb}{a_{r+1}} \end{matrix}; q, z\right), \tag{6}$$

where $\sqrt{b}, \frac{qb}{a_4}, ..., \frac{qb}{a_{r+1}} \notin \Omega_q$.

The following notation ${}_{r+1}\phi_s^m$, $m \in \mathbb{Z}$ (originally due to van de Bult & Rains [17, p. 4]), for basic hypergeometric series with zero parameter entries. Consider $p \in \mathbb{N}_0$. Then define

$${}_{r+1}\phi_s^{-p}\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := {}_{r+p+1}\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1}, \overbrace{0, \ldots, 0}^{p} \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right), \tag{7}$$

$${}_{r+1}\phi_s^{p}\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := {}_{r+1}\phi_{s+p}\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_s, \underbrace{0, \ldots, 0}_{p} \end{matrix}; q, z\right), \tag{8}$$

where $b_1, \ldots, b_s \notin \Omega_q \cup \{0\}$, and ${}_{r+1}\phi_s^0 = {}_{r+1}\phi_s$. The terminating basic hypergeometric series ${}_{r+1}\phi_s^m(q^{-n}, \mathbf{a}; \mathbf{b}; q, z)$, for some $n \in \mathbb{N}_0$, $\mathbf{a} := \{a_1, \ldots, a_r\}$, $\mathbf{b} := \{b_1, \ldots, b_s\}$, is well-defined for all $r, s \in \mathbb{N}_0$, $m \in \mathbb{Z}$. In [7, Exercise 1.4ii] one finds the inversion formula for terminating basic hypergeometric series.

Theorem 2 *(Gasper and Rahman's (2004) Inversion Theorem). Let $m, n, k, r, s \in \mathbb{N}_0$, $a_k \in \mathbb{C}$, $1 \le k \le r$, $b_m \notin \Omega_q^n$, $1 \le m \le s$, $q \in \mathbb{C}^\dagger$. Then,*

$$
{}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1} \times {}_{s+1}\phi_r^{s-r}\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_s} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_r} \end{matrix}; q, \frac{q^{n+1}}{z}\frac{b_1 \cdots b_s}{a_1 \cdots a_r}\right). \tag{9}
$$

Corollary 3. *Let $n, r \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a_k, b_k \notin \Omega_q^n \cup \{0\}$, $1 \le k \le r$. Then,*

$$
{}_{r+1}\phi_r\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; q, z\right) = q^{-\binom{n}{2}} \left(-\frac{z}{q}\right)^n \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_r; q)_n} {}_{r+1}\phi_r\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_r} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_r} \end{matrix}; q, \frac{q^{n+1}}{z}\frac{b_1 \cdots b_r}{a_1 \cdots a_r}\right). \tag{10}
$$

Proof. Take $r = s$, in (9) and using the definition (4) completes the proof. □

Note that in Corollary 3 if the terminating basic hypergeometric series on the left-hand side is balanced then the argument of the terminating basic hypergeometric series on the right-hand side is q^2/z.

Applying Corollary 3 to the definition of ${}_{r+1}W_r$, we obtain the following result for a terminating very-well-poised basic hypergeometric series ${}_{r+1}W_r$.

Corollary 4. *Let $n \in \mathbb{N}_0$, $b, a_k, q, z \in \mathbb{C}^*$, $\sqrt{b}, q^{n+1}b, \frac{qb}{a_k}, \frac{q^{1-n}}{b}, \frac{q^{1-n}}{a_k} \notin \Omega_q^n$, $k = 5, \ldots, r+1$. Then, one has the following transformation formula for a very-well-poised terminating basic hypergeometric series:*

$$
{}_{r+1}W_r\left(b; q^{-n}, a_5, \ldots, a_{r+1}; q, z\right) = q^{-\binom{n}{2}} \left(\frac{-z}{q}\right)^n \frac{(\pm q\sqrt{b}, b, a_5, \ldots, a_{r+1}; q)_n}{\left(\pm\sqrt{b}, q^{n+1}b, \frac{qb}{a_5}, \ldots, \frac{qb}{a_{r+1}}; q\right)_n} \times {}_{r+1}W_r\left(\frac{q^{-2n}}{b}; q^{-n}, \frac{q^{-n}a_5}{b}, \ldots, \frac{q^{-n}a_{r+1}}{b}; q, \frac{q^{2n+r-3}b^{r-3}}{(a_5 \cdots a_{r+1})^2 z}\right). \tag{11}
$$

Proof. Use Corollary 3 and (6). □

An interesting and useful consequence of this formula is the $r = 7$ special case,

$$
{}_8W_7\left(b; q^{-n}, c, d, e, f; q, z\right) = q^{-\binom{n}{2}} \left(\frac{-z}{q}\right)^n \frac{\left(\pm q\sqrt{b}, b, c, d, e, f; q\right)_n}{\left(\pm\sqrt{b}, q^{n+1}b, \frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f}; q\right)_n} \times {}_8W_7\left(\frac{q^{-2n}}{b}; q^{-n}, \frac{q^{-n}c}{b}, \frac{q^{-n}d}{b}, \frac{q^{-n}e}{b}, \frac{q^{-n}f}{b}; q, \frac{q^{2n+4}b^4}{z(cdef)^2}\right). \tag{12}
$$

Note that when one obtains an ${}_8W_7$ from a balanced ${}_4\phi_3$ using Watson's q-analogue of Whipple's theorem [6, (17.9.15)] with (11), then $q^{2n+4}b^4/(z(cdef)^2) = z$.

We will obtain new transformations for basic hypergeometric orthogonal polynomials by taking advantage of the following remark.

Remark 5. Since $x = \cos\theta$ is an even function of θ, all polynomials in $\cos\theta$ will be invariant under the map $\theta \mapsto -\theta$.

Remark 6. Observe in the following discussion we will often be referring to a collection of constants a, b, c, d, e, f. In such cases, which will be clear from context, then the constant e should not be confused with Euler's number e, the base of the natural logarithm, i.e., $\log \mathrm{e} = 1$. Observe the different (roman) typography for Euler's number.

3. The Askey–Wilson polynomials

Define the sets $\mathbf{4} := \{1,2,3,4\}$, $\mathbf{a} := \{a_1,a_2,a_3,a_4\}$, $a_k \in \mathbb{C}^*$, $k \in \mathbf{4}$, and $x = \cos\theta \in [-1,1]$. The Askey–Wilson polynomials $p_n(x;\mathbf{a}|q)$ are a family of polynomials symmetric in the four free parameters a_1, a_2, a_3 and a_4. These polynomials have a long and in-depth history and their properties have been studied in detail. The basic hypergeometric series representations of the Askey–Wilson polynomials fall into four main categories: (1) terminating ${}_4\phi_3$ representations; (2) terminating ${}_8W_7$ representations; (3) nonterminating ${}_8W_7$ representations; and (4) nonterminating ${}_4\phi_3$ representations. One may obtain the alternative nonterminating representations of the Askey–Wilson polynomials using [7, (2.10.7)] and [6, 17.9.16]. However, these nonterminating representations will not be further discussed in this paper.

3.1. The Askey–Wilson polynomial terminating series representations and transformations

The terminating series representations of the Askey–Wilson polynomials are given in terms of terminating balanced ${}_4\phi_3$ and terminating very-well-poised ${}_8W_7$ basic hypergeometric series. The full result was presented in [4, Theorem 3]. The symmetric structure of the mapping properties of the terminating basic hypergeometric functions which appear in this result are the essential ingredients for the remainder of the paper. We will reproduce the following theorem which was originally given in [4, Theorem 3] because we will refer to the details of it so often below.

Theorem 7. *Let $n \in \mathbb{N}_0$, $p,s,r,t,u \in \mathbf{4}$, p,r,t,u distinct and fixed, $q \in \mathbb{C}^\dagger$. Then, the Askey–Wilson polynomials have the following terminating basic hypergeometric series representations given by:*

$$p_n(x;\mathbf{a}|q) := a_p^{-n}\left(\{a_{ps}\}_{s\neq p};q\right)_n\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}a_{1234},a_p\mathrm{e}^{\pm i\theta}\\ \{a_{ps}\}_{s\neq p}\end{matrix};q,q\right) \tag{13}$$

$$= q^{-\binom{n}{2}}(-a_p)^{-n}\frac{\left(\frac{a_{1234}}{q};q\right)_{2n}\left(a_p\mathrm{e}^{\pm i\theta};q\right)_n}{\left(\frac{a_{1234}}{q};q\right)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},\left\{\frac{q^{1-n}}{a_{ps}}\right\}_{s\neq p}\\ \frac{q^{2-2n}}{a_{1234}},\frac{q^{1-n}\mathrm{e}^{\pm i\theta}}{a_p}\end{matrix};q,q\right) \tag{14}$$

$$= \mathrm{e}^{in\theta}\left(a_{pr},a_t\mathrm{e}^{-i\theta},a_u\mathrm{e}^{-i\theta};q\right)_n\,{}_4\phi_3\left(\begin{matrix}q^{-n},a_p\mathrm{e}^{i\theta},a_r\mathrm{e}^{i\theta},\frac{q^{1-n}}{a_{tu}}\\ a_{pr},\frac{q^{1-n}\mathrm{e}^{i\theta}}{a_t},\frac{q^{1-n}\mathrm{e}^{i\theta}}{a_u}\end{matrix};q,q\right) \tag{15}$$

$$= \mathrm{e}^{in\theta}\frac{\left(\frac{a_{1234}}{q};q\right)_{2n}\left(\{a_s\mathrm{e}^{-i\theta}\}_{s\neq p},\frac{a_{1234}\,\mathrm{e}^{-i\theta}}{qa_p};q\right)_n}{\left(\frac{a_{1234}}{q};q\right)_n\left(\frac{a_{1234}\,\mathrm{e}^{-i\theta}}{qa_p};q\right)_{2n}}$$
$$\times\,{}_8W_7\left(\frac{q^{1-2n}a_p\mathrm{e}^{i\theta}}{a_{1234}};q^{-n},\left\{\frac{q^{1-n}a_{ps}}{a_{1234}}\right\}_{s\neq p},a_p\mathrm{e}^{i\theta};q,\frac{q\mathrm{e}^{i\theta}}{a_p}\right) \tag{16}$$

$$= \mathrm{e}^{in\theta}\frac{\left(a_p\mathrm{e}^{-i\theta},\{\frac{a_{1234}}{a_{ps}}\}_{s\neq p};q\right)_n}{\left(\frac{a_{1234}\,\mathrm{e}^{i\theta}}{a_p};q\right)_n}{}_8W_7\left(\frac{a_{1234}\mathrm{e}^{i\theta}}{qa_p};q^{-n},\{a_s\mathrm{e}^{i\theta}\}_{s\neq p},q^{n-1}a_{1234};q,\frac{q\mathrm{e}^{-i\theta}}{a_p}\right) \tag{17}$$

$$= a_p^{-n}\frac{\left(a_{pt},a_{pu},a_r\mathrm{e}^{\pm i\theta};q\right)_n}{\left(\frac{a_r}{a_p};q\right)_n}\,{}_8W_7\left(\frac{q^{-n}a_p}{a_r};q^{-n},\frac{q^{1-n}}{a_{rt}},\frac{q^{1-n}}{a_{ru}},a_p\mathrm{e}^{\pm i\theta};q,q^na_{tu}\right) \tag{18}$$

$$= \mathrm{e}^{in\theta}\frac{\left(\{a_s\mathrm{e}^{-i\theta}\};q\right)_n}{(\mathrm{e}^{-2i\theta};q)_n}\,{}_8W_7\left(q^{-n}\mathrm{e}^{2i\theta};q^{-n},\{a_s\mathrm{e}^{i\theta}\};q,\frac{q^{2-n}}{a_{1234}}\right). \tag{19}$$

Proof. See the proof of [4, Theorem 3]. □

Corollary 8. *Let $n \in \mathbb{N}_0$, $b, c, d, e, f \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$. Then, one has the following transformation formulas for a terminating ${}_8W_7$ to a terminating ${}_8W_7$:*

$${}_8W_7\left(b;q^{-n},c,d,e,f;q,\frac{q^{n+2}b^2}{cdef}\right) \tag{20}$$

$$=q^{\binom{n}{2}}\left(\frac{-q^2b^2}{cdef}\right)^n\frac{(qb,b,c,d,e,f;q)_n}{(b;q)_{2n}\left(\frac{qb}{c},\frac{qb}{d},\frac{qb}{e},\frac{qb}{f};q\right)_n}{}_8W_7\left(\frac{q^{-2n}}{b};q^{-n},\frac{q^{-n}c}{b},\frac{q^{-n}d}{b},\frac{q^{-n}e}{b},\frac{q^{-n}f}{b};q,\frac{q^{n+2}b^2}{cdef}\right) \tag{21}$$

$$=\frac{\left(\frac{qb}{ce},\frac{qb}{cf},qb,d;q\right)_n}{\left(\frac{qb}{c},\frac{qb}{e},\frac{qb}{f},\frac{d}{c};q\right)_n}{}_8W_7\left(\frac{q^{-n}c}{d};q^{-n},\frac{q^{-n}c}{b},\frac{qb}{de},\frac{qb}{df},c;q,\frac{ef}{b}\right) \tag{22}$$

$$=\frac{\left(\frac{qb}{de},\frac{qb}{df},\frac{qb}{ef},qb;q\right)_n}{\left(\frac{qb}{def},\frac{qb}{d},\frac{qb}{e},\frac{qb}{f};q\right)_n}{}_8W_7\left(\frac{q^{-n-1}def}{b};q^{-n},d,e,f,\frac{q^{-n-1}cdef}{b^2};q,\frac{q}{c}\right) \tag{23}$$

$$=\frac{\left(\frac{q^2b^2}{cdef},qb,d,e,f;q\right)_n}{\left(\frac{def}{qb},\frac{qb}{c},\frac{qb}{d},\frac{qb}{e},\frac{qb}{f};q\right)_n}{}_8W_7\left(\frac{q^{1-n}b}{def};q^{-n},\frac{q^{-n}c}{b},\frac{qb}{de},\frac{qb}{df},\frac{qb}{ef};q,\frac{q}{c}\right) \tag{24}$$

$$=\frac{\left(\frac{q^2b^2}{cdef},qb;q\right)_n}{\left(\frac{qb}{c},\frac{q^2b^2}{def};q\right)_n}{}_8W_7\left(\frac{qb^2}{def};q^{-n},\frac{qb}{de},\frac{qb}{df},\frac{qb}{ef},c;q,\frac{q^{n+1}b}{c}\right) \tag{25}$$

$$=q^{\binom{n}{2}}\left(-\frac{qb}{c}\right)^n\frac{\left(\frac{qb^2}{def},\frac{qb}{ef},\frac{qb}{de},\frac{qb}{df},qb,c;q\right)_n}{\left(\frac{qb^2}{def};q\right)_{2n}\left(\frac{qb}{c},\frac{qb}{d},\frac{qb}{e},\frac{qb}{f};q\right)_n}$$
$$\times\,{}_8W_7\left(\frac{q^{-2n-1}def}{b^2};q^{-n},\frac{q^{-n}d}{b},\frac{q^{-n}e}{b},\frac{q^{-n}f}{b},\frac{q^{-n-1}cdef}{b^2};q,\frac{q^{n+1}b}{c}\right). \tag{26}$$

Proof. This corollary was presented in [4, Corollary 3], but some of the representations were missing, so we present the full result here. Start with [4, Theorem 3] and set $\mathrm{e}^{2i\theta}=q^nb$, $a_p=q^{-\frac{n}{2}}\frac{c}{\sqrt{b}}$, $a_r=q^{-\frac{n}{2}}\frac{d}{\sqrt{b}}$, $a_t=q^{-\frac{n}{2}}\frac{e}{\sqrt{b}}$, $a_u=q^{-\frac{n}{2}}\frac{f}{\sqrt{b}}$, setting $\theta\mapsto-\theta$ where necessary. Then, multiply every formula by the factor

$$\mathsf{A}_n(b,c,d,e,f|q):=\frac{q^{2\binom{n}{2}}(-1)^n(qb)^{\frac{5n}{2}}(qb;q)_n}{(cdef)^n\left(\frac{qb}{c},\frac{qb}{d},\frac{qb}{e},\frac{qb}{f};q\right)_n}.$$

With simplification, this completes the proof. □

The above corollary relates a terminating very-well-poised ${}_8W_7$ to six other representations of terminating very-well-poised ${}_8W_7$s. The following corollary which results from comparing the symmetric ${}_8W_7$ representation of the Askey–Wilson polynomials to the ${}_4\phi_3$ representations of the Askey–Wilson polynomials is directly connected to Watson's q-analog of Whipple's theorem [6, (17.9.15)]. However, beyond the single representation which is usually displayed, we are able to extend this to a total of four representations of terminating balanced ${}_4\phi_3$s.

Corollary 9. *(Watson's q-analog of Whipple's theorem [6, (17.9.15)]). Let $n \in \mathbb{N}_0$, $b, c, d, e, f \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$. Then*

$$ {}_8W_7\left(b; q^{-n}, c, d, e, f; q, \frac{q^{n+2}b^2}{cdef}\right) = \frac{(qb, \frac{qb}{ef}; q)_n}{(\frac{qb}{e}, \frac{qb}{f}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{qb}{cd}, e, f \\ \frac{q^{-n}ef}{b}, \frac{qb}{c}, \frac{qb}{d} \end{matrix}; q, q\right) \tag{27}$$

$$ = \left(\frac{qb}{cd}\right)^n \frac{(\frac{qb}{ef}, qb, c, d; q)_n}{(\frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{q^{-n}e}{b}, \frac{q^{-n}f}{b}, \frac{qb}{cd} \\ \frac{q^{-n}ef}{b}, \frac{q^{1-n}}{c}, \frac{q^{1-n}}{d} \end{matrix}; q, q\right) \tag{28}$$

$$ = \frac{\left(\frac{q^2b^2}{cdef}, qb, e; q\right)_n}{\left(\frac{qb}{c}, \frac{qb}{d}, \frac{qb}{f}; q\right)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{qb}{ec}, \frac{qb}{ed}, \frac{qb}{ef} \\ \frac{q^2b^2}{cdef}, \frac{q^{1-n}}{e}, \frac{qb}{e} \end{matrix}; q, q\right) \tag{29}$$

$$ = e^n \frac{\left(\frac{qb}{ec}, \frac{qb}{ed}, \frac{qb}{ef}, qb; q\right)_n}{\left(\frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f}; q\right)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{q^{-n-1}cdef}{b^2}, \frac{q^{-n}e}{b}, e \\ \frac{q^{-n}ec}{b}, \frac{q^{-n}ed}{b}, \frac{q^{-n}ef}{b} \end{matrix}; q, q\right). \tag{30}$$

Note that the above terminating ${}_4\phi_3$s are balanced.

Proof. Same as in the proof of Corollary 8 except applying the transformation to the terminating balanced ${}_4\phi_3$s in [4, Theorem 3]. This completes the proof. □

Remark 10. The Askey–Wilson polynomials are symmetric in their four parameters, the ${}_8W_7$ representation in which this symmetry is evident demonstrates this symmetry. On the other hand, the polynomial nature of the Askey–Wilson polynomials is not clearly evident from the ${}_8W_7$ representation. In the first ${}_4\phi_3$ representation, the polynomial nature is evident.

3.2. Converse for Watson's q-analog of Whipple's theorem

One important transformation for terminating basic hypergeometric series related to the Askey–Wilson polynomials is Watson's q-analog of Whipple's theorem [6, (17.9.15)]. This result relates a terminating balanced ${}_4\phi_3$ to a terminating very-well-poised ${}_8W_7$. The following corollary, an extension of this theorem, is a direct consequence of Corollary 9. Both of the following results directly relate a terminating balanced ${}_4\phi_3$ to a terminating very-well-poised ${}_8W_7$. The balancing condition for the terminating ${}_4\phi_3$ is $q^{1-n}abc = def$.

Corollary 11. *(Converse for Watson's q-analog of Whipple's theorem).*
Let $n \in \mathbb{N}_0$, $a, b, c, d, e, f \in \mathbb{C}^*$, $q \in \mathbb{C}^\dagger$, such that $q^{1-n}abc = def$ (balancing condition for the terminating ${}_4\phi_3$). Then

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q\right) = \frac{(\frac{f}{b}, \frac{f}{c}; q)_n}{(\frac{f}{bc}, f; q)_n} {}_8W_7\left(\frac{q^{-n}bc}{f}; q^{-n}, \frac{e}{a}, \frac{d}{a}, b, c; q, \frac{qa}{f}\right) \tag{31}$$

$$ = \frac{(\frac{ef}{bc}, \frac{e}{a}, b, c; q)_n}{(\frac{ef}{abc}, \frac{bc}{f}, e, f; q)_n} {}_8W_7\left(\frac{q^{-n}f}{bc}; q^{-n}, \frac{q^{1-n}}{d}, \frac{q^{1-n}}{e}, \frac{f}{b}, \frac{f}{c}; q, \frac{qa}{f}\right) \tag{32}$$

$$ = c^n \frac{(\frac{d}{c}, \frac{e}{c}, \frac{f}{c}, b; q)_n}{(\frac{b}{c}, d, e, f; q)_n} {}_8W_7\left(\frac{q^{-n}c}{b}; q^{-n}, \frac{d}{b}, \frac{e}{b}, \frac{f}{b}, c; q, \frac{q}{a}\right) \tag{33}$$

$$ = \frac{(\frac{e}{a}, \frac{e}{b}, \frac{e}{c}; q)_n}{(\frac{de}{abc}, \frac{e}{d}, e; q)_n} {}_8W_7\left(\frac{q^{-n}d}{e}; q^{-n}, \frac{q^{1-n}}{e}, \frac{d}{a}, \frac{d}{b}, \frac{d}{c}; q, q^n f\right). \tag{34}$$

Proof. Consider (27), then solving the following set of algebraic equations

$$\left(A, B, C, D, E, \frac{q^{1-n}BC}{DE}\right) = \left(\frac{qb}{ef}, c, d, \frac{q^{-n}cd}{b}, \frac{qb}{e}, \frac{qb}{f}\right), \tag{35}$$

gives the solution

$$(b, c, d, e, f) = \left(\frac{q^{-n}BC}{D}, B, C, \frac{q^{1-n}BC}{DE}, \frac{F}{A}\right). \tag{36}$$

Now make these replacements in (20)–(26), and solve for the ${}_4\phi_3$ in (27), while replacing $(A, B, C, D, E, F) \mapsto (a, b, c, d, e, f)$, and utilizing the balancing condition $q^{1-n}abc = def$. Note that one can write (31), (32) as equivalent expressions using the balancing condition as follows

$${}_4\phi_3\left(\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q\right) = \frac{\left(\frac{de}{ab}, \frac{de}{ac}; q\right)_n}{\left(\frac{de}{a}, \frac{de}{abc}; q\right)_n} {}_8W_7\left(\frac{de}{qa}; q^{-n}, \frac{d}{a}, \frac{e}{a}, b, c; q, \frac{qa}{f}\right) \tag{37}$$

$$= q^{\binom{n}{2}} \left(\frac{-de}{bc}\right)^n \frac{(\frac{de}{qa}, \frac{d}{a}, \frac{e}{a}, b, c; q)_n}{(\frac{de}{qa}; q)_{2n} (\frac{de}{abc}, e, d; q)_n}$$

$$\times {}_8W_7\left(\frac{q^{1-2n}a}{de}; q^{-n}, \frac{q^{1-n}}{d}, \frac{q^{1-n}}{e}, \frac{q^{1-n}ab}{de}, \frac{q^{1-n}ac}{de}; q, \frac{qa}{f}\right), \tag{38}$$

which reduces the number of inequivalent expressions by two. This completes the proof. □

4. The symmetric structure of terminating representations of the Askey–Wilson polynomials

In this section we describe the symmetric nature of the equivalence classes of expressions for the terminating basic hypergeometric representations which correspond to the Askey–Wilson polynomials.

Consider the 11 equivalence classes of terminating ${}_4\phi_3$ and ${}_8W_7$ expressions in Corollaries 8-9, namely (20)–(30). There are four equivalence classes of balanced terminating ${}_4\phi_3$ expressions (27)–(30) and 7 equivalence classes of very-well-poised terminating ${}_8W_7$ expressions (20)–(26). Equivalent expressions within an equivalence class are obtained by compositions of the trivial interchange of positions for numerator and/or denominator parameters in the basic hypergeometric series and under the 4!=24 permutations of the symmetric parameter c, d, e, f labeling.

The above described 11 equivalence classes in Corollaries 8-9 correspond to a total of 7 equivalence classes of terminating basic hypergeometric series representations of the Askey–Wilson polynomials. These are represented by 3 ${}_4\phi_3$ equivalence classes and 4 ${}_8W_7$ equivalence classes which are given in [4, Theorem 3]. Note that each of these equivalence classes are equivalent under the map $\theta \mapsto -\theta$.

In this section we describe the symmetric nature of these equivalence classes under the mapping of inversion (9) and that due to a theorem due to Van der Jeugt and Rao [19] which provides the symmetry group of nonterminating very well poised ${}_8W_7$ basic hypergeometric functions, namely Theorem 12 below. The symmetry groups of several relevant basic hypergeometric functions have been studied in the literature [10,12,13,19]. For terminating balanced ${}_4\phi_3$ expressions, the following surprisingly simple result has been established in [19, Proposition 2].

Theorem 12 *(Van der Jeugt and Rao (1999)). Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\mathbf{x} := \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $x_k \in \mathbb{C}^*$, $k \in \{1, 2, 3, 4, 5, 6\}$, be six parameters satisfying $x_{123456} = q^{1-n}$, with $f : {\mathbb{C}^*}^6 \times \mathbb{C}^\dagger \to \mathbb{C}$ defined by*

$$f(\mathbf{x}; q) := q^{\binom{n}{2}} \frac{(x_{1234}, x_{1235}, x_{1236}; q)_n}{x_{123}^n} {}_4\phi_3\left(\begin{matrix} q^{-n}, x_{23}, x_{13}, x_{12} \\ x_{1234}, x_{1235}, x_{1236} \end{matrix}; q, q\right). \tag{39}$$

Then $f(\mathbf{x})$ is symmetric in the variables x_k.

From Van der Jeugt and Rao's (1999) result, it is clear that the symmetry group of the terminating balanced ${}_4\phi_3$ is S_6, the symmetric group of degree six, $|S_6| = 720$. This was originally established by [2], although the 720 transformations were explicitly written out by Bailey [1, Chapter VII]. Upon examination of the terminating balanced ${}_4\phi_3$ expressions in Corollary 9, we see that there are four equivalence classes of basic hypergeometric representations for these expressions (27)–(30).

Remark 13. The Van der Jeugt and Rao (1999) result [19, Proposition 2] clearly indicates that the symmetry group structure of the terminating balanced ${}_4\phi_3$ is S_6, which has order equal to 720. It is interesting to make comparison of this result with the four terminating balanced expressions in Corollary 9, namely (27)–(30).

Proposition 14. *The number of allowed permutations and rearrangements of the terminating balanced ${}_4\phi_3$s (27)–(30) in Corollary 9 is $|S_6| = 720$ (where $|\cdot|$ represents the cardinality).*

Proof. There are 6 possible variable pair product combinations (cd, ce, cf, de, df, ef). In what proceeds, we ignore the positioning of the numerator factor q^{-n}. For (27), (28) there are 6 possible numerator positionings for each pair combination and 6 possible denominator positionings for each pair combination, so $|(27)| = 6^3 = 216$. Therefore $|(27), (28)| = 432$. For (29), (30), there are four variables, (c, d, e, f) and again 6 possible numerator positionings and 6 possible denominator positionings, so $|(29)| = 6 \times 6 \times 4 = 144$. Since (30) is the inversion of (29), the counting is the same. Hence, $|(29), (30)| = 288$. Finally we have $|(27), (28), (29), (30)| = 432 + 288 = 720 = |S_6|$. □

Remark 15. There is no symmetry analysis for a terminating ${}_8W_7$ which corresponds to the Van der Jeugt and Rao (1999) result for a terminating balanced ${}_4\phi_3$. They do however have a symmetry proposition for a nonterminating ${}_8W_7$, namely [19, Proposition 5], see Theorem 21 below. It is important to note that the nonterminating ${}_8W_7$ does not possess Gasper and Rahman's inversion symmetry, Theorem 2, and there is no nonterminating analog of this symmetry, so the group structure of the terminating ${}_8W_7$ is not necessarily clear. On the other hand, one has the Watson q-analog of Whipple's theorem [6, (17.9.16)] which relates a terminating balanced ${}_4\phi_3$ to a terminating very-well-poised ${}_8W_7$, so one expects there to be a one-to-one relation between these functions.

We now prove this result.

Proposition 16. *The number of allowed permutations and rearrangements of the terminating balanced ${}_8W_7$s (20)–(26) in Corollary 8 is $|S_6| = 720$.*

Proof. As in Proposition 14, ignore the positioning of the numerator factor q^{-n}. For (20), (21), there are $4! = 24$ permutations of the variables c, d, e, f. For (25), there are four triple-variable product combinations (cde, cdf, cef, def), and therefore the number of possibilities for each of the 24 possibilities is 4. Hence $|(25)| = 24 \times 4 = 96$. Its inversion pair (26) has the same number of possibilities, namely 96. For (23) one has 4 variables with four possible three-variable product combinations, for each of the four three-variable product combinations, there are 4 possible numerator parameter positions for the $cdef$ term, and 6 possible arrangements of the three remaining variables. Hence there are 24 possible positionings of the numerator parameters. Again with four possible three-variable product combinations (cde, cef, cfd, def), we arrive again at 96, and as well for its inversion pair (24), so $|(23), (24), (25), (26)| = 96 \times 4 = 384$. For (22), which is its own self-inverse, we have 48 possibilities. Since there are 6 two-variable product combinations (cd, ce, cf, de, df, ef), then one has $|(22)| = 48 \times 6 = 288$. Summing up the contributions one has $|(20), (21), (22), (23), (24), (25), (26)| = 24 \times 2 + 96 \times 4 + 288 = 720 = |S_6|$. This completes the proof. □

Table 1
Total number of arrangements for terminating balanced ${}_4\phi_3$s (27)–(30) and terminating very-well-poised ${}_8W_7$s (20)–(26) expressions (in bold) in Corollaries 8-9. The total number of possibilities, namely the possible arrangements and relabelings, sum separately to the order $|S_6| = 720$, namely for each set of equivalence classes of ${}_4\phi_3$s and ${}_8W_7$s separately. See Propositions 14, 16.

Expression equivalence class	(27)	(28)	(29)	(30)	(20)	(21)	(22)	(23)	(24)	(25)	(26)
Number of possibilities	216	216	144	144	**24**	**24**	**288**	**96**	**96**	**96**	**96**

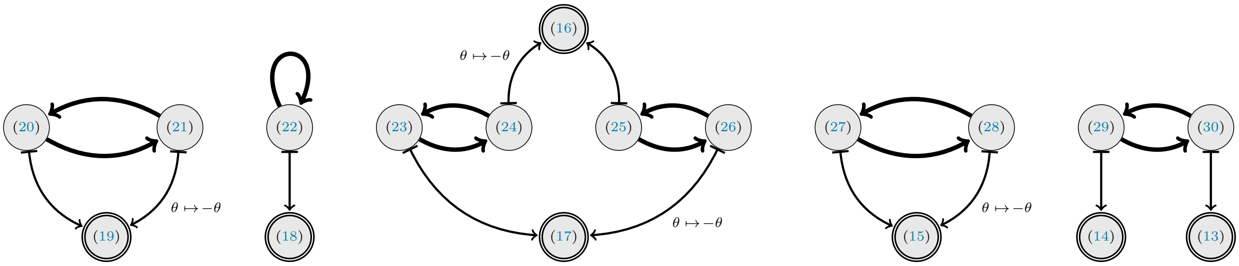

Fig. 1. This figure depicts the equivalence classes of terminating ${}_8W_7$ (20)–(26) and ${}_4\phi_3$ (27)–(30) expressions in Corollaries 8-9 and their corresponding equivalence classes of terminating Askey–Wilson basic hypergeometric representations in [4, Theorem 3]. The expressions (20)–(30) are paired (using thick arrows) using Gasper and Rahman's inversion formula (9). More specifically, to verify the inversion pairings for the ${}_4\phi_3$ expressions, one can use (10), and for the ${}_8W_7$ expressions, one can use (12), or more explicitly the equality of (20) and (21). Note that (22) is the sole expression which is its own self-inverse. For the terminating Askey–Wilson hypergeometric representation equivalence classes (13)–(19), arrows indicate which expressions in Corollaries 8-9 are mapped to under the standard map (41) to the terminating representations of the Askey–Wilson polynomials in [4, Theorem 3]. Arrows marked $\theta \mapsto -\theta$ indicate that the expressions in Corollaries 8-9 map to the same terminating Askey–Wilson basic hypergeometric representation equivalence class under this mapping.

See Table 1 for a delineation of the total number of possibilities of expressions in Corollaries 8-9.

Remark 17. Even though the set of transformations for the terminating balanced ${}_4\phi_3$s and ${}_8W_7$s each correspond to the symmetric group S_6, the breakdown of equivalence classes does not appear to be isomorphic to any of the subgroups of S_6 that the authors investigated. However there are many subgroups of S_6 (1455) [8], so future investigations may provide some insight here.

Remark 18. A straightforward analysis of the transformations implied by Theorem 12 indicates that under these transformations, each of the four equivalence classes of the balanced ${}_4\phi_3$ expressions in Corollary 9 maps using Theorem 12 separately to all three other equivalence classes.

Remark 19. Observe that the ${}_4\phi_3$ equivalence classes of expressions (27)–(30) in Corollary 9 are paired (27) $\leftrightarrow$ (28) and (29) $\leftrightarrow$ (30) under Gasper and Rahman's inversion formula, $z = q$, $r = 3$ in (10), for a terminating basic hypergeometric ${}_4\phi_3$ representation of the Askey–Wilson polynomial,

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{matrix}; q, q\right) = q^{-\binom{n}{2}}(-1)^n \frac{(a_1, a_2, a_3; q)_n}{(b_1, b_2, b_3; q)_n} \, {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \frac{q^{1-n}}{b_2}, \frac{q^{1-n}}{b_3} \\ \frac{q^{1-n}}{a_1}, \frac{q^{1-n}}{a_2}, \frac{q^{1-n}}{a_3} \end{matrix}; q, q\right), \tag{40} $$

where $q^{1-n}a_{123} = b_{123}$. Furthermore, the ${}_8W_7$ equivalence classes of expressions (20)–(21) are paired using Gasper and Rahman's inversion formula, namely the equality (20) = (21). See the thick arrows in Fig. 1 for a pictorial representation of these inversion pairings.

Remark 20. One can study the mappings of the equivalence classes of expressions in Corollaries 8-9 to the terminating representations of the Askey–Wilson polynomials in [4, Theorem 3] by using the standard map

$$ (b, c, d, e, f) \mapsto \left(q^{-n}\mathrm{e}^{2i\theta}, a_p\mathrm{e}^{i\theta}, a_r\mathrm{e}^{i\theta}, a_t\mathrm{e}^{i\theta}, a_u\mathrm{e}^{i\theta}\right). \tag{41} $$

Both expressions (27), (28), map to the basic hypergeometric representation (15), except with (28), one has $\theta \mapsto -\theta$. For the ${}_4\phi_3$ expressions under the standard map (41), the expression (29) maps to (14) and the expression (30) maps to (13). Similarly for the ${}_8W_7$ expressions using (41), then (20), (21) ($\theta \mapsto -\theta$) map to (19); (22) maps to (18); (25), (24) ($\theta \mapsto -\theta$) maps to (16); and (23), (26) ($\theta \mapsto -\theta$) maps to (17). See Fig. 1 for a pictorial representation of these mappings from Corollaries 8-9 to the terminating representations of the Askey–Wilson polynomials in Theorem 7.

Now consider the equivalence classes of terminating ${}_8W_7$ expressions in Corollary 8, namely (20)–(26). There is a surprising structure to the behavior under mappings of these equivalence classes. Let us start this discussion by reviewing what is known about the symmetry of the nonterminating ${}_8W_7$. For nonterminating very-well-poised ${}_8W_7$ expressions, the following result has been previously established in [19, Proposition 5].

Theorem 21 *(Van der Jeugt and Rao (1999)). Let* $q \in \mathbb{C}^\dagger$, $\mathbf{x} := \{x_1, x_2, x_3, x_4, x_5\}$, $x_k \in \mathbb{C}^*$, $k \in \{0, 1, 2, 3, 4, 5\}$, *be six parameters with* $f : {\mathbb{C}^*}^6 \times \mathbb{C}^\dagger \to \mathbb{C}$ *defined by*

$$f(x_0; \mathbf{x}; q) := w\left(q^{-1}x_0^3 x_{12345}; \frac{x_{012345}}{x_1^2}, \frac{x_{012345}}{x_2^2}, \frac{x_{012345}}{x_3^2}, \frac{x_{012345}}{x_4^2}, \frac{x_{012345}}{x_5^2}; q\right), \tag{42}$$

where

$$w(b; a, c, d, e, f; q) = \frac{\left(\frac{q^2b^2}{acdef}, \frac{qb}{a}, \frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f}; q\right)_\infty}{(qb; q)_\infty} {}_8W_7\left(b; a, c, d, e, f; q, \frac{q^2b^2}{acdef}\right). \tag{43}$$

Then $f(x_0; \mathbf{x}; q)$ *satisfies* $f(x_0; \mathbf{x}; q) = f(x_0; p \cdot \mathbf{x}; q)$, *for every element* $p \in WB_5$ *that has an even number of minus signs in its matrix representation. Hence the invariance group of the very-well-poised nonterminating* ${}_8W_7$ *is the group* WD_5.

Note that the groups WB_n and WD_n are the Weyl groups of the root systems of types B_n and D_n (see [9, Chapter III]). It is clear from Van der Jeugt and Rao's (1999) discussion that the symmetry group of the nonterminating very-well-poised ${}_8W_7$ is WD_5, $|WD_5| = 5!2^4 = 1920$. According to Zudilin [22] this transformation group was clear in Bailey (1964) [1, Section 7.5] which focused on a study of the transformations of the very-well-poised nonterminating ${}_7F_6$, whose q-analog is the nonterminating very-well-poised ${}_8W_7$. (See Zudilin [21, Lemma 8] for a discussion of the computation of the order and some properties of this symmetry group which is connected to the group structure of the Riemann zeta value $\zeta(3)$.)

Now we discuss the symmetric nature of the terminating ${}_8W_7$s in Corollary 8. Terminating ${}_8W_7$ expressions may be obtained from nonterminating ${}_8W_7$ expressions by setting one of the numerator parameters equal to q^{-n}, $n \in \mathbb{N}_0$. If you apply Van der Jeugt and Rao's Theorem 21 with one of the numerator parameters equal to some q^{-n}, then some subset of the transformations maps to equivalence classes for terminating expressions, and the complement maps to equivalence classes of nonterminating expressions (not explicitly treated in this paper). The result of the mappings using Theorem 21 from terminating ${}_8W_7$ equivalence classes to terminating ${}_8W_7$ equivalence classes is listed in Table 2 and displayed pictorially in Fig. 2.

Upon examination of the terminating balanced ${}_8W_7$ expressions in Corollary 8, we see that there are seven equivalence classes of terminating ${}_8W_7$ expressions (20)–(26). A straightforward computer algebra analysis of the transformations implied by Van der Jeugt and Rao's nonterminating Proposition, Theorem 21 (where we have selected only those expressions which result in terminating expressions), has indicated that under these transformations, each of the seven equivalence classes of the very-well-poised ${}_8W_7$ expressions split into three separate associations of terminating very-well-poised ${}_8W_7$ equivalence classes.

Table 2
Given that the original and mapped ${}_8W_7$ expressions are terminating, this table provides the mapping properties of the ${}_8W_7$ equivalence classes (20)–(26) under the action of Theorem 21. The numbers on the right-part of the table indicate the total number of ${}_8W_7$ expressions which are mapped using Theorem 21, given a specific choice of parameter labeling (dashes represent zero). See Fig. 2 for a graphical representation of these mapping properties.

Original ${}_8W_7$ Expression Equivalence Class	Mapped ${}_8W_7$ Expression Equivalence Classes	(20)	(25)	(21)	(24)	(22)	(23)	(26)
{(20)}	{(20), (25)}	**120**	**480**	–	–	–	–	–
{(25)}	{(20), (25)}	**120**	**480**	–	–	–	–	–
{(21)}	{(21), (24)}	–	–	**120**	**480**	–	–	–
{(24)}	{(21), (24)}	–	–	**120**	**480**	–	–	–
{(22)}	{(22), (23), (26)}	–	–	–	–	**120**	**360**	**120**
{(23)}	{(22), (23), (26)}	–	–	–	–	**120**	**360**	**120**
{(26)}	{(22), (23), (26)}	–	–	–	–	**120**	**360**	**120**

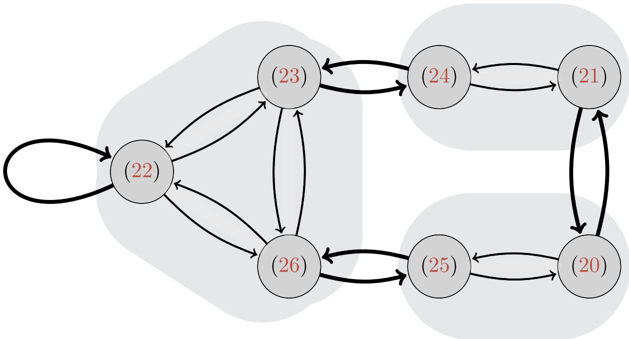

Fig. 2. This figure provides a graphical representation of Table 2 together with the action of inversion (9). More specifically, it depicts the equivalence classes of terminating very-well-poised ${}_8W_7$ expressions (20)–(26) in Corollary 8, with thick arrows indicating pairings using inversion (9), and thin arrows indicating mappings using Theorem 21. The shaded regions indicate equivalence class grouping under Theorem 21.

Table 3
This table lists the mappings and their total number of occurrences which occur if one applies the converse for Watson's q-analog of Whipple's theorem, namely Corollary 11 to the terminating balanced ${}_4\phi_3$ expressions in Corollary 9. For each ${}_4\phi_3$ expression, terminating very-well-poised ${}_8W_7$ expressions are produced when you include all permutations of the numerator parameters and denominator parameters. The numbers on the right-part of the table indicate the total number of expression equivalence classes (out of $3!^2 = 36$ permutations) mapped to for a given choice of parameter labeling. Dotted lines represent boundaries of inversion pairs.

Original ${}_4\phi_3$ Expression Equivalence Class	(20)	(21)	(22)	(23)	(24)	(25)	(26)
{(27)}	**4**	**4**	**56**	**20**	**20**	**20**	**20**
{(28)}	**4**	**4**	**56**	**20**	**20**	**20**	**20**
{(29)}	**6**	**6**	**60**	**18**	**18**	**18**	**18**
{(30)}	**6**	**6**	**60**	**18**	**18**	**18**	**18**

Remark 22. The three separate associations of equivalence classes for terminating very-well-poised ${}_8W_7$s in Corollary 8 which are obtained by applying Theorem 21 for nonterminating ${}_8W_7$ misses the connections between the three associations. In order to connect these associations, one must rely on Gasper and Rahman's inversion formula, which have no nonterminating counterpart, so therefore would be undiscoverable using Van der Jeugt and Rao's (1999) analysis [19, Proposition 5] (see Tables 3, 4 and 5).

Table 4
This table lists the mappings which occur if one applies Watson's q-analog of Whipple's theorem, namely Corollary 9, including all permutations of the numerator parameters, to the terminating very-well-poised ${}_8W_7$ expressions in Corollary 8. This results in the production of terminating balanced ${}_4\phi_3$s for each ${}_8W_7$ expression. The numbers on the right-part of the table indicate the total number of expression equivalence classes (out of $2 \cdot 4! = 48$ permutations) mapped to a given choice of parameter labeling. Dotted lines represent boundaries of inversion pairs.

Original ${}_8W_7$ Expression Equivalence Class	(27)	(28)	(29)	(30)
{(20)}	**24**	**24**	**24**	**24**
{(21)}	**24**	**24**	**24**	**24**
{(22)}	**28**	**28**	**20**	**20**
{(23)}	**30**	**30**	**16**	**16**
{(24)}	**30**	**30**	**16**	**16**
{(25)}	**30**	**30**	**16**	**16**
{(26)}	**30**	**30**	**16**	**16**

Table 5
This table describes mapping properties of the converse for Watson's q-analog of Whipple's theorem, Corollary 11. It first provides the mapping properties for the ${}_8W_7$ equivalence classes (31)–(34) which are mapped if one applies Van der Jeugt and Rao's nonterminating Proposition, Theorem 21 (where we have selected only those expressions which result in terminating expressions). The numbers on the right-part of the table indicate the total number of ${}_8W_7$ expression equivalence classes mapped to for a given choice of parameter labeling. Dashes indicate zero mappings. See Fig. 3.

Original ${}_8W_7$ Expression Equivalence Class	Mapped ${}_8W_7$ Expression Equivalence Classes	(31)	(34)	(32)	(33)
{(31)}	{(31), (34)}	360	240	–	–
{(34)}	{(31), (34)}	360	240	–	–
{(32)}	{(32), (33)}	–	–	360	240
{(33)}	{(32), (33)}	–	–	360	240

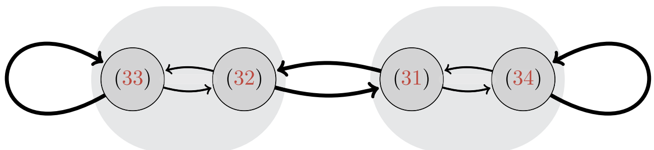

Fig. 3. This figure depicts the relation of equivalence classes of terminating very-well-poised ${}_8W_7$ expressions (31)–(34) in the converse for Watson's q-analog of Whipple's theorem, Corollary 11. Thick arrows indicate equivalence classes which are paired using Gasper and Rahman's inversion formula for terminating basic hypergeometric series (9). Thin arrows indicate which nodes map terminating ${}_8W_7$ equivalence classes to terminating ${}_8W_7$ equivalence classes using Van der Jeugt and Rao's nonterminating Proposition, Theorem 21 (where we have selected only those expressions which result in terminating expressions), see Table 2. Shaded regions indicate which equivalence classes are grouped using Theorem 21.

5. A broader perspective on the symmetric structure of the terminating representations of the Askey–Wilson polynomials

One can consider a broader interpretation of the symmetric structure of the terminating series representations of Askey–Wilson polynomials. For instance, it is understood that the Askey–Wilson polynomials are limits of Rahman's family of biorthogonal rational functions [14, (3.1)]

$$R_n(\tfrac{1}{2}(z+z^{-1}); a, b, c, d, e|q) := {}_{10}W_9\left(\frac{a^2bcde}{q}; q^{-n}, q^{n-1}abcd, az^{\pm}, abce, abde, acde; q, q\right), \qquad (44)$$

which are balanced and very-well-poised and symmetric in b, c, d, e. It was shown in [13] that the symmetry group of transformations for Rahman's biorthogonal rational functions (terminating balanced very-well-poised ${}_{10}W_9$s) is the Weyl group of the exceptional Lie group of the root system E_6, namely $W(E_6)$, where $|W(E_6)| = 51840$. If the limit of (44) is taken as $e \to 0$, then one obtains the unnormalized Askey–Wilson polynomials

$$\lim_{e\to 0} R_n(\tfrac{1}{2}(z+z^{-1}); a, b, c, d, e|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az^{\pm} \\ ab, ac, ad \end{matrix}; q, q\right) = \frac{a^n p_n(\frac{1}{2}(z+z^{-1}); a, b, c, d|q)}{(ab, ac, ad; q)_n}.$$

As stated previously, the symmetry group for the terminating series representations of the Askey–Wilson polynomials is the Weyl group of the root system (alternating group) A_5, namely $W(A_5)$ which is isomorphic to the symmetric group S_6, namely $|W(A_5)| = |S_6| = 720$ (again see [19]).

The broader perspective we seek is that of the particular elliptic hypergeometric functions which generalize Askey–Wilson polynomials. We will not go into the details regarding the definition of elliptic hypergeometric functions here, these can be found in many references including [15], [20] and see also Chapter 4 by Rosengren in [5]. If one considers Spiridonov's elliptic extension of Rahman's family of biorthogonal rational functions [16]

$$R_n(z; B, C, D, E, F; q, p) := {}_{12}V_{11}\left(\frac{BC}{q}; q^{-n}, \frac{q^{n-1}BC^3}{DEF}, Bz^{\pm}, D, E, F; q, p\right), \tag{45}$$

and one makes the substitution $(B, C, D, E, F) = (a, abcde, abce, abde, acde)$, then we produce the following elliptic representation:

$$R_n(z; a, abcde, abce, abde, acde; q, p) = {}_{12}V_{11}\left(\frac{a^2bcde}{q}; q^{-n}, q^{n-1}abcd, az^{\pm}, abce, abde, acde; q, p\right).$$

Taking the limit as $p \to 0$ using [7, (11.2.22)]

$$\lim_{p\to 0} {}_{r+1}V_r(a_1; a_6, \ldots, a_{r+1}; q, p) = {}_{r-1}W_{r-2}(a_1; a_6, \ldots, a_{r+1}; q, q)\,, \tag{46}$$

produces

$$\lim_{p\to 0} R_n(z; a, abcde, abce, abde, acde; q, p) = R_n(\tfrac{1}{2}(z+z^{-1}); a, b, c, d, e|q).$$

Hence, it is evident that Spiridonov's elliptic extension of Rahman's biorthogonal rational functions is an elliptic generalization of the Askey–Wilson polynomials. It was shown in [10] that the symmetry group for Spiridonov's elliptic extension of Rahman's biorthogonal rational functions $R_n(z; a, abcde, abce, abde, acde; q, p)$ is isomorphic to $W(E_6)$, just like in the non-elliptic case.

One obvious follow-up study would be the identification of the elliptic equivalence class sub-structure, similar to what is presented in the analysis above for terminating series representations of the Askey–Wilson polynomials. It has been pointed out to us by Ole Warnaar that the inversion transformation for elliptic hypergeometric series ${}_rE_s$ for the case $s = r - 1$ completely holds. By combining the elliptic inversion transformation along with the well-studied two-term transformations which are known for the very-well-poised ${}_{12}V_{11}$, one could further investigate the division of the action of $W(E_6)$ into equivalence classes.

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