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Research Article

Two-dimensional contiguous relations for the linearization coefficients of classical orthogonal polynomials

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ABSTRACT

By using the three-term recurrence relation for orthogonal polynomials, we produce a collection of two-dimensional contiguous relations for certain generalized hypergeometric functions. These generalized hypergeometric functions arise through linearization coefficients for some classical orthogonal polynomials in the Askey-scheme, namely Gegenbauer (ultraspherical), Hermite, Jacobi and Laguerre polynomials.

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1. Introduction

In this paper, we investigate some implications of the linearization coefficients for classical orthogonal polynomials. It might be stated that the importance of linearization formulas in the study of orthogonal polynomials was superbly highlighted in Richard Askey's famous National Science Foundation Regional Conference lecture series at Virginia Polytechnic Institute in June 1974. These lectures resulted in a beautiful set of lecture notes that Askey lovingly assembled in [1], and in particular in his beautiful chapter, Lecture 5: *linearization of products* where he discusses the importance and history of linearization formulas for Chebyshev, Gegenbauer (ultraspherical), Jacobi, Krawtchouk, Meixner, Laguerre and Hermite polynomials. One of the beautiful things about linearization coefficients for orthogonal polynomials is that they are surprisingly connected with some beautiful combinatorial problems. These combinatorial connections have been generalized and exploited by Askey and many others (for some nice reviews of this beautiful connection with linearization formulas for orthogonal polynomials, see [2,3]). Other interesting applications of linearization formulas include duality [4], positivity [5], moments [6], and addition theorems [7,8] to just scratch the surface of this deep topic.

Our application of linearization, encroaches on the study of the properties of the generalized hypergeometric series which arise in linearization coefficients. In particular, we produce two-dimensional contiguous relations for linearization coefficients of hypergeometric orthogonal polynomials in the Askey-scheme. These contiguous relations are

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derived by using integrals of products of these orthogonal polynomials. These integrals are related to linearization coefficients for the polynomials. The idea for the two-dimensional contiguous relations goes back to a paper by Ismail, Kasraoui & Zeng (2013) [9]. In this paper we re-derive the general expression for the two-dimensional contiguous relations and then apply this relation to several specific examples, namely for the linearization of a product of two and three Gegenbauer polynomials, a product of two and three Hermite polynomials, linearization of two Jacobi polynomials and for a product of two unscaled and two scaled Laguerre polynomials.

2. Preliminaries

We adopt the following list conventions as follows. Within a list of items, we define

$$a+\begin{Bmatrix}x_1\\ \vdots\\ x_n\end{Bmatrix}:=\{a+x_1,\ldots,a+x_n\},$$

and when $\pm$ is used within a list of values, we define $\pm a:=\{a,-a\}$. Let $z\in\mathbb{C}$, $n,k\in\mathbb{N}_0$ unless otherwise stated. The definition that we use for the Pochhammer symbol (shifted factorial) is given by

$$(z)_n:=(z)(z+1)\cdots(z+n-1),\ (z)_0:=1,\quad z\in\mathbb{C},$$
$$(z_1,\ldots,z_k)_n:=(z_1)_n\cdots(z_k)_n.$$

We will also adopt the following compact notation for the minimum and maximum of any two integers, $m,n\in\mathbb{Z}$,

$$m\vee n:=\max(m,n),\quad m\wedge n:=\min(m,n).$$

Define the generalized hypergeometric series [10, Chapter 16]

$${}_rF_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r)_n}{(b_1,\ldots,b_s)_n}\frac{x^n}{n!},$$

and the Kampé de Fériet double hypergeometric series [11, (28)]

$$F^{p:q;k}_{l:m;n}\left(\begin{matrix}a_1,\ldots,a_p:b_1,\ldots,b_q;c_1,\ldots,c_k\\ \alpha_1,\ldots,\alpha_l:\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n\end{matrix};x,y\right)$$
$$=\sum_{r,s=0}^{\infty}\frac{(a_1,\ldots,a_p)_{r+s}(b_1,\ldots,b_q)_r(c_1,\ldots,c_k)_s}{(\alpha_1,\ldots,\alpha_l)_{r+s}(\beta_1,\ldots,\beta_m)_r(\gamma_1,\ldots,\gamma_n)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{2.1}$$

3. Two-dimensional contiguous relations which correspond to the linearization of a product of orthogonal polynomials

In this section we derive the general relation for two-dimensional contiguous relations for linearization coefficients of unscaled orthogonal polynomials, namely Theorem 3.1 below. This theorem is similar to [9, Theorem 2.1], albeit for unscaled polynomials $p_n(x)$. In the remainder of this paper, we compute explicitly the linearization coefficients of the products of two and three orthogonal polynomials. From this we show examples of these contiguous relations for certain continuous hypergeometric orthogonal polynomials in the Askey scheme. However, in Theorem 7.7 we extend this theorem for scaled Laguerre polynomials. The general result presented in [9, Theorem 2.1] is for integrals of products of scaled orthogonal polynomials $p_n(\lambda x)$, for some $\lambda \in \mathbb{C}$.

Consider a sequence of continuous orthogonal polynomials $p_n(x)$, $m, n \in \mathbb{N}_0$, $x \in \mathbb{C}$, which satisfy the orthogonality relation

$$\int_{\mathcal{I}} p_m(x;\boldsymbol{\alpha}) p_n(x;\boldsymbol{\alpha})\,\mathrm{d}\mu = h_n(\boldsymbol{\alpha})\delta_{m,n}, \tag{3.1}$$

where $\mathcal{I}$ is the support of the measure μ and $\boldsymbol{\alpha}$ is a set of parameters. Orthogonal polynomials satisfy the following three-term recurrence relation, with the assumption $p_{-1}(x;\boldsymbol{\alpha}) := 0$,

$$p_{n+1}(x;\boldsymbol{\alpha}) = (A_n x + B_n) p_n(x;\boldsymbol{\alpha}) - C_n p_{n-1}(x;\boldsymbol{\alpha}). \tag{3.2}$$

The linearization coefficients $\mathsf{a}_{k,\mathbf{n}} := \mathsf{a}_{\mathbf{m}}$, where $\mathbf{n} := \{n_1, \dots, n_N\} \in \mathbb{N}_0^N$, $N \geq 2$, $k \in \mathbb{N}_0$, $\mathbf{m} := \{k\} \cup \mathbf{n}$, are defined using

$$p_{n_1}(x;\boldsymbol{\alpha}) \cdots p_{n_N}(x;\boldsymbol{\alpha}) = \sum_{k=0}^{n_1+\cdots+n_N} \mathsf{a}_{\mathbf{m}}\, p_k(x;\boldsymbol{\alpha}). \tag{3.3}$$

Note that even though it is true that $k \in \{0, \dots, n_1 + \cdots + n_N\}$, it may be that k has a more restricted range depending on the specific orthogonal polynomials involved.

Now consider the integral over $N+1$ orthogonal polynomials $\mathsf{P}(\mathbf{m};\boldsymbol{\alpha}) \in \mathbb{R}$, where $\mathbf{m} := \{n_1, \dots, n_{N+1}\}$ by

$$\mathsf{P}(\mathbf{m};\boldsymbol{\alpha}) := \int_{\mathcal{I}} p_{n_1}(x;\boldsymbol{\alpha}) \cdots p_{n_N}(x;\boldsymbol{\alpha}) p_{n_{N+1}}(x;\boldsymbol{\alpha})\,\mathrm{d}\mu. \tag{3.4}$$

We can formally see that integral is associated with the linearization of a product of N orthogonal polynomials as follows. Without loss of generality choose

$$n_1 \leq n_2 \leq \cdots \leq n_N \leq n_{N+1}.$$

Then after substituting (3.3) in (3.4) and using (3.1), one obtains

$$\mathsf{P} := \mathsf{P}(\mathbf{m};\boldsymbol{\alpha}) = \int_{\mathcal{I}} \sum_{k=0}^{n_1+\cdots+n_N} \mathsf{a}_{\mathbf{m}}\, p_{n_{N+1}}(x;\boldsymbol{\alpha}) p_k(x;\boldsymbol{\alpha})\,\mathrm{d}\mu = h_{N+1}(\boldsymbol{\alpha})\mathsf{a}_{\mathbf{m}}, \tag{3.5}$$

and since overlap in the orthogonality only occurs if $n_{N+1} \leq n_1 + \cdots + n_N$, the integral $\mathsf{P}(\mathbf{m};\boldsymbol{\alpha})$ will vanish otherwise. Further, define

$$\mathsf{P}_j^{\pm} := \mathsf{P}_j^{\pm}(\mathbf{m};\boldsymbol{\alpha}) := \mathsf{P}(n_1, \dots, n_{j-1}, n_j \pm 1, n_{j+1}, \dots, n_{N+1};\boldsymbol{\alpha}). \tag{3.6}$$

Theorem 3.1: *Let $N, j, k \in \mathbb{N}_0$, $1 \leq j < k \leq N+1$, $n_j, n_k \in \mathbf{m}$, $x \in \mathcal{I}$, $A_{n_j}, B_{n_j}, C_{n_j}$ satisfy the three-term recurrence relation (3.2). Then the definite integral $\mathsf{P}(\mathbf{m};\boldsymbol{\alpha})$ corresponding to* *a product of $N+1$ orthogonal polynomials defined by (3.4), satisfies the following sequence of $\binom{N+1}{2}$ contiguous relations*

$$(B_{n_j}A_{n_k} - A_{n_j}B_{n_k})\mathsf{P} - A_{n_k}\mathsf{P}_j^+ - C_{n_j}A_{n_k}\mathsf{P}_j^- + A_{n_j}\mathsf{P}_k^+ + C_{n_k}A_{n_j}\mathsf{P}_k^- = 0,$$

where the P, $\mathsf{P}_j^{\pm}$ are given by (3.5), (3.6) respectively.

Proof: Choose a quantum number $m := n_j \in \mathbf{m}$, and then evaluate P_j^+ using (3.4). Then use (3.2) to replace $p_{m+1}(x)$ with $(A_m x + B_m)p_m(x) - C_m p_{m-1}(x)$ in the integrand. Now choose a different quantum number $n := n_k \in \mathbf{m}$ and replace $xp_n(x)$ in the resulting expression with

$$xp_n(x) = \frac{1}{A_n}\left(p_{n+1}(x) - B_n p_n(x) + C_n p_{n-1}(x)\right). \tag{3.7}$$

Multiplying both sides of the resulting expression by A_n produces a five term two-dimensional contiguous relation for P involving $\mathsf{P}_j^{\pm}$ and $\mathsf{P}_k^{\pm}$. Repeating this process for all $\binom{N+1}{2}$ unique combinations of quantum numbers in $\mathbf{m}$ completes the proof. ■

Remark 3.2: For a linearization of a product of N orthogonal polynomials, choose $j \neq k \in \{1, \ldots, N\}$. The three-term recurrence relation (3.7) is symmetric under permutation of n_j and n_k. Hence each unique contiguous relation is the result of choosing two quantum numbers from $N+1$ possibilities and therefore one will obtain a sequence of $\binom{N+1}{2}$ contiguous relations for the definite integral of a product of $N+1$ orthogonal polynomials.

In the remainder of the paper, we compute examples of these contiguous relations for certain continuous hypergeometric orthogonal polynomials in the Askey scheme.

4. Gegenbauer (ultraspherical) polynomials

The Gegenbauer (or ultraspherical) polynomials can be defined as [12, (9.8.19)]

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+2\lambda \\ \lambda+\frac{1}{2}\end{matrix}; \frac{1-x}{2}\right),$$

where $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C} \setminus \{0\}$, $x \in \mathbb{C}$. Gegenbauer polynomials are orthogonal on $(-1,1)$, with orthogonality relation [12, (9.8.20)]

$$\int_{-1}^{1} C_m^{\lambda}(x)C_n^{\lambda}(x)(1-x^2)^{\lambda-\frac{1}{2}}\,\mathrm{d}x = \frac{\pi\,\Gamma(2\lambda+n)\delta_{m,n}}{2^{2\lambda-1}(n+\lambda)n!\Gamma(\lambda)^2} =: h_n^{\lambda}\delta_{m,n}, \tag{4.1}$$

where $\lambda \in (-\frac{1}{2}, \infty)\setminus\{0\}$. The three-term recurrence relation for Gegenbauer polynomials is described via (3.2) with [10, Table 18.9.1]

$$A_n = \frac{2(n+\lambda)}{n+1}, \quad B_n = 0, \quad C_n = \frac{n+2\lambda-1}{n+1}. \tag{4.2}$$

4.1. Linearization of a product of two Gegenbauer polynomials

The linearization formula for a product of two Gegenbauer polynomials is given by [10, (18.18.22)]

$$C_m^\lambda(x)C_n^\lambda(x) = \sum_{k=0}^{m} B_{k,m,n}^\lambda C_{m+n-2k}^\lambda(x), \tag{4.3}$$

where $m, n \in \mathbb{N}_0$, without loss of generality $n \geq m$, and

$$B_{k,m,n}^\lambda := \frac{(m+n+\lambda-2k)(m+n-2k)!(\lambda)_k(\lambda)_{m-k}(\lambda)_{n-k}(2\lambda)_{m+n-k}}{(m+n+\lambda-k)k!(m-k)!(n-k)!(\lambda)_{m+n-k}(2\lambda)_{m+n-2k}}. \tag{4.4}$$

According to Askey [1, (5.7)], Dougall [13] (in 1919) first stated (4.3), but did not give his proof, and Hsü [14] (in 1938) was able to prove it by induction. Using (4.1), we see that (4.3) is equivalent to the following integral of a product of three Gegenbauer polynomials

$$\mathsf{C}(k,m,n;\lambda) := \int_{-1}^{1} C_m^\lambda(x)C_n^\lambda(x)C_{m+n-2k}^\lambda(x)(1-x^2)^{\lambda-\frac{1}{2}}\,\mathrm{d}x = h_{m+n-2k}^\lambda B_{k,m,n}^\lambda. \tag{4.5}$$

Remark 4.1: Using the recurrence relations given by (4.2) with Theorem 3.1, one obtains the following contiguous relations:

$$(m+\lambda)(p+1)\mathsf{C}_1^+ + (p+2\lambda-1)(m+\lambda)\mathsf{C}_1^- \\ = (p+\lambda)(m+1)\mathsf{C}_2^+ + (p+\lambda)(m+2\lambda-1)\mathsf{C}_2^-, \tag{4.6}$$

$$(n+\lambda)(m+1)\mathsf{C}_2^+ + (m+2\lambda-1)(n+\lambda)\mathsf{C}_2^- \\ = (m+\lambda)(n+1)\mathsf{C}_3^+ + (m+\lambda)(n+2\lambda-1)\mathsf{C}_3^-, \tag{4.7}$$

$$(n+\lambda)(p+1)\mathsf{C}_1^+ + (p+2\lambda-1)(n+\lambda)\mathsf{C}_1^- \\ = (p+\lambda)(n+1)\mathsf{C}_3^+ + (p+\lambda)(n+2\lambda-1)\mathsf{C}_3^-, \tag{4.8}$$

where the subscripts j of C_j where $j \in \{1,2,3\}$, correspond to the parameters p, m, n respectively. For (4.6), (4.7), (4.8) to be satisfied, there are conditions on the parameters, namely,

$$p \in \{n-m, \ldots, n+m\}, \quad m+n-p \pm 1 \in 2\mathbb{Z}. \tag{4.9}$$

To use the contiguous relations in Remark 4.1 with (4.5), we can alter it by taking $p = m+n-2k$, to obtain $\mathsf{C}(p,m,n;\lambda)$ in terms of the linearization coefficients $B_{k,m,n}^\lambda$, namely

$$\mathsf{C}(p,m,n;\lambda) := h_p^\lambda B_{\frac{1}{2}(m+n-p),m,n}^\lambda. \tag{4.10}$$

The contiguous relations given in Remark 4.1 can then be combined with (4.10) and the conditions (4.9) to be expressed as

$$\frac{(\lambda+m)(2\lambda+p)}{(\lambda+p+1)} B_{\frac{1}{2}(m+n-p-1),m,n} + \frac{p(\lambda+m)}{(\lambda+p-1)} B_{\frac{1}{2}(m+n-p+1),m,n}$$

$$
\begin{aligned}
&= (m+1)B_{\frac{1}{2}(m+n-p+1),m+1,n} + (m+2\lambda-1)B_{\frac{1}{2}(m+n-p-1),m-1,n},\\
&(n+\lambda)(m+1)B_{\frac{1}{2}(m+n-p+1),m+1,n} + (m+2\lambda-1)(n+\lambda)B_{\frac{1}{2}(m+n-p-1),m-1,n}\\
&= (m+\lambda)(n+1)B_{\frac{1}{2}(m+n-p+1),m,n+1} + (m+\lambda)(n+2\lambda-1)B_{\frac{1}{2}(m+n-p-1),m,n-1},\\
&\frac{(\lambda+n)(2\lambda+p)}{(\lambda+p+1)}B_{\frac{1}{2}(m+n-p-1),m,n} + \frac{p(\lambda+n)}{(\lambda+p-1)}B_{\frac{1}{2}(m+n-p+1),m,n}\\
&= (n+1)B_{\frac{1}{2}(m+n-p+1),m,n+1} + (n+2\lambda-1)B_{\frac{1}{2}(m+n-p-1),m,n-1}.
\end{aligned}
\tag{4.11}
$$

Take for instance for (4.11) which follows from (4.6). Using the identities (4.1), (4.4), it reduces to

$$
\begin{aligned}
&(m+\lambda)(p+1)\mathsf{C}_1^{+} + (p+2\lambda-1)(m+\lambda)\mathsf{C}_1^{-} - (p+\lambda)(m+1)\mathsf{C}_2^{+} - (p+\lambda)(m+2\lambda-1)\mathsf{C}_2^{-}\\
&= \frac{2p!(\lambda)_{\frac{1}{2}(n+m-p-1)}(\lambda)_{\frac{1}{2}(m+p-n-1)}(\lambda)_{\frac{1}{2}(n+p-m-1)}(2\lambda)_{\frac{1}{2}(n+m+p-1)}}{\Gamma(\frac{1}{2}(n+m-p+1))\Gamma(\frac{1}{2}(m-n+p+1))\Gamma\left(\frac{1}{2}(n-m+p+1)\right)(\lambda)_{\frac{1}{2}(n+m+p-1)}(2\lambda)_{p-1}}\\
&\quad\times\left(\frac{(p+1)(\lambda+m)(2\lambda+m+p-n-1)(2\lambda+n+p-m-1)(4\lambda+n+m+p-1)}{(m+p-n+1)(n+p-m+1)(2\lambda+p-1)(2\lambda+m+n+p-1)(2\lambda+m+n+p+1)}\right.\\
&\quad-\frac{(m+1)(p+\lambda)(2\lambda+m+n-p-1)(2\lambda+m+p-n-1)(4\lambda+m+n+p-1)}{(m+n-p+1)(m+p-n+1)(2\lambda+p-1)(2\lambda+m+n+p-1)(2\lambda+m+n+p+1)}\\
&\quad-\frac{(\lambda+p)(2\lambda+m-1)(2\lambda+n+p-m-1)}{(n+p-m-1)(2\lambda+p-1)(2\lambda+m+n+p-1)}\\
&\quad\left.+\frac{(\lambda+m)(2\lambda+m+n-p-1)}{(m+n-p+1)(2\lambda+m+n+p-1)}\right).
\end{aligned}
$$

Since the rational coefficient multiplying the factorials and shifted factorials vanish, we can see that this identity is trivially satisfied. So this is a clear validation of the contiguous relations implied by Theorem 3.1. Note that similar validations can be obtained by using (4.7), (4.8), which we leave to the reader.

4.2. *Linearization of a product of three Gegenbauer polynomials*

Now we present the linearization formula for a product of three Gegenbauer polynomials.

Theorem 4.2: *Let $p, m, n \in \mathbb{N}_0$ and without loss of generality $p \le m \le n$, $\lambda \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $x \in \mathbb{C}$. Then*

$$
C_p^{\lambda}(x)C_m^{\lambda}(x)C_n^{\lambda}(x) = \sum_{k=0}^{\lfloor\frac{p+m+n}{2}\rfloor} \mathsf{F}_{k,p,m,n}^{\lambda} C_{p+m+n-2k}^{\lambda}(x), \tag{4.12}
$$

where

$$
\mathsf{F}_{k,p,m,n}^{\lambda} := \begin{cases} \mathsf{D}_{k,p,m,n}^{\lambda}, & \text{if } 0 \le k \le p-1,\\ \mathsf{E}_{k,p,m,n}^{\lambda}, & \text{if } p \le k \le \left\lfloor \dfrac{p+m+n}{2} \right\rfloor, \end{cases}
$$

$$\mathsf{D}^{\lambda}_{k,p,m,n} := \frac{(m+n)!(\lambda)_m(\lambda)_n(\lambda+m+n+p-2k)(m+n+p-2k)!(\lambda)_k(\lambda)_{p-k}(\lambda)_{m+n-k}(2\lambda)_{m+n+p-k}}{m!n!(\lambda)_{m+n}(\lambda+m+n+p-k)k!(p-k)!(m+n-k)!(\lambda)_{m+n+p-k}(2\lambda)_{m+n+p-2k}}$$
$$\times\ {}_{11}F_{10}\left(\begin{array}{c} \left\{\begin{matrix}\lambda\\ \lambda+p-k\end{matrix}\right\}, \left\{\begin{matrix}-k\\ -m\\ -n\\ -m-n+k\end{matrix}\right\}, \left\{\begin{matrix}-\lambda-m-n\\ -\lambda-m-n-p+k\end{matrix}\right\}, \frac{-\lambda+2-m-n}{2}, \frac{-2\lambda+\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}-m-n}{2} \\ 1+p-k, \left\{\begin{matrix}-\lambda+1-k\\ -\lambda+1-m\\ -\lambda+1-n\\ -\lambda+1-m-n+k\end{matrix}\right\}, \left\{\begin{matrix}-2\lambda+1-m-n\\ -2\lambda+1-m-n-p+k\end{matrix}\right\}, \frac{-\lambda-m-n}{2}, \frac{\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}-m-n}{2}\end{array};1\right),$$

$$\mathsf{E}^{\lambda}_{k,p,m,n} := \frac{(\lambda+m+n+p-2k)(m+n+2p-2k)!(\lambda)_p(\lambda)_{k-p}(\lambda)_{m+p-k}(\lambda)_{n+p-k}(\lambda)_{m+n+p-2k}(2\lambda)_{m+n+p-k}}{(\lambda+m+n+p-k)p!(k-p)!(m+p-k)!(n+p-k)!(\lambda)_{m+n+p-k}(\lambda)_{m+n+2p-2k}(2\lambda)_{m+n+p-2k}}$$
$$\times\ {}_{11}F_{10}\left(\begin{array}{c} \left\{\begin{matrix}\lambda\\ \lambda+k-p\end{matrix}\right\}, \left\{\begin{matrix}-p\\ -m-p+k\\ -n-p+k\\ -m-n-p+2k\end{matrix}\right\}, \left\{\begin{matrix}-\lambda-m-n-p+k\\ -\lambda-m-n-2p+2k\end{matrix}\right\}, \frac{-\lambda+2k-2p-m-n+2}{2}, \frac{-2\lambda+\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}+2k-2p-m-n}{2} \\ 1+k-p, \left\{\begin{matrix}-\lambda+1-p\\ -\lambda+1-m-p+k\\ -\lambda+1-n-p+k\\ -\lambda+1-m-n-p+2k\end{matrix}\right\}, \left\{\begin{matrix}-2\lambda+1-m-n-p+k\\ -2\lambda+1-m-n-2p+2k\end{matrix}\right\}, \frac{-\lambda+2k-2p-m-n}{2}, \frac{\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}+2k-2p-m-n}{2}\end{array};1\right).$$

Proof: Consider (4.12), where $\mathsf{F}^{\lambda}_{k,p,m,n}$ is to be determined. Using (4.5), one obtains the following integral of a product of four Gegenbauer polynomials which is equivalent to the linearization of a product of three Gegenbauer polynomials, namely

$$\mathsf{F}^{\lambda}_{k,p,m,n} = \frac{1}{h^{\lambda}_{m+n+p-2k}} \int_{-1}^{1} C^{\lambda}_p(x)C^{\lambda}_m(x)C^{\lambda}_n(x)C^{\lambda}_{p+m+n-2k}(x)(1-x^2)^{\lambda-\frac{1}{2}}\,\mathrm{d}x. \tag{4.13}$$

Now use (4.3) to write the product of Gegenbauer polynomials of degree m and n as a single sum over $l \in \mathbb{N}_0$, $0 \le l \le m$. This converts (4.13) into a single sum of an integral of a product of three Gegenbauer polynomials whose terms can be evaluated using (4.5). After avoiding the factor $l+p-k$ becoming negative (in which case $\mathsf{F}^{\lambda}_{k,p,m,n}$ vanishes), we obtain

$$\mathsf{F}^{\lambda}_{k,p,m,n} = \sum_{l=\max(0,k-p)}^{m} B^{\lambda}_{l,m,n} B^{\lambda}_{k-l,p,m+n-2l},$$

which breaks the linearization formula into two regions depending on k, namely

$$C^{\lambda}_p(x)C^{\lambda}_m(x)C^{\lambda}_n(x) = \sum_{k=0}^{p-1} C^{\lambda}_{p+m+n-2k}(x)\mathsf{D}^{\lambda}_{k,p,l,m} + \sum_{k=p}^{\lfloor\frac{p+m+n}{2}\rfloor} C^{\lambda}_{p+m+n-2k}(x)\mathsf{E}^{\lambda}_{k,p,l,m},$$

where

$$\mathsf{D}^{\lambda}_{k,p,m,n} := \sum_{l=0}^{m} B^{\lambda}_{l,m,n} B^{\lambda}_{k-l,p,m+n-2l}, \quad \mathsf{E}^{\lambda}_{k,p,m,n} := \sum_{l=0}^{p+m-k} B^{\lambda}_{l+k-p,m,n} B^{\lambda}_{p-l,p,m+n+2p-2k-2l}.$$

Factoring the products of linearization coefficients in terms of shifted factorials and factorials completes the proof. ∎

Remark 4.3: Setting $p=0$ in (4.12) straightforwardly produces (4.3) since the first sum vanishes and the ${}_{11}F_{10}(1)$ in the second sum is unity.

Corollary 4.4: *Let $p, m, n \in \mathbb{N}_0$ and without loss of generality $p \le m \le n$, $\lambda \in (-\frac{1}{2}, \infty) \setminus \{0\}$. Then*

$$\frac{(2\lambda)_p(2\lambda)_m(2\lambda)_n}{p!m!n!} = \frac{(2\lambda)_{p+m+n}}{(p+m+n)!} \sum_{k=0}^{\lfloor \frac{p+m+n}{2} \rfloor} \mathsf{F}^{\lambda}_{k,p,m,n} \frac{\left(\frac{-p-m-n}{2}\right)_k \left(\frac{-p-m-n+1}{2}\right)_k}{\left(\frac{-2\lambda-p-m-n+1}{2}\right)_k \left(\frac{-2\lambda-p-m-n+2}{2}\right)_k}.$$

Proof: Let $x = 1$ in (4.12) and [10, Table 18.6.1] $C^{\lambda}_n(1) = (2\lambda)_n/n!$. ■

Define the following integral of the product of four Gegenbauer polynomials

$$\mathsf{C}(p, m, n, l; \lambda) := \int_{-1}^{1} C^{\lambda}_p(x) C^{\lambda}_m(x) C^{\lambda}_n(x) C^{\lambda}_l(x) (1-x^2)^{\lambda-\frac{1}{2}}\, \mathrm{d}x,$$

where $l = p+m+n-2k$. We now present the contiguous relations for the integral of a product of four Gegenbauer polynomials.

Theorem 4.5: *Let $p, m, n \in \mathbb{N}_0$, and without loss of generality $p \le m \le n$, and $l \in \{0, \ldots, p+m+n\}$ such that $p+m+n-l \pm 1$ is even. Then*

$$\begin{aligned}
&(m+\lambda)(p+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p+1,m,n} + (p+2\lambda-1)(m+\lambda)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p-1,m,n} \\
&\quad = (p+\lambda)(m+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m+1,n} + (p+\lambda)(m+2\lambda-1) \\
&\qquad \times \mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m-1,n}, \\
&(n+\lambda)(p+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p+1,m,n} + (p+2\lambda-1)(n+\lambda)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p-1,m,n} \\
&\quad = (p+\lambda)(n+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m,n+1} + (p+\lambda)(n+2\lambda-1) \\
&\qquad \times \mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m,n-1}, \\
&(p+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p+1,m,n} + (p+2\lambda-1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p-1,m,n} \\
&\quad = \frac{(2\lambda+l)(p+\lambda)}{(\lambda+l+1)}\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m,n} + \frac{l(p+\lambda)}{(\lambda+l-1)}\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m,n}, \\
&(n+\lambda)(m+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m+1,n} + (m+2\lambda-1)(n+\lambda)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m-1,n} \\
&\quad = (m+\lambda)(n+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m,n+1} + (m+\lambda)(n+2\lambda-1) \\
&\qquad \times \mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m,n-1}, \\
&(m+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m+1,n} + (m+2\lambda-1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m-1,n} \\
&\quad = \frac{(2\lambda+l)(m+\lambda)}{(\lambda+l+1)}\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m,n} + \frac{l(m+\lambda)}{(\lambda+l-1)}\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m,n}, \\
&(n+1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m,n+1} + (n+2\lambda-1)\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m,n-1} \\
&\quad = \frac{(2\lambda+l)(n+\lambda)}{(\lambda+l+1)}\,\mathsf{F}_{\frac{1}{2}(p+m+n-l-1),p,m,n} + \frac{l(n+\lambda)}{(\lambda+l-1)}\,\mathsf{F}_{\frac{1}{2}(p+m+n-l+1),p,m,n},
\end{aligned}$$

where

$$\mathsf{F}^{\lambda}_{\frac{1}{2}(p+m+n-l\pm1),p,m,n} := \begin{cases} \mathsf{D}^{\lambda}_{\frac{1}{2}(p+m+n-l\pm1),p,m,n}, & \text{if } m+n-p+2 \le l \le m+n+p, \\ \mathsf{E}^{\lambda}_{\frac{1}{2}(p+m+n-l\pm1),p,m,n}, & \text{if } 0 \le l \le m+n-p. \end{cases}$$

Proof: Aside from the contiguous relations (4.6)–(4.8), there are three more

$$(l+\lambda)(p+1)\mathsf{C}_1^+ + (p+2\lambda-1)(l+\lambda)\mathsf{C}_1^- \\ = (p+\lambda)(l+1)\mathsf{C}_4^+ + (p+\lambda)(l+2\lambda-1)\mathsf{C}_4^-, \tag{4.14}$$

$$(l+\lambda)(m+1)\mathsf{C}_2^+ + (m+2\lambda-1)(l+\lambda)\mathsf{C}_2^- \\ = (m+\lambda)(l+1)\mathsf{C}_4^+ + (m+\lambda)(l+2\lambda-1)\mathsf{C}_4^-,$$

$$(l+\lambda)(n+1)\mathsf{C}_3^+ + (n+2\lambda-1)(l+\lambda)\mathsf{C}_3^- \\ = (n+\lambda)(l+1)\mathsf{C}_4^+ + (n+\lambda)(l+2\lambda-1)\mathsf{C}_4^-, \tag{4.15}$$

because we now have an integral of a product of four orthogonal polynomials and therefore there will be $\binom{4}{2} = 6$ contiguous relations. The parameters p, m, n and l are associated with the subscripts 1, 2, 3 and 4 respectively, and $l = p+m+n-2k$. Applying this to the six contiguous relations (4.6)–(4.8), (4.14)–(4.15), and using Theorem 4.2 completes the proof. ■

5. Hermite polynomials

The Hermite polynomials can be defined as [12, (9.15.1)]

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix}-\frac{1}{2}n, -\frac{1}{2}(n-1) \\ -\end{matrix}; -\frac{1}{x^2}\right), \tag{5.1}$$

where $n \in \mathbb{N}_0$, $x \in \mathbb{C}$. Hermite polynomials are orthogonal on $(-\infty, \infty)$, with orthogonality relation [12, (9.15.2)]

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,2^n n!\delta_{m,n} =: h_n\delta_{m,n}. \tag{5.2}$$

The recurrence relation (3.2) for Hermite polynomials is given through

$$A_n = 2, \quad B_n = 0, \quad C_n = 2n. \tag{5.3}$$

The definite integral of a product of $N \in \mathbb{N}_0$, $N \ge 2$, Hermite polynomials is defined by

$$\mathsf{H}(\mathbf{n}) := \int_{-\infty}^{\infty} H_{n_1}(x)\cdots H_{n_{N+1}}(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x,$$

where $\mathbf{n} := \{n_1, \ldots, n_{N+1}\}$. Note that $\mathsf{H}(\mathbf{n})$ has a generating function given by [15, Exercise 4.11]

$$\exp\left(2\sum_{1\le i<j\le k} t_it_j\right) = \frac{1}{\sqrt{\pi}}\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{t_1^{n_1}\cdots t_k^{n_k}}{n_1!\cdots n_k!}\mathsf{H}(\mathbf{n}). \tag{5.4}$$

5.1. Linearization of a product of two Hermite polynomials

The Hermite polynomials (5.1) have a linearization formula given by [10, (18.18.23)]

$$H_m(x)H_n(x) = \sum_{k=0}^{m} \mathsf{b}_{k,m,n} H_{m+n-2k}(x), \tag{5.5}$$

where $m, n \in \mathbb{N}_0$, without loss of generality $n \geq m$, and

$$\mathsf{b}_{k,m,n} := \frac{2^k m! n!}{k!(m-k)!(n-k)!}.$$

According to [16], the linearization formula for Hermite polynomials (5.5) is referred to as the Feldheim-Watson linearization formula for the Hermite polynomials. This formula can be traced back to papers of Feldheim and Watson in 1938 [17,18]. The definite integral corresponding to the linearization of a product of two Hermite polynomials is given by

$$\mathsf{H}(p, m, n) := \int_{-\infty}^{\infty} H_p(x)H_m(x)H_n(x)\, \mathrm{e}^{-x^2}\, \mathrm{d}x.$$

Using (5.5) and orthogonality (5.2), we see that $\mathsf{H}(p, m, n)$ is given by

$$\mathsf{H}(p, m, n) = \begin{cases} 0, & \text{if } p > n+m \text{ or } m > n+p \text{ or } n > m+p \\ & \text{or } ((p+m+n) \mod 2) = 1, \\ \dfrac{\sqrt{\pi}\, m!n!p!2^{\lfloor \frac{p+m+n}{2} \rfloor}}{\lfloor \frac{m+n-p}{2} \rfloor! \lfloor \frac{m+p-n}{2} \rfloor! \lfloor \frac{n+p-m}{2} \rfloor!}, & \text{otherwise.} \end{cases} \tag{5.6}$$

Corollary 5.1: *Let $|t_1|, |t_2|, |t_3| < 1$. Then $\mathsf{H}(p, m, n)$ has the following multilinear generating function (5.4), given by*

$$\exp(t_1t_2 + t_1t_3 + t_2t_3) = \sum_{n,m=0}^{\infty} \sum_{p=0}^{m\wedge n} \frac{2^{p+m\vee n} t_1^{2p+|n-m|} t_2^m t_3^n}{(m\wedge n - p)!\, (p + |n-m|)!\, p!}.$$

Proof: Starting with (5.4), (5.6), for a fixed $m, n \in \mathbb{N}_0$, p is non-vanishing for $|n - m| \leq p \leq n + m$. Shifting the p-index by $|n - m|$ and scaling p by a power of two to remove the remaining vanishing values of $\mathsf{H}(p, n, m)$ completes the proof. ■

Remark 5.2: Let $x \in \mathbb{C}$, $p, m, n \in \mathbb{N}_0$, and without loss of generality assume $m \leq n$. Further, let $p \in \{n - m, \ldots, n + m\}$ and $p + m + n \pm 1$ even. Combining Theorem 3.1 with the coefficients of the recurrence relation given by (5.3) yields three contiguous relations for the integral of the product of three Hermite polynomials:

$$\mathsf{H}_2^+ - \mathsf{H}_1^+ - 2p\mathsf{H}_1^- + 2m\mathsf{H}_2^- = 0, \tag{5.7}$$

$$\mathsf{H}_3^+ - \mathsf{H}_1^+ - 2p\mathsf{H}_1^- + 2n\mathsf{H}_3^- = 0, \tag{5.8}$$

$$\mathsf{H}_3^+ - \mathsf{H}_2^+ - 2m\mathsf{H}_2^- + 2n\mathsf{H}_3^- = 0. \tag{5.9}$$

Using (5.6) with (5.7) the following identity is obtained:

$$\begin{aligned}
&(m+1)\left\lfloor\frac{m+n-p-1}{2}\right\rfloor!\left\lfloor\frac{n+p-m+1}{2}\right\rfloor!\left\lfloor\frac{m+p-n-1}{2}\right\rfloor! \\
&\quad-(p+1)\left\lfloor\frac{m+n-p+1}{2}\right\rfloor!\left\lfloor\frac{n+p-m-1}{2}\right\rfloor!\left\lfloor\frac{m+p-n-1}{2}\right\rfloor! \\
&\quad-\left\lfloor\frac{m+p-n+1}{2}\right\rfloor!\left\lfloor\frac{m+n-p-1}{2}\right\rfloor!\left\lfloor\frac{n+p-m+1}{2}\right\rfloor! \\
&\quad+\left\lfloor\frac{m+n-p+1}{2}\right\rfloor!\left\lfloor\frac{m+p-n+1}{2}\right\rfloor!\left\lfloor\frac{n+p-m-1}{2}\right\rfloor! = 0.
\end{aligned}$$

This can be rearranged to the following

$$\begin{aligned}
&\left(\frac{n+p-m+1}{2}\left((m+1)-\frac{m+p-n+1}{2}\right)+\frac{m+n-p+1}{2}\left(-(p+1)+\frac{m+p-n+1}{2}\right)\right) \\
&\times\left\lfloor\frac{m+n-p-1}{2}\right\rfloor!\left\lfloor\frac{n+p-m-1}{2}\right\rfloor!\left\lfloor\frac{m+p-n-1}{2}\right\rfloor! = 0.
\end{aligned}$$

Since the coefficient multiplying the factorials vanishes, we can see that the identity is trivially satisfied. So this is a clear validation of the contiguous relations implied by Theorem 3.1. Note that similar validations can be obtained by using (5.8), (5.9), which we leave to the reader.

5.2. Linearization of a product of three Hermite polynomials

Theorem 5.3: *Let $p, m, n \in \mathbb{N}_0$ and without loss of generality $p \le m \le n$, $x \in \mathbb{C}$. Then*

$$H_p(x)H_m(x)H_n(x) = \sum_{k=0}^{\left\lfloor\frac{p+m+n}{2}\right\rfloor} \mathsf{f}_{k,p,m,n} H_{p+m+n-2k}(x), \tag{5.10}$$

where

$$\mathsf{f}_{k,p,m,n} := \begin{cases} \mathsf{d}_{k,p,m,n}, & \text{if } 0 \le k \le p-1, \\ \mathsf{e}_{k,p,m,n}, & \text{if } p \le k \le \left\lfloor\frac{p+m+n}{2}\right\rfloor, \end{cases}$$

$$\mathsf{d}_{k,p,m,n} := \frac{(m+n)!p!2^k}{k!(p-k)!(m+n-k)!}\,{}_4F_3\left(\begin{matrix}-k, -m, -n, -m-n+k, \\ 1+p-k, \frac{\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}-m-n}{2}\end{matrix};\frac{1}{4}\right),$$

$$\begin{aligned}
\mathsf{e}_{k,p,m,n} &:= \frac{m!n!(m+n+2p-2k)!2^k}{(k-p)!(m+p-k)!(n+p-k)!(p+m+n-2k)!} \\
&\quad\times {}_4F_3\left(\begin{matrix}-p, -m-p+k, -n-p+k, -m-n-p+2k, \\ 1+k-p, \frac{\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}+2k-2p-m-n}{2}\end{matrix};\frac{1}{4}\right).
\end{aligned}$$

Proof: Using (cf. [12, (9.8.34)])

$$H_n(x) = n! \lim_{\alpha\to\infty} \alpha^{-\frac{n}{2}} C_n^{\alpha}\left(\frac{x}{\sqrt{\alpha}}\right)$$

four times in (4.12), and [10, (5.11.12)], the result follows. ■

Remark 5.4: Setting $p = 0$ in (5.10) straightforwardly produces (5.5) since the first sum vanishes and the ${}_4F_3(\frac{1}{4})$ in the second sum is unity.

Define the following definite integral of a product of four Hermite polynomials

$$\mathsf{H}(k,p,m,n) := \int_{-\infty}^{\infty} H_k(x)H_p(x)H_m(x)H_n(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x.$$

Using orthogonality and the linearization relation (5.10) one produces the following corollary.

Corollary 5.5: *Let $k,p,m,n \in \mathbb{N}_0$. Then*

$$\mathsf{H}(k,p,m,n) = \begin{cases} \mathsf{E}_{k,p,m,n}, & \textit{if } 0 \le k \le m+n-p, \\ \mathsf{D}_{k,p,m,n}, & \textit{if } m+n-p+2 \le k \le m+n+p, \\ 0, & \textit{otherwise}, \end{cases} \tag{5.11}$$

where

$$\mathsf{E}_{k,p,m,n} := \frac{\sqrt{\pi}\, m!n!(p+k)!2^{\lfloor\frac{k+p+m+n}{2}\rfloor}}{\lfloor\frac{m+n-p-k}{2}\rfloor!\lfloor\frac{k+p+n-m}{2}\rfloor!\lfloor\frac{k+p+m-n}{2}\rfloor!} \times {}_4F_3\left(\begin{matrix} -k,-p,\lfloor\frac{n-m-p-k}{2}\rfloor,\lfloor\frac{m-n-p-k}{2}\rfloor \\ 1+\lfloor\frac{m+n-p-k}{2}\rfloor,\frac{\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}-k-p}{2} \end{matrix};\frac{1}{4}\right),$$

$$\mathsf{D}_{k,p,m,n} := \frac{\sqrt{\pi}\, k!p!(m+n)!2^{\lfloor\frac{p+k+m+n}{2}\rfloor}}{\lfloor\frac{k+p-m-n}{2}\rfloor!\lfloor\frac{m+n+p-k}{2}\rfloor!\lfloor\frac{m+n+k-p}{2}\rfloor!} \times {}_4F_3\left(\begin{matrix} -m,-n,\lfloor\frac{p-k-m-n}{2}\rfloor,\lfloor\frac{k-p-m-n}{2}\rfloor \\ 1+\lfloor\frac{p+k-m-n}{2}\rfloor,\frac{\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}-m-n}{2} \end{matrix};\frac{1}{4}\right).$$

Proof: Start with Theorem 5.3 and then integrate both sides using the property of orthogonality for Hermite polynomials (5.2). This leads to the following integral for the linearization coefficients, namely

$$\mathsf{f}_{k,p,m,n} = \frac{1}{h_{m+n+p-2k}} \int_{-\infty}^{\infty} H_p(x)H_m(x)H_n(x)H_{p+m+n-2k}(x)\,\mathrm{e}^{-x^2}\mathrm{d}x.$$

This allows us to write the integral of a product of four Hermite polynomials as

$$\mathsf{H}(p,m,n,l) = h_l\,\mathsf{f}_{\frac{1}{2}(p+m+n-l),p,m,n}.$$

■

Theorem 5.6: *Let $p, m, n \in \mathbb{N}_0$ and without loss of generality $p \leq m \leq n$, $k \in \{0, \ldots, p + m + n\}$. Then the contiguous relations for the integral for a product of four Hermite polynomials are given by*

$$\begin{aligned}
&\mathsf{F}_{k,p,m+1,n} - \mathsf{F}_{k,p+1,m,n} + 2m\mathsf{F}_{k,p,m-1,n} - 2p\mathsf{F}_{k,p-1,m,n} = 0,\\
&\mathsf{F}_{k,p,m,n+1} - \mathsf{F}_{k,p+1,m,n} + 2n\mathsf{F}_{k,p,m,n-1} - 2p\mathsf{F}_{k,p-1,m,n} = 0,\\
&\mathsf{F}_{k+1,p,m,n} - \mathsf{F}_{k,p+1,m,n} + 2k\mathsf{F}_{k-1,p,m,n} - 2p\mathsf{F}_{k,p-1,m,n} = 0,\\
&\mathsf{F}_{k,p,m,n+1} - \mathsf{F}_{k,p,m+1,n} + 2n\mathsf{F}_{k,p,m,n-1} - 2m\mathsf{F}_{k,p,m-1,n} = 0,\\
&\mathsf{F}_{k+1,p,m,n} - \mathsf{F}_{k,p,m+1,n} + 2k\mathsf{F}_{k-1,p,m,n} - 2m\mathsf{F}_{k,p,m-1,n} = 0,\\
&\mathsf{F}_{k+1,p,m+1,n} - \mathsf{F}_{k,p,m,n+1} + 2k\mathsf{F}_{k-1,p,m,n} - 2n\mathsf{F}_{k,p,m,n-1} = 0,
\end{aligned}$$

where

$$\mathsf{F}_{k,p,m,n} = \begin{cases} \mathsf{E}_{k,p,m,n}, & \text{if } 0 \leq k \leq m+n-p,\\ \mathsf{D}_{k,p,m,n}, & \text{if } m+n-p+2 \leq k \leq m+n+p,\\ 0, & \text{otherwise.} \end{cases}$$

Proof: In addition to the contiguous functions shown in Remark 5.2, there are three additional ones:

$$\begin{aligned}
\mathsf{H}_2^+ - \mathsf{H}_1^+ - 2p\mathsf{H}_1^- + 2m\mathsf{H}_2^- &= 0,\\
\mathsf{H}_3^+ - \mathsf{H}_1^+ - 2p\mathsf{H}_1^- + 2n\mathsf{H}_3^- &= 0,\\
\mathsf{H}_3^+ - \mathsf{H}_2^+ - 2m\mathsf{H}_2^- + 2n\mathsf{H}_3^- &= 0.
\end{aligned}$$

Combining these with (5.11), the contiguous relations of the theorem are obtained. ■

6. Jacobi polynomials

The Jacobi polynomials can be defined as [10, (18.5.7)]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right),$$

where $n \in \mathbb{N}_0$, $\alpha, \beta > -1$, $x \in \mathbb{C}$. Jacobi polynomials are orthogonal on $(-1, 1)$, with the orthogonality relation [10, Table 18.3.1]

$$\begin{aligned}
&\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\\
&\quad = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{n,m} =: h_n\delta_{n,m}.
\end{aligned} \tag{6.1}$$

The coefficients for the three-term recurrence relation for the Jacobi polynomials are given by [10, (18.9.2)]

$$\begin{aligned} A_n &:= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \\ B_n &:= \frac{(\alpha^2-\beta^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \\ C_n &:= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \end{aligned} \tag{6.2}$$

6.1. Linearization of a product of two Jacobi polynomials

The linearization formula for Jacobi polynomials was given in a fundamental work by Rahman [19, (1.9)]. Rahman expressed the linearization coefficient in terms of a ${}_9F_8(1)$. His result was given in terms of some other variables $s = n-m, j = k-n+m$, and also contained a typographical error. The typographical error was that the term $(2s-2n-\alpha-\beta)$ in Rahman's original publication should have been written as $(2s-2n-\alpha-\beta)_j$. (*Note that in [20, (2.1.1)], it was realized that Rahman's result contained a typographical error.*) The corrected and further simplified version of Rahman's result is given as follows.

Theorem 6.1 ([19]): *Let $m, n \in \mathbb{N}_0$ and without loss of generality, $n \geq m$, $\alpha, \beta \in \mathbb{C}$. Then*

$$P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{2m} \mathsf{a}_{k,m,n}^{\alpha,\beta} P_{k+n-m}^{(\alpha,\beta)}(x), \tag{6.3}$$

where

$$\begin{aligned} \mathsf{a}_{k,n,m}^{\alpha,\beta} :=& \frac{(\alpha+1,\beta+1)_n(\alpha+\beta+1)_{2n-2m}(\alpha+\beta+1)_{2m}(\alpha+\beta+1+2n-2m+2k)}{m!(\alpha+\beta+1)_m(\alpha+1,\beta+1)_{n-m}(\alpha+\beta+2)_{2n}(\alpha+\beta+1)} \\ &\times \frac{(n-m+1,\alpha+\beta+2n-2m+1,2\alpha+2\beta+2n+2,-2m,\alpha-\beta)_k}{k!(2\beta+2n-2m+2,\alpha+n-m+1,\alpha+\beta+2n+2,-\alpha-\beta-2m)_k} \\ &\times {}_9F_8\left(\begin{matrix} \beta+n-m+\frac12, \frac{\beta+n-m+\frac52}{2}, \beta+\frac12, \beta+n+1, -\alpha-m, \frac{\alpha+\beta+k+\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}{2}+n-m, \frac{-k+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ \frac{\beta+n-m+\frac12}{2}, n-m+1, \frac12-m, \alpha+\beta+n+\frac32, \frac{\beta-\alpha-k+\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}{2}, \frac{k+\left\{\begin{smallmatrix}2\\3\end{smallmatrix}\right\}}{2}+\beta+n-m \end{matrix};1\right). \end{aligned}$$

Proof: See Rahman (1981) [19, (1.9)] and [21, (3.3)] for description of correction. ■

Corollary 6.2: *Let $l, m, n \in \mathbb{N}_0$ and without loss of generality $m \leq n$ and $l \in \{n-m, \ldots, n+m\}$ using the linearization formula for Jacobi polynomials, it is possible to write the integral of the product of three Jacobi polynomials as follows:*

$$\mathsf{P}(l,m,n;\alpha,\beta) := \int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)P_l^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \,\mathrm{d}x = \mathsf{a}_{l+m-n,n,m}^{\alpha,\beta} h_l,$$

where h_l is defined in (6.1).

Theorem 6.3: *Let $l, m, n \in \mathbb{N}_0$ and without loss of generality $m \leq n$ and $l \in \{n - m, \dots, n+m\}$*

$$\begin{aligned}
&\frac{2(\alpha-\beta)(\alpha+\beta)(m-n)(\alpha+\beta+m+n+1)(\alpha+\beta+2n+1)}{(\alpha+\beta+2m)(\alpha+\beta+2n)}\mathsf{a}^{\alpha,\beta}_{l+m-n,n,m}\\
&\quad+\frac{(\alpha+\beta+m+1)(m+1)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}{(\alpha+\beta+2m+1)}\mathsf{a}^{\alpha,\beta}_{l+m-n+1,n,m+1}\\
&\quad+\frac{(\alpha+m)(\beta+m)(\alpha+\beta+2m+2)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}{(\alpha+\beta+2m)(\alpha+\beta+2m+1)}\mathsf{a}^{\alpha,\beta}_{l+m-n-1,n,m-1}\\
&\quad-(\alpha+\beta+2m+2)(\alpha+\beta+n+1)(n+1)\mathsf{a}^{\alpha,\beta}_{l+m-n-1,n+1,m}\\
&\quad-\frac{(\alpha+\beta+2m+2)(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)}{(\alpha+\beta+2n)}\mathsf{a}^{\alpha,\beta}_{l+m-n+1,n-1,m}=0,\\
&\frac{2(\alpha-\beta)(\alpha+\beta)(l-n)(\alpha+\beta+n+l+1)}{(\alpha+\beta+2l)(\alpha+\beta+2n)}\mathsf{a}^{\alpha,\beta}_{l+m-n,n,m}\\
&\quad-\frac{(\alpha+\beta+2l+2)(\alpha+\beta+n+1)(n+1)}{(\alpha+\beta+2n+1)}\mathsf{a}^{\alpha,\beta}_{l+m-n-1,n+1,m}\\
&\quad-\frac{(\alpha+n)(\beta+n)(\alpha+\beta+2l+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n)}\mathsf{a}^{\alpha,\beta}_{l+m-n+1,n-1,m}\\
&\quad+\frac{(\alpha+\beta+2n+2)(\alpha+l+1)(\beta+l+1)(\alpha+\beta+2l+1)}{(\alpha+\beta+2l+3)}\mathsf{a}^{\alpha,\beta}_{l+m-n+1,n,m}\\
&\quad+\frac{l(\alpha+\beta+l)(\alpha+\beta+2l+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+2l-1)(\alpha+\beta+2l)}\mathsf{a}^{\alpha,\beta}_{l+m-n-1,n,m}=0,\\
&\frac{2(\alpha-\beta)(\alpha+\beta)(l-m)(\alpha+\beta+m+l+1)}{(\alpha+\beta+2l)(\alpha+\beta+2l+2)(\alpha+\beta+2m)}\mathsf{a}^{\alpha,\beta}_{l+m-n,n,m}\\
&\quad-\frac{(m+1)(\alpha+\beta+m+1)}{(\alpha+\beta+2m+1)}\mathsf{a}^{\alpha,\beta}_{l+m-n+1,n,m+1}\\
&\quad-\frac{(\alpha+m)(\beta+m)(\alpha+\beta+2m+2)}{(\alpha+\beta+2m)(\alpha+\beta+2m+1)}\mathsf{a}^{\alpha,\beta}_{l+m-n-1,n,m-1}\\
&\quad+\frac{(\alpha+l+1)(\beta+l+1)(\alpha+\beta+2m+2)}{(\alpha+\beta+2l+2)(\alpha+\beta+2l+3)}\mathsf{a}^{\alpha,\beta}_{l+m-n+1,n,m}\\
&\quad+\frac{l(\alpha+\beta+l)(\alpha+\beta+2m+2)}{(\alpha+\beta+2l-1)(\alpha+\beta+2l)}\mathsf{a}^{\alpha,\beta}_{l+m-n-1,n,m}=0.
\end{aligned}$$

Proof: To determine the contiguous relations, the coefficients of the three-term recurrence relation defined in (6.2) are used. Combining these coefficients with the general form of the contiguous relation of Theorem 3.1 results in

$$\begin{aligned}
&\frac{(\alpha-\beta)(\alpha+\beta)(\alpha+\beta+2n_j)(n_j-n_k)(\alpha+\beta+n_j+n_k+1)(\alpha+\beta+2n_j+1)}{(n_j+1)(\alpha+\beta+n_j+1)(\alpha+\beta+2n_j)(n_k+1)(\alpha+\beta+n_k+1)(\alpha+\beta+2n_k)}\mathsf{P}\\
&\quad+\frac{(\alpha+\beta+2n_k+1)(\alpha+\beta+2n_k+2)}{2(n_k+1)(\alpha+\beta+n_k+1)}\mathsf{P}^{+}_{n_j}\\
&\quad+\frac{(\alpha+n_j)(\beta+n_j)(\alpha+\beta+2n_j+2)(\alpha+\beta+2n_k+1)(\alpha+\beta+2n_k+2)}{2(n_j+1)(\alpha+\beta+n_k+1)(\alpha+\beta+2n_j)(n_k+1)(\alpha+\beta+n_k+1)}\mathsf{P}^{-}_{n_j}
\end{aligned}$$

$$= \frac{(\alpha+\beta+2n_j+1)(\alpha+\beta+2n_j+2)(\alpha+n_k)(\beta+n_k)(\alpha+\beta+2n_k+2)}{2(n_j+1)(\alpha+\beta+n_j+1)(n_k+1)(\alpha+\beta+n_k+1)(\alpha+\beta+2n_k)}\mathsf{P}^+_{n_k}$$
$$+ \frac{(\alpha+\beta+2n_j+1)(\alpha+\beta+2n_j+2)}{2(n_j+1)(\alpha+\beta+n_j+1)}\mathsf{P}^-_{n_k}.$$

Then using Corollary 6.2 with the definition of the coefficient $\mathsf{a}^{\alpha,\beta}_{k,n,m}$ in Theorem 6.1 completes the proof. ■

7. Laguerre polynomials

We now consider Laguerre polynomials $L_n^\alpha(x)$ and we restrict the values of the polynomials such that $x \in \mathbb{C}$, $\Re\alpha > -1$. One may define the Laguerre polynomials in terms of its generalized hypergeometric series representation [12, Section 9.12], as follows

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right).$$

The Laguerre polynomials have the following orthogonality relation [12, (9.12.2)]

$$\int_0^\infty L_m^\alpha(x)L_n^\alpha(x)x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x = \frac{\Gamma(\alpha+1+n)}{n!}\delta_{m,n} =: h_m\delta_{m,n},$$

where $\Re\alpha > -1$ is imposed so that the orthogonality integral exists. For Laguerre polynomials, one has the following three-term recurrence relation coefficients (3.2),

$$A_n := \frac{-1}{n+1},\quad B_n := \frac{2n+\alpha+1}{n+1},\quad C_n := \frac{n+\alpha}{n+1}. \tag{7.1}$$

Define $n_1,\ldots,n_{N+1} \in \mathbb{N}_0$, $\Re\alpha > -1$,

$$\mathsf{L}(\mathbf{n};\alpha) := \int_0^\infty L^\alpha_{n_1}(x)\cdots L^\alpha_{n_{N+1}}(x)x^\alpha\mathrm{e}^{-x}\,\mathrm{d}x. \tag{7.2}$$

Using the generating function for Laguerre polynomials [12, (9.12.10)]

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^\infty L_n^\alpha(x)t^n,$$

one finds that (7.2) has the following generating function [15, (9.3.7)]

$$\sum_{n_1,\ldots,n_k=0}^\infty \mathsf{L}(\mathbf{n};\alpha)t_1^{n_1}\cdots t_k^{n_k}$$
$$= \Gamma(\alpha+1)\left((1-t_1)\cdots(1-t_k)\right)^{-\alpha+1}\left(1+\sum_{j=1}^k\frac{t_j}{1-t_j}\right)^{-\alpha-1}. \tag{7.3}$$

7.1. Linearization of a product of two Laguerre polynomials

Theorem 7.1: *Let $x \in \mathbb{C}$, $\Re\alpha > -1$, $n, m \in \mathbb{N}_0$, and without loss of generality $n \geq m$. Then*

$$L_m^\alpha(x)L_n^\alpha(x) = \frac{2^{2m}(\frac{1}{2})_m(\alpha+1)_n}{(n-m)!} \times \sum_{k=n-m}^{n+m} \frac{(-1)^{k+n+m}k!}{(\alpha+1)_k(m+n-k)!(k-n+m)!} \times {}_3F_2\left(\begin{matrix}-m-\alpha, \frac{n-m-k+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ n-m+1, \frac{1}{2}-m\end{matrix}; 1\right) L_k^\alpha(x).$$

Proof: Let $x = 1 - 2x\beta^{-1}$ in (6.3). Applying the relevant limiting relation [12, (9.8.16)]

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = L_n^\alpha(x),$$

produces

$$\lim_{\beta\to\infty} P_m^{(\alpha,\beta)}(1-2x\beta^{-1})P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = \lim_{\beta\to\infty} \sum_{k=n-m}^{n+m} P_k^{(\alpha,\beta)}(1-2x\beta^{-1})\mathsf{a}_{k,m,n}^{\alpha,\beta}.$$

One also has the following useful asymptotic result. Let $k \in \mathbb{N}_0$, $b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0\}$. Then

$$\lim_{a\to\infty} \frac{(ca+b)_k}{a^k} = c^k. \tag{7.4}$$

Using (7.4) to evaluate the limit on the right-hand side completes the proof. ∎

Define $p, m, n \in \mathbb{N}_0$, $\Re\alpha > -1$,

$$\mathsf{L}_{p,m,n}^\alpha := \int_0^\infty L_p^\alpha(x)L_m^\alpha(x)L_n^\alpha(x)x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x,$$

and using (3.5), (7.1), we have for $|n-m| \leq p \leq m+n$,

$$\mathsf{L}_{p,m,n}^\alpha = \frac{\Gamma(\alpha+1+m\vee n)(-1)^{p+n+m}2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{|n-m|!(m+n-p)!(p-|n-m|)!} {}_3F_2\left(\begin{matrix}-\alpha-m\wedge n, \frac{|n-m|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m|, \frac{1}{2}-m\wedge n\end{matrix}; 1\right). \tag{7.5}$$

Hence we have the following generating function (7.3)

$$\sum_{n,m=0}^\infty \frac{t_m^m t_n^n 2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}(\alpha+1)_{m\vee n}}{|n-m|!} \sum_{p=0}^{2(m\wedge n)} \frac{(-1)^p t_p^{p+|n-m|}}{(2(m\wedge n)-p)!\,p!}$$

$$\times\,{}_3F_2\left(\begin{matrix}-\alpha-m\wedge n,\frac{-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\1+|n-m|,\frac{1}{2}-m\wedge n\end{matrix};1\right)$$

$$=\big((1-t_p)(1-t_m)(1-t_n)+t_p(1-t_m)(1-t_n)+t_m(1-t_p)(1-t_n)$$
$$+\,t_n(1-t_p)(1-t_m)\big)^{-\alpha-1}.$$

Theorem 7.2: *Let $p,m,n\in\mathbb{N}_0, a>1$ such that $|n-m|\le p\le m+n$. Then, one has three contiguous relations for the product of the three Laguerre polynomials given by*

$$\frac{(m+\alpha)(m+n-p)\Gamma(\alpha+1+(m-1)\vee n)2^{2((m-1)\wedge n)}(\frac{1}{2})_{(m-1)\wedge n}}{|n-m+1|!(p-|n-m+1|)!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-\alpha-(m-1)\wedge n,\frac{|n-m+1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\1+|n-m+1|,\frac{1}{2}-(m+1)\wedge n\end{matrix};1\right)$$
$$-\frac{(p+1)(m+n-p)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{|n-m|!(p-|n-m|+1)!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-\alpha-m\wedge n,\frac{|n-m|-p-1+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\1+|n-m|,\frac{1}{2}-m\wedge n\end{matrix};1\right)$$
$$+\frac{(m+1)\Gamma(\alpha+1+(m+1)\vee n)2^{2((m+1)\wedge n)}(\frac{1}{2})_{(m+1)\wedge n}}{(m+n-p+1)|n-m-1|!(p-|n-m-1|)!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-\alpha-(m+1)\wedge n,\frac{|n-m-1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\1+|n-m-1|,\frac{1}{2}-(m+1)\wedge n\end{matrix};1\right)$$
$$-\frac{(p+\alpha)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{(m+n-p+1)|n-m|!(p-|n-m-1|)!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-\alpha-m\wedge n,\frac{|n-m|-p+1+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\1+|n-m|,\frac{1}{2}-m\wedge n\end{matrix};1\right)$$
$$-\frac{2(p-m)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{|n-m|!(p-|n-m|)!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-\alpha-m\wedge n,\frac{|n-m|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\1+|n-m|,\frac{1}{2}-m\wedge n\end{matrix};1\right)=0,$$

$$\frac{(n+\alpha)(m+n-p)\Gamma(\alpha+1+m\vee(n-1))4^{m\wedge(n-1)}(\frac{1}{2})_{m\wedge(n-1)}}{|n-m-1|!(p-|n-m-1|)!}$$

$$
\begin{aligned}
&\times {}_3F_2\left(\begin{matrix}-\alpha - m\wedge(n-1), \frac{|n-m-1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m-1|, \frac{1}{2} - m\wedge(n-1)\end{matrix}; 1\right) \\
&- \frac{(p+1)(m+n-p)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{|n-m|!(p-|n-m|+1)!} \\
&\times {}_3F_2\left(\begin{matrix}-\alpha - m\wedge n, \frac{|n-m|-p-1+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m|, \frac{1}{2} - m\wedge n\end{matrix}; 1\right) \\
&+ \frac{(n+1)\Gamma(\alpha+1+m\vee(n+1))2^{2(m\wedge(n+1))}(\frac{1}{2})_{m\wedge(n+1)}}{(m+n-p)|n-m+1|!(p-|n-m+1|)!} \\
&\times {}_3F_2\left(\begin{matrix}-\alpha - m\wedge(n+1), \frac{|n-m+1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m+1|, \frac{1}{2} - m\wedge(n+1)\end{matrix}; 1\right) \\
&- \frac{(p+\alpha)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{(m+n-p)|n-m|!(p-|n-m|+1)!} \\
&\times {}_3F_2\left(\begin{matrix}-\alpha - m\wedge n, \frac{|n-m|-p+1+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m|, \frac{1}{2} - m\wedge n\end{matrix}; 1\right) \\
&- \frac{2(p-n)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{|n-m|!(p-|n-m|)!} \\
&\times {}_3F_2\left(\begin{matrix}-\alpha - m\wedge n, \frac{|n-m|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m|, \frac{1}{2} - m\wedge n\end{matrix}; 1\right) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(n+\alpha)(m+n-p)\Gamma(\alpha+1+(m-1)\vee n)4^{((m-1)\wedge n)}(\frac{1}{2})_{(m-1)\wedge n}}{|n-m+1|!(p-|n-m+1|)!} \\
&\times {}_3F_2\left(\begin{matrix}-\alpha - (m-1)\wedge n, \frac{|n-m+1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m+1|, \frac{1}{2} - (m+1)\wedge n\end{matrix}; 1\right) \\
&- \frac{(m+\alpha)(m+n-p)\Gamma(\alpha+1+m\vee(n-1))4^{(m\wedge(n-1))}(\frac{1}{2})_{m\wedge(n-1)}}{|n-m-1|!(p-|n-m-1|)!} \\
&\times {}_3F_2\left(\begin{matrix}-\alpha - m\wedge(n-1), \frac{|n-m-1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2} \\ 1+|n-m-1|, \frac{1}{2} - m\wedge(n-1)\end{matrix}; 1\right) \\
&- \frac{(m+1)\Gamma(\alpha+1+(m+1)\vee n)2^{2((m+1)\wedge n)}(\frac{1}{2})_{(m+1)\wedge n}}{(m+n-p+1)|n-m-1|!(p-|n-m+1|)!}
\end{aligned}
$$

$$\times\ {}_3F_2\left(\begin{matrix}-\alpha-(m+1)\wedge n,\ \frac{|n-m-1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\ 1+|n-m-1|,\ \frac{1}{2}-(m+1)\wedge n\end{matrix};1\right)$$

$$+\frac{(n+1)\Gamma(\alpha+1+m\vee(n+1))2^{2(m\wedge(n+1))}(\frac{1}{2})_{m\wedge(n+1)}}{(m+n-p+1)|n-m+1|!(p-|n-m+1|)!}$$

$$\times\ {}_3F_2\left(\begin{matrix}-\alpha-m\wedge(n+1),\ \frac{|n-m+1|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\ 1+|n-m+1|,\ \frac{1}{2}-m\wedge(n+1)\end{matrix};1\right)$$

$$-\frac{2(m-n)\Gamma(\alpha+1+m\vee n)2^{2(m\wedge n)}(\frac{1}{2})_{m\wedge n}}{|n-m|!(p-|n-m|)!}$$

$$\times\ {}_3F_2\left(\begin{matrix}-\alpha-m\wedge n,\ \frac{|n-m|-p+\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}}{2}\\ 1+|n-m|,\ \frac{1}{2}-m\wedge n\end{matrix};1\right)=0.$$

Proof: Using Theorem 3.1, one has the following three contiguous relations:

$$(p+1)\mathsf{L}_1^+-(m+1)\mathsf{L}_2^+=(m+\alpha)\mathsf{L}_2^--(p+\alpha)\mathsf{L}_1^-+2(p-m)\mathsf{L},$$
$$(p+1)\mathsf{L}_1^+-(n+1)\mathsf{L}_3^+=(n+\alpha)\mathsf{L}_3^--(p+\alpha)\mathsf{L}_1^-+2(p-n)\mathsf{L},$$
$$(m+1)\mathsf{L}_2^+-(n+1)\mathsf{L}_3^+=(n+\alpha)\mathsf{L}_3^--(m+\alpha)\mathsf{L}_2^-+2(m-n)\mathsf{L}.$$

Using the linearization of a product of two Laguerre polynomials (expressed as an integral of a product of three Laguerre polynomials) written as a hypergeometric function ${}_3F_2(1)$ (7.5) and substituting it into the above equations completes the proof. ■

7.2. Linearization of a product of two scaled Laguerre polynomials

Consider the integral associated with linearization coefficients for a product of two scaled Laguerre polynomials

$$\mathcal{L}^\alpha_{p,m,n}(a,b):=\int_0^\infty L_p^\alpha(x)L_m^\alpha(ax)L_n^\alpha(bx)x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x, \tag{7.6}$$

where $\Re\alpha>-1$ and $\mathcal{L}^\alpha_{p,n,n}(a,b)=0$ if $p\ge n+m+1$.

Remark 7.3: One way to see that $\Re\alpha>-1$ is the necessary condition for (7.6) is as follows. The above integral can be written out in terms of hypergeometric series to become

$$\mathcal{L}^\alpha_{p,m,n}(a,b)=\frac{(\alpha+1)_p(\alpha+1)_m(\alpha+1)_n}{p!m!n!}$$
$$\times\sum_{s=0}^{p}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-p)_s(-m)_k(-n)_l a^k b^l}{(\alpha+1)_s(\alpha+1)_k(\alpha+1)_l s!k!l!}\int_0^\infty x^{s+k+l+\alpha}\mathrm{e}^{-x}\mathrm{d}x.$$

The integral then becomes $\Gamma(\alpha+1+s+k+l)$, where it is required that $\Re(\alpha+s+k+l)>-1$. Since this must be true for all values that s, k, and l take in the summation, this requirement takes the form $\Re\alpha>-1$.

We now present a theorem which describes double sum representations for this integral.

Theorem 7.4: *Let $p,m,n\in\mathbb{N}_0$, $a,b>0$, $\Re\alpha>-1$. Then*

$$\mathcal{L}^{\alpha}_{p,m,n}(a,b)=\frac{\Gamma(\alpha+1)(\alpha+1)_n(-\alpha-p)_m}{p!m!(n+m-p)!}\left(-\frac{a}{b}\right)^m b^p$$
$$\times\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-m)_k(-\alpha-m)_k(p+1)_l(\alpha+1+p)_l(p-n-m)_{k+l}}{(\alpha+p-m+1)_{k+l}(p+1)_{k+l-m}k!l!}\left(-\frac{b}{a}\right)^k b^l.$$

Proof: By inserting the definition of the Laguerre polynomials, this integral can be converted to a triple sum, which can be evaluated using the integral definition of the gamma function, namely

$$\mathcal{L}^{\alpha}_{p,m,n}(a,b)=\frac{(\alpha+1)_p(\alpha+1)_m(\alpha+1)_n}{p!m!n!}$$
$$\times\sum_{s=0}^{p}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-p)_s(-m)_k(-n)_l\Gamma(\alpha+1+s+k+l)a^kb^l}{(\alpha+1)_s(\alpha+1)_k(\alpha+1)_l\, s!k!l!}.$$

By re-writing $\Gamma(\alpha+1+k+l+s)=\Gamma(\alpha+1)(\alpha+1)_{k+l}(\alpha+1+k+l)_s$, one can write the sum over s as a terminating Gauss hypergeometric series at unity. The hypergeometric series can be evaluated using the Chu-Vandermonde identity [10, (15.4.24)]. This converts the triple sum into a double sum which produces

$$\mathcal{L}^{\alpha}_{p,m,n}(a,b)=\frac{\Gamma(\alpha+1+m)\Gamma(\alpha+1+n)}{p!m!\,\Gamma(\alpha+1)}$$
$$\times\sum_{k=0}^{m}\sum_{l=0}^{n-p+k}\frac{(-1)^{l+k}(-m)_k(p+1)_l(\alpha+1+p)_la^kb^{p+l-k}}{(\alpha+1)_k(p+1)_{l-k}(\alpha+1+p)_{l-k}(n-p+k-l)!k!l!}.$$

In order to extend the sum indices to infinity, one can reverse the order of the k index by setting $k\mapsto m-k$. Performing a series of standard manipulations for Pochhammer symbols produces the final form which completes the proof. ■

An alternative form of the integral $\mathcal{L}^{\alpha}_{p,m,n}(a,b)$ follows by utilizing the identity in Theorem 7.4, namely

$$(p+1)_{r-m}=\frac{(p-m)!}{p!}(p-m+1)_r, \tag{7.7}$$

which is valid for $p\geq m$ (see also [22]).

Corollary 7.5: *Let $p, m, n \in \mathbb{N}_0$, $p \geq m$, $a, b > 0$, $\Re\alpha > -1$. Then*

$$\mathcal{L}^{\alpha}_{p,m,n}(a,b) = \frac{\Gamma(\alpha+1+n)(-\alpha-p)_m}{m!(p-m)!(n+m-p)!}\left(-\frac{a}{b}\right)^m b^p\, Z^{\alpha}_{p,m,n}(a,b),$$

where

$$Z^{\alpha}_{p,m,n}(a,b) = \sum_{l,k=0}^{\infty} \frac{(p-n-m)_{k+l}(-m,-\alpha-m)_k(p+1,\alpha+1+p)_l}{(p-m+1,\alpha+p-m+1)_{k+l}\,k!l!}\left(-\frac{b}{a}\right)^k b^l$$
$$= F^{1:2;2}_{2:0;0}\left(\begin{matrix} p-n-m : -m, -\alpha-m; p+1, \alpha+p+1 \\ p-m+1, \alpha+p-m+1 : -; - \end{matrix} ; -\frac{b}{a}, b\right), \tag{7.8}$$

where $F^{1:2;2}_{2:0;0}$ is terminating Kampé de Fériet double hypergeometric series (2.1).

Proof: The terminating double hypergeometric Kampé de Fériet form (2.1) of the integral $\mathcal{L}^{\alpha}_{p,m,n}(a,b)$ in Corollary 7.5 follows from Theorem 7.4 by utilizing the identity (7.7) ($p \geq m$) with $r = k + l$ and (2.1). This completes the proof. ■

Remark 7.6: Note that the $a = b = 1$ linearization formula corresponding to (7.8) may be found [18, p. 32] whose linearization coefficients are given in terms of a terminating ${}_3F_2(1)$. However Watson's formula must be taken with special care as the terminating ${}_3F_2(1)$ may be undefined. This is demonstrated by the case $n = 3, m = 2$. Watson sums the ${}_3F_2(1)$ over a sum index $M \in \{n-m, \ldots, n+m\} = \{1, \ldots, 5\}$. Then for $M = 1$, both denominator parameters will be less than or equal to zero.

Theorem 7.7: *Let $p, m, n \in \mathbb{N}_0$, $a, b, > 0$, $\Re\alpha > -1$. Then the contiguous relations for the integral of the product of three Laguerre polynomials, which two are scaled is the following:*

$$a(p+\alpha)\mathcal{L}_1^- - (m+\alpha)\mathcal{L}_2^- + (2m+\alpha+1)\mathcal{L}$$
$$- a(2p+\alpha+1)\mathcal{L} - (m+1)\mathcal{L}_2^+ + a(p+1)\mathcal{L}_1^+ = 0,$$
$$b(p+\alpha)\mathcal{L}_1^- - (n+\alpha)\mathcal{L}_3^- + (2n+\alpha+1)\mathcal{L}$$
$$- b(2p+\alpha+1)\mathcal{L} - (n+1)\mathcal{L}_3^+ + b(p+1)\mathcal{L}_1^+ = 0,$$
$$b(m+\alpha)\mathcal{L}_2^- - a(n+\alpha)\mathcal{L}_3^- + a(2n+\alpha+1)\mathcal{L} - b(2m+\alpha+1)\mathcal{L}$$
$$- a(n+1)\mathcal{L}_3^+ + b(m+1)\mathcal{L}_2^+ = 0,$$

where

$$\mathcal{L} := \mathcal{L}(p,m,n;\alpha;a,b),$$
$$\mathcal{L}_1^{\pm} := \mathcal{L}(p\pm 1,m,n;\alpha;a,b),$$
$$\mathcal{L}_2^{\pm} := \mathcal{L}(p,m\pm 1,n;\alpha;a,b),$$
$$\mathcal{L}_3^{\pm} := \mathcal{L}(p,m,n\pm 1;\alpha;a,b).$$

Proof: Again Theorem 3.1 will be used with the coefficients of the three-term recurrence relations of Laguerre polynomials. However, because we are now working with scaled Laguerre polynomials, one of these coefficients is now modified to the following:

$$A_n = -\frac{a}{n+1}.$$

Where a is the scaling factor of the Laguerre polynomial $L_n^\alpha(ax)$. The other two coefficients seen in (7.1) are unchanged. This can be found by inserting the scaled Laguerre polynomial into the three-term recurrence relation and solving for the coefficients. ■

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